Fermi Questions

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Solutions for Fermi Questions, January 2012

Question 1: Shoveling snow

What volume of snow will be shoveled, plowed, or otherwise removed from paved surfaces in the United States this winter?

Answer: In order to answer this question, we need to estimate the area to be cleared and the depth of the snow. Let's start with the area to be cleared. We will ignore Alaska since, despite its huge area, it is very sparsely populated and only a tiny fraction of it is plowed. We will assume that all paved areas (roads, driveways, parking lots, and airports) are cleared, so we need to estimate that area.

We can estimate this in a few ways. We can try bounding the answer. The paved area is less than 10% of the total available surface area and more than 0.1%, so we can estimate that it is 1%. Since the continental United States is about 4×10^3 km wide and 2×10^3 km "high," the total area is 10^7 km^2. This gives a paved area of 10^5 km^2.

Alternatively, we can estimate the paved area per person. The street in front of the average detached house will be 30 m long (more than 10 m and less than 100 m) and 10 m wide (more than 3 m and less than 30 m). Note that there will be a lot less street per house in cities and a lot more in rural areas. This street is shared with the house opposite, but we also need to include major roads, parking lots, and the roads in front of shops and businesses. Those two effects will offset each other, so we will use 300 m^2 of street per household. With 3×10^8 people in the United States and about 3 people per household, this gives a total paved area of

$$A = (3\times10^2 \text{ m}^2/\text{house})(0.3 \text{ house/people})(3\times10^8 \text{ people})$$
$$= 3\times10^{10} \text{ m}^2 = 3\times10^4 \text{ km}^2$$

so the two estimates agree within a factor of 10.

Estimating the paved area per car will be left as an exercise for the reader.

Now we need to estimate the depth of the annual snow fall. About half of the continental United States receives a significant amount of snow. (This leaves out cities whose snow removal "plan" is to wait for spring.) They receive more than 10 cm of snow and less than 10 m (except for Buffalo, NY) each year so we will estimate an average snow burden of 1 m.

This means that the volume of snow shoveled, plowed, or otherwise removed each year in the continental United States is (using the first area estimate)

$$V = Ah = (0.5)(10^{11} \text{ m}^2)(1 \text{ m}) = 5\times10^{10} \text{ m}^3 = 50 \text{ km}^3.$$

Note that this is approximately equal to the total volume of all buildings in the United States (Fermi Column, *TPT*, March 2008).

Question 2: Orbiting a black hole

We know that the center of the Milky Way contains a supermassive black hole because we have observed a star orbiting it with a very small minimum distance ("perihole"). What is the maximum speed of that star?

Answer: In order to estimate the maximum speed of the star, we need to estimate the mass of the black hole and the minimum distance to the black hole, "perihole." Since the mass of a galaxy is about 10^{11} $M_\odot$, the mass of the black hole will be more than 10^6 $M_\odot$ and less than 10^{10} $M_\odot$, giving a geometric mean of 10^8 $M_\odot$.

If you forget that the mass of the Sun is $M_\odot = 2\times10^{30}$ kg, or that $M_\odot = 10^6$ M_E, you can recalculate it from the period and radius of Earth's orbit. This gives

$$M_{BH} = 2\times10^{38} \text{ kg}.$$

The minimum distance from the star to the black hole will be more than 1 light-year and less than 10^3 l-y so we will estimate that perihole is 10^2 l-y. Since

$$1 \text{ l-y} = (3\times10^8 \text{ m/s})(\pi\times10^7 \text{s}) = 10^{16} \text{ m},$$

at perihole, $r = 10^{18}$ m.

The speed of the star at perihole will be greater than its speed in a circular orbit and less than escape velocity. Since the difference between circular orbit speed and escape velocity is only $\sqrt{2}$, we will just use escape velocity:

$$\frac{1}{2}mv^2 = \frac{GMm}{R}$$
$$v = \sqrt{2GM/R}$$
$$= [2(7\times10^{-11}\text{Nm}^2/\text{kg}^2)(2\times10^{38}\text{kg})/(10^{18}\text{m})]^{0.5}$$
$$= [3\times10^{10}(\text{m/s})^2]^{0.5}$$
$$= 2\times10^5 \text{m/s} = 200 \text{ km/s}.$$

This is about six times faster than Earth's orbital speed. Note that, because of the square root, factors of 10 errors in the mass or the distance only cause factors of three errors in the velocity.