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ON HOW TO ALLOCATE THE FIXED COST OF TRANSPORT SYSTEMS*

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Abstract

In this study, we consider different cities located along a tram line. Each city may have one or several stations and information is available about the flow of passengers between any pair of stations. A fixed cost (salaries of the executive staff, repair facilities, or fixed taxes) must be divided among the cities. This cost is independent of the number of passengers and the length of the line. We propose a mathematical model to identify suitable mechanisms for sharing the fixed cost. In the proposed model, we propose, and characterize axiomatically, three rules, which include the uniform split, the proportional allocation and a intermediate situation. The analyzed axioms represent the basic requirements for fairness and elemental properties of stability.

Keywords: axiom, cost game, cost sharing, fairness, transport networks.

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1 Introduction

Determining how to divide the costs of constructing and maintaining different types of infrastructure has become increasingly important because it requires the cooperation of several institutions, states, regions, or countries. Special cases such as transport networks are: railroads that are planned at European level and cross more than one country, highways that involve several regions, or metro and tram lines that span across different cities. The problem of how to distribute the cost among the participating agents (countries, regions, cities,...) has been studied in many works. These papers focused on costs that could vary according to the intensity of use or the length of the infrastructure. In other words, the cost is a function of the rest of the elements of the problem. However, in this paper we present a novel method for analyzing the allocation of fixed costs. For example, the salaries of the executive staff, the maintenance of the railway yard, the payment of some fixed local taxes, and other expenses that do not depend on the usage, the length, the number of stations, etc. Our goal is to develop several schemes in order to determine the contribution of each agent to this total fixed cost.

Let us assume that a tram line passes through different cities and each city may have one or several stations. Information is available about the number of passengers between any pair of stations, and thus how many people use each station (which can be treated as an indicator of its importance). Finally, a cost must be divided among the cities involved on the line. In summary, the *problem* has four elements comprising the set of cities located along the line, the sets of stations that belong to each city, a flow matrix that indicates the number of users between any pair of stations, and a cost that needs to be split (which is not a function of the previous elements, unlike the problems considered in other studies).

The present study addresses the axiomatic analysis of cost allocation rules in networks, where a *rule* is simply a mechanism for distributing the cost among the cities. Economic networks and the axiomatic methodology were surveyed by Sharkey (1995) and Thomson (2001), respectively. In the axiomatic method, the rules are justified in terms of the axioms or properties that they satisfy. In general, suitable combinations of properties are imposed as the desirable or minimal requirements that the rule must satisfy. The goal is then to identify the solutions or unique solution that satisfy these axioms. Thus, in this study, we introduce a collection of properties that are suitable for the framework considered.

The first group of properties imposes the basic requirements in terms of equity in the allocation of the cost. In particular, *null municipality* and *weak null municipality* state the conditions under which a city should be exempted from contributing. *Symmetry* and *adjacent symmetry* require that cities which can be consider as equals and they should pay equal amounts. The second group of properties are related to the stability of the rule with respect to changes in the problem. Thus, the *weighted additivity* requires that the final distribution of the cost is not altered if we split the problem into several subproblems (e.g., distributing the cost yearly is equivalent to spreading it monthly and then aggregating). The *bilateral ratio consistency* requires that the relative ratio of the contributions by two cities does not change if a third leaves the consortium. Finally, *trip decomposition* requires that the distribution of the cost is not altered when passengers split the same long trip into smaller ones.

We show that if symmetry, bilateral ratio consistency, and weighted additivity are required, then we must distribute the cost uniformly among the cites, regardless of the number of stations and the flows of passengers. We also show that a unique rule exists that is compatible with null municipality, symmetry, and weighted additivity. This rule is the *station-based proportional rule*, which divides the cost proportionally according to the number of users in each city. Finally, we prove that the *track-based proportional rule* is the only method that satisfies adjacent symmetry, weak null municipality, trip decomposition, and weighted additivity.

Note that we have also considered the cooperative game theory approach. However, due to the nature of the problem where the cost is fixed, the natural characteristic function for the corresponding cost game is constant (assigning the same fixed cost to any non-empty coalition). Thus, the standard solution concepts such as the Shapley value, the core, or the nucleolus yield unsatisfactory allocations. Therefore, we employ a more direct approach. However, as we explain in the final comments, the rules that we characterize and the axioms considered are related to those for characterizing the Shapley value.

Related literature

The problem of allocating the cost of building or maintaining a facility among all agents who are involved in providing a service is a classic problem addressed in cost sharing research. In the well-known airport problem, the Shapley value (Shapley (1953)) is obtained with low computational effort (see Littlechild & Owen (1973)). In this problem, the costs of using an airstrip must be shared among the planes that operate in the airport while considering their different sizes. The infrastructure of this problem is a line, as found in our problem, that is shared from the start by all of the planes. The different sizes of planes need different lengths of tracks, so the value function for the associated cost game varies according to the coalition of players. By contrast, the cost is always the same and the value function is constant in our model. A broad survey of airport problems was provided by Thomson (2014). The idea proposed by Littlechild & Owen (1973) was generalized as the so-called "painting stories" by Maschler et al. (2010) and Bergantiños et al. (2014).

The costs associated with highway profit sharing were considered by Kuipers et al. (2013) and Sudhölter & Zarzuelo (2017), where each agent used consecutive sections of a highway. This problem differs from airport problems because the sections used by an agent do not need to start from the beginning of the line.

Several studies have addressed cost sharing for the problem of cleaning a polluted river. In this problem, the river is treated as a segment divided into subsegments where each belongs to a region/municipality. A central agency determines the cost of cleaning each segment. The seminal study of this problem was conducted by Ni & Wang (2007) who proposed two methods (*local responsibility sharing* and *upstream equal sharing*) for determining the allocations, which are the Shapley values of two transferable utility (TU) games. van den Brink & van der Laan (2008) showed that this problem is essentially an airport problem. Gómez-Rúa (2011) provided axiomatic characterizations for a family of rules by using properties based on water taxes, where one of these rules matched with the weighted Shapley value. Alcalde-Unzu et al. (2015) characterized an *upstream responsibility rule* for assigning a region located in a segment with a value in terms of its responsibility by taking as the transfer rate as the mid-point between its lower and upper limits. The remaining cost was divided among the upstream regions. The problem of cleaning a polluted river essentially considers the line along which the agents are located, as found in our problem. However, the river only flows downstream, whereas tram lines carry passengers in both directions. In addition, the overall cost of cleaning the river may be related to the pollution generated by each region.

An interesting application of cost allocation in public transport was provided by Sánchez-Soriano et al. (2002) who considered how to share the cost of transport for university students. Other studies also investigated the distribution of transportation. Thus, Algaba et al. (2017) studied the problem of sharing revenues among transport companies in a multimodal transport system that cooperates by offering tickets for using all transport modes. In this model, multiple arcs between two nodes represent the different companies that provide services between each pair of stations. Similar to our model, the intensity of use is represented by a matrix of flows. They introduced the colored egalitarian solution, which is the Shapley value of a convex TU-game and it is located in the core. Another interesting situation was studied by Slikker & Nouweland (2000) and Norde et al. (2002) who investigated the allocation of the fixed and variable costs of a railway network among the trains using this infrastructure.

Research into revenue and cost sharing in networks has also provided satisfactory results in other situations such as analyses of a terrestrial flight telephone system (see van den Nouweland et al. (1996)) and power networks (see Bergantiños & Martínez (2014)). A key feature of our model is that we always consider the same cost. Thus, in contrast to all of the aforementioned studies, in our model, if we consider the related TU-game, the characteristic function is constant and it does not depend on coalitions. Therefore we fill a gap in the literature of cost allocation in transport networks and similar.

The remainder of this paper is organized as follows. In Section 2, we present the model, the elements of the problem, and several rules. In Section 3, we introduce the properties used throughout this study. In Section 4, we present characterizations of the problem components. We conclude by giving some final remarks in Section 6.

2 Mathematical model and rules for the allocation of fixed costs in a transport system

Let $M = \{1, \ldots, m\}$ $(m \geq 3)$ be the set of **municipalities**. Let $S = \{s_1, \ldots, s_n\}$ be an ordered set of **stations**, which are located on a line. For a given station s_h we assume that s_{h-1} and s_{h+1} are located to the left and right of s_h, respectively. Each station belongs to one (and only one) municipality. We denote by S_i the set of stations in municipality i. We assume that all S_i are connected with respect to the line, i.e., if two stations belong to i, then any intermediate station also belongs to i. More formally, if $s_h, s_{h+l} \in S_i$ then $s_g \in S_i$ for all $g \in [h, h+l]$.

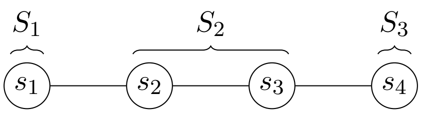
The flows of passengers is described by a **flow matrix** (denoted by OD), which specifies the number of people that use the line between each pair of stations.

$$OD = \begin{pmatrix} 0 & \omega_{12} & \dots & \omega_{1n} \\ \omega_{21} & 0 & \dots & \omega_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n1} & \omega_{n2} & \dots & 0 \end{pmatrix} \in \mathbb{R}_+^{n \times n},$$

where ω_{gh} is a measure of the number of passengers whose trip starts in station s_g and end in station s_h. We assume that at least one entry of OD is different from zero. Finally, the network has a **fixed cost**, $C \in \mathbb{R}_+$, which must be distributed among the municipalities in M

The allocation problem, or simply the **problem**, is defined by the 4-tuple $a = (M, S, OD, C)$. The class of all these allocation problems is denoted by $\mathbb{A}$.

Example 1. *Consider the case of a trolley line that passes across three municipalities* $M = \{1, 2, 3\}$ *with four stations* $S = \{s_1, s_2, s_3, s_4\}$ *that are distributed as follows:*



The fixed cost is $C = 12$ *and the flow matrix is*

$$OD = \begin{pmatrix} 0 & 4 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 5 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where, $\omega_{12} = 4$ *means that four people are traveling from* s_1 *to* s_2*, and* $\omega_{21} = 1$ *indicates that only one person is traveling from* s_2 *to* s_1*.*

For a given flows matrix OD and a given pair of municipalities $\{i, j\} \subseteq M$, the number of passengers whose trip starts in one of the stations in municipality i and ends in one of the stations in municipality j is denoted by $\Omega_{ij}(OD)$, i.e.:

$$\Omega_{ij}(OD) = \sum_{s_g \in S_i} \sum_{s_h \in S_j} \omega_{gh}$$

Note that when $i = j$, $\Omega_{ii}(OD)$ gives the number of people who travel within municipality i. Similarly, we define $\Omega_i^+(OD)$ and $\Omega_i^-(OD)$ as the number of passengers that start and end in any of the stations of municipality i, respectively:

$$\Omega_i^+(OD) = \sum_{s_g \in S_i} \sum_{s_h \in S} \omega_{gh} = \sum_{j \in M} \Omega_{ij}(OD)$$

and

$$\Omega_i^-(OD) = \sum_{s_g \in S_i} \sum_{s_h \in S} \omega_{hg} = \sum_{j \in M} \Omega_{ji}(OD)$$

Thus, the use of the stations in i is given by:

$$\Omega_i(OD) = \Omega_i^+(OD) + \Omega_i^-(OD)$$

Finally, $\Omega(OD)$ denotes the total number of passengers involved in the flow matrix:

$$\Omega(OD) = \|OD\|_1 = \frac{1}{2} \sum_{i=1}^{m} \Omega_i(OD)$$

Example 2. *For Example 1, we find that* $\Omega(OD) = 15$*,*

$$\begin{pmatrix} \Omega_{11}(OD) & \Omega_{21}(OD) & \Omega_{31}(OD) \\ \Omega_{12}(OD) & \Omega_{22}(OD) & \Omega_{32}(OD) \\ \Omega_{13}(OD) & \Omega_{23}(OD) & \Omega_{33}(OD) \end{pmatrix} = \begin{pmatrix} 0 & 6 & 0 \\ 6 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

i	$\Omega_i^+(OD)$	$\Omega_i^-(OD)$	$\Omega_i(OD)$
1	*6*	*6*	*12*
2	*9*	*9*	*18*
3	*0*	*0*	*0*

An **allocation** for $a \in \mathbb{A}$ is a distribution of the fixed cost among the municipalities, i.e., a vector $x \in \mathbb{R}_+^M$ such that $\sum_{i \in M} x_i = C$. Let $X(a)$ be the set of all allocation vectors for $a \in \mathbb{A}$. A **rule** is a procedure for selecting allocation vectors, i.e., a function, $R : \mathbb{A} \longrightarrow \bigcup_{a \in \mathbb{A}} X(a)$, that selects a unique allocation vector $R(a) \in X(a)$ for each problem $a \in \mathbb{A}$.

Now, we present some examples of rules. The first is the *uniform rule* and it simply requires that the cost is split equally among the municipalities.

Uniform rule. For each $a \in \mathbb{A}$ and each $i \in M$,

$$U_i(a) = \frac{C}{m}$$

This is the only rule we propose that does not take into account the information available on passenger traffic on the tram line, but it is used as a benchmark because it coincides with the Shapley value and the nucleolus of the associated cooperative cost game. The following two rules do use all the information, the first uses the stations and the second uses the segments between stations as structural elements that define the network.

The next rule allocates the cost in proportion to the number of passengers who use the stations (to board or to get off) in a municipality.

Station-based proportional rule. For each $a \in \mathbb{A}$ and each $i \in M$,

$$SP_i(a) = \frac{C}{2\Omega(OD)} \cdot \Omega_i(OD)$$

The last rule allocates the fixed cost in proportion to the use of segments of the line. To do that, it assumes that each passenger is divided into as many parts as the number of tracks used in her trip, and then each part of the passenger corresponding to a track is distributed equally between the two stations delimiting that track.

Track-based proportional rule. For each $a \in \mathbb{A}$ and each $i \in M$,

$$TP_i(a) = \frac{C}{\Omega(OD)} \cdot \sum_{s_f \in S_i} \sum_{\substack{s_g, s_h \in S, s_g \neq s_h \\ f \in [g,h] \text{ or } f \in [h,g]}} \frac{\omega_{gh}}{\left(2 - \left\lceil \frac{|f-g| \cdot |h-f|}{(h-g)^2} \right\rceil \right) |h-g|},$$

where $\lceil z \rceil = min \{k \in \mathbb{Z} : k \geq z\}$.

In the next example we illustrate the functioning of the rules defined above.

Example 3. *We consider the problem described in Example 1.*

(a) Uniform rule:

$$U(a) = \left(\frac{12}{3}, \frac{12}{3}, \frac{12}{3}\right) = (4, 4, 4)$$

(b) Station-based proportional rule:

$$SP(a) = \left(\frac{12}{30} \cdot 12, \frac{12}{30} \cdot 18, \frac{12}{30} \cdot 0\right) = \left(\frac{24}{5}, \frac{36}{5}, 0\right)$$

(c) Track-based proportional rule: We use this example to illustrate the function of the formula in the definition of this rule. We start by counting the number of passengers between each pair of stations. As illustrated in Figure 1, five passengers are traveling between s_1 and s_2, seven between s_1 and s_3, and so on. Second, we compute the use of each track. In the case of the track between s_1 and s_2, it is natural to assign to this track, the five travelers between s_1 and s_2, half of the travelers between s_1 and s_3 (because it is a half of their whole trip), and a third of the travelers between s_1 and s_3 (because this track represents only a third of

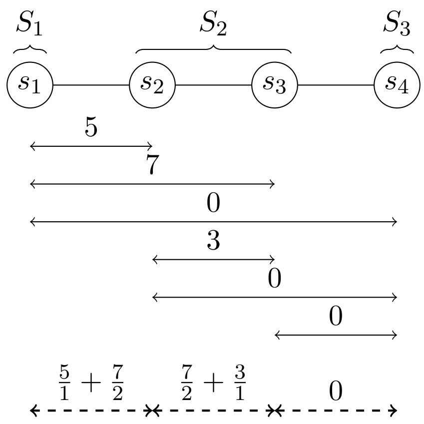


Figure 1: Computation of the track-based proportional rule (first step).

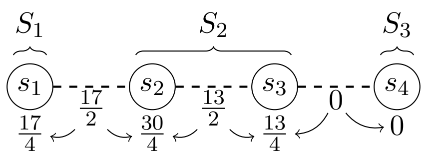


Figure 2: Computation of the track-based proportional rule (second step).

their whole journey). The dashed line in the figure shows the use of each of the three tracks in this example.

Now, we must allocate the utilization of the tracks to the stations. The simplest approach is to assume that such the use is split equally between the two stations at the extreme ends of the track (Figure 2).

Finally, the cost is divided in proportion to the use of the tram line by each municipality, which is the sum of the uses of its stations. Hence,

$$TP(a) = \left(\frac{12}{15} \cdot \frac{17}{4}, \frac{12}{15} \cdot \frac{43}{4}, \frac{12}{15} \cdot 0\right) = \left(\frac{51}{15}, \frac{129}{5}, 0\right).$$

3 Fairness properties of the rules

The first requirement, *null municipality*, states that a municipality does not have to contribute to the fixed cost if none of its stations is used by passengers. Thus, if nobody departs from or arrives at any of the stations located in a municipality, then this municipality is exempted from payment.

Null municipality. For each $a \in \mathbb{A}$ and each $i \in M$, if $\omega_{gh} = \omega_{hg} = 0$ for all $s_g \in S_i$ and all $s_h \in S$, then $R_i(a) = 0$.

In Example 1, Municipality 3 is null because nobody uses its single station.

The second property is a weakening of the previous one. Now, we consider that a municipality is null if no traveler uses its stations and no train passes through its stations and tracks. We note that only municipalities at the extreme ends of the line may potentially satisfy this requirement.

Weak null municipality. For each $a \in \mathbb{A}$ and each $i \in M$, if one of the following two conditions holds

- $\omega_{gh} = \omega_{hg} = 0$, for all $j \leq i$, for all $s_g \in S_j$, and all $s_h \in S$;
- $\omega_{gh} = \omega_{hg} = 0$, for all $j \geq i$, for all $s_g \in S_j$, and all $s_h \in S$;

then $R_i(a) = 0$.

Clearly, null municipality implies weak null municipality but the converse is not true.

Our third property imposes a minimal criterion of equity, where it requires that symmetric municipalities must contribute equally. We say that two municipalities are *symmetric* if the same number of passengers travel within them and between them and any other third municipality.

Symmetry. For each $a \in \mathbb{A}$ and each $\{i, j\} \subseteq M$, if $\Omega_{ii}(OD) = \Omega_{jj}(OD)$, and $\Omega_{ik}(OD) = \Omega_{jk}(OD)$, and $\Omega_{ki}(OD) = \Omega_{kj}(OD))$, for all $k \in M \setminus \{i, j\}$. Then $R_i(a) = R_j(a)$.

The next property states that if all traffic is on the line between two adjacent stations that belong to two different municipalities, then both municipalities must contribute equally. We note that this requirement is very weak because it only imposes a minimal condition of fairness in a very specific situation.

Adjacent symmetry. For each $a \in \mathbb{A}$ and each $\{i, j\} \subseteq M$, if $\omega_{gh} + \omega_{hg} = \Omega(OD)$, such that $|g - h| = 1$, and $g \in S_i, h \in S_j$, then $R_i(a) = R_j(a)$.

The next property requires a certain type of independence for the rule with respect to changes in the set of municipalities. More precisely, given a rule, we consider a problem and apply the rule to the problem. Imagine now that all of the municipalities but two leave the consortium, and thus these two municipalities must meet all the costs themselves. The *bilateral ratio consistency* requires that when the new situation is re-evaluated and the cost is divided between these two municipalities, the ratio between their allocations for the new problem is the same as the ratio between their allocations for the original problem. The reduced problem is defined in a natural manner, where the set of municipalities still comprises the same two in the original situation (say $N = \{i, j\}$), the set of stations is formed by all the stations that belong to either i or j (and only those), the fixed cost C is the same as in the original problem, and the reduced flow matrix $OD_{\{i,j\}}$ is obtained by removing in OD all the columns and rows that correspond to stations not in N.

Bilateral ratio consistency. For each $a = (M, S, OD, C) \in \mathbb{A}$ and each pair of municipalities $\{i, j\} \subseteq M$ we have that

$$\frac{R_i(a)}{R_j(a)} = \frac{R_i(a_{\{i,j\}})}{R_j(a_{\{i,j\}})},$$

where $a_{\{i,j\}} = \big(\{i, j\}, S_i \cup S_j, OD_{\{i,j\}}, C\big)$.

The following requirement states that the distribution of the cost is not altered by splitting a long trip into small trips. Thus, the allocation must be the same regardless of whether an individual goes from station s_g to station s_h directly or indirectly (from s_g to an intermediate station and from there to s_h).

Example 4. *For Example 1, the flow matrix is as follows*

$$OD = \begin{pmatrix} 0 & 4 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 5 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

According to OD, there are two travelers from s_1 to s_3. Now, the two passengers split, one that goes from s_1 to s_2 and the other goes from s_2 to s_3. In this case, the flow matrix becomes as follows

$$OD' = \begin{pmatrix} 0 & 4+1 & 2-2 & 0 \\ 1 & 0 & 1+1 & 0 \\ 5 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 5 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 5 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If the rule satisfies trip decomposition, then the distribution of the fixed cost is the same for both OD and OD'.

Trip decomposition. For each $(M, S, OD, C), (M, S, OD', C) \in \mathbb{A}$. If $s_g, s_h \in S$, are stations such that $h - g > 1$, and

either

1. $\omega'_{g(g+1)} = \omega_{g(g+1)} + \frac{\omega_{gh}}{|h-g|}; \omega'_{(g+1)(g+2)} = \omega_{(g+1)(g+2)} + \frac{\omega_{gh}}{|h-g|}, \ldots, \omega'_{(h-1)h} = \omega_{(h-1)h} + \frac{\omega_{gh}}{|h-g|}$; and $\omega'_{gh} = 0$;
2. $\omega'_{ef} = \omega_{ef}$, if $(ef) \neq (gh)$,

or

1. $\omega''_{h(h-1)} = \omega_{h(h-1)} + \frac{\omega_{hg}}{|h-g|}; \omega''_{(h-1)(h-2)} = \omega_{(h-1)(h-2)} + \frac{\omega_{hg}}{|h-g|}, \ldots, \omega'_{(g+1)g} = \omega_{(g+1)g} + \frac{\omega_{hg}}{|h-g|}$; and $\omega''_{hg} = 0$;
2. $\omega''_{ef} = \omega_{ef}$, if $(ef) \neq (hg)$,

then, $R(M, S, OD, C) = R(M, S, OD', C)$ and $R(M, S, OD, C) = R(M, S, OD'', C)$

Next, instead of allocating the fixed cost C for a whole year according to the traffic in that period, we may solve the problem month by month. Thus, for each month, we distribute $\frac{C}{12}$ by considering the passengers flows only in that month and we then aggregate for the 12 months. Additivity requires that the final allocation must be the same regardless of whether we solve the problem yearly or monthly (and we then aggregate).

Additivity. For each $(M, S, OD, C) \in \mathbb{A}$ and each $i \in M$,

$$R_i(M, S, OD, C) = \sum_{t=1}^{T} R_i(M, S, OD_t, C_t),$$

where $OD = \sum_{t=1}^{T} OD_t$ and $C = \sum_{t=1}^{T} C_t$.

Proposition 1. *If a rule satisfies additivity, then it does not depend on the flow matrix.*

Proof. Let $OD, OD' \in \mathbb{R}^{n \times n}$ be two different flow matrices. Let us define $\overline{OD} \in \mathbb{R}^{n \times n}$ such that

$$\overline{\omega}_{gh} = \min \left\{ \omega_{gh}, \omega'_{gh} \right\}.$$

Then $OD = \overline{OD} + (OD - \overline{OD})$ and $OD' = \overline{OD} + (OD' - \overline{OD})$. By applying *additivity*,

$$\begin{aligned} R(M,S,OD,C) &= R(M,S,\overline{OD},C) + R(M,S,OD-\overline{OD},0) \\ &= R(M,S,\overline{OD},C) \\ &= R(M,S,\overline{OD},C) + R(M,S,OD-\overline{OD'},0) \\ &= R(M,S,OD',C) \end{aligned}$$

and thus, the allocation is the same for any flow matrix. □

As shown in the previous proposition, additivity is very demanding because it leads to rules that do not depend on the flow matrix, obviating all of the information provided by one of the key elements of the problem. If additivity is imposed, then the problem is reduced to simply dividing the fixed cost among the municipalities by only considering the stations in each of them and, eventually, its identity. To avoid that, we introduced a weighted version of additivity, where the weights are the inverse of the cost per passenger in each period.

Weighted additivity. For each $(M,S,OD,C) \in \mathbb{A}$ and each $i \in M$,

$$\frac{\Omega(OD)}{C} R_i\left(M,S,OD,C\right) = \sum_{t=1}^{T} \frac{\Omega(OD_t)}{C_t} R_i\left(M,S,OD_t,C_t\right),$$

where $OD = \sum_{t=1}^{T} OD_t$ and $C = \sum_{t=1}^{T} C_t$.

4 Axiomatic characterization of the rules

Now, we present the characterizations of the rules introduced in Section 2. Our first result states that the unique rule that fulfills null municipality, symmetry, and weighted additivity is the rule that divides the cost in proportion to the number of users in each municipality.

Theorem 1. *A rule satisfies null municipality, symmetry, and weighted additivity if and only if it is the station-based proportional rule.*

Proof. First, we show that the station-based proportional rule satisfies the three properties in this statement.

(a) Null municipality. Let $i \in M$ such that $\omega_{hf} = \omega_{fg} = 0$ for all $s_h, s_g \in S_i$ and all $s_f \in S \setminus S_i$. Then $\Omega_i(OD) = 0$, and hence $SP_i(a) = 0$

(b) Symmetry. Let $\{i,j\} \subseteq M$ such that $\Omega_{ii}(OD) = \Omega_{jj}(OD)$, and $\Omega_{ik}(OD) = \Omega_{jk}(OD)$ and $\Omega_{ki}(OD) = \Omega_{kj}(OD)$ for all $k \in M \setminus \{i,j\}$. Then, it holds that $\Omega_i(OD) = \Omega_j(OD)$. By definition of the station-based proportional rule, we immediately find that $SP_i(a) = SP_j(a)$.

(c) Weighted additivity. First, it is easy to prove that $\Omega_{ij}(OD) = \sum_{t=1}^{T} \Omega_{ij}(OD_t)$. Thus, we can find that $\Omega_i(OD) = \sum_{t=1}^{T} \Omega_i(OD_t)$. Now, for each $i \in M$, we have the following.

$$\begin{aligned}
\sum_{t=1}^{T} \frac{\Omega(OD_t)}{C_t} SP_i\left(M,S,OD_t,C_t\right) &= \sum_{t=1}^{T} \frac{\Omega(OD_t)}{C_t} \frac{C_t}{2\Omega(OD_t)} \cdot \Omega_i(OD_t) \\
&= \frac{\Omega(OD)}{C} \cdot \frac{C}{\Omega(OD)} \cdot \frac{1}{2} \sum_{t=1}^{T} \Omega_i(OD_t) \\
&= \frac{\Omega(OD)}{C} \cdot \frac{C}{\Omega(OD)} \cdot \frac{1}{2} \Omega_i(OD) \\
&= \frac{\Omega(OD)}{C} \cdot SP_i\left(M,S,OD,C\right).
\end{aligned}$$

Let us prove the converse, i.e., let R be a rule that satisfies null municipality, symmetry, and weighted additivity. We show that $R = SP$. Let $a^{gh} = (M,S,OD^{gh},C) \in \mathbb{A}$ be a problem such that all of the entries of the flow matrix are null, except for the entry gh that is equal to one ($\omega_{ef} = 0$ for all $(e,f) \neq (g,h)$, and $\omega_{gh} = 1$). In this case, there are two municipalities (not necessarily different) $i,j \in M$ to which these stations belong, i.e., $s_g \in S_i$ and $s_h \in S_j$. The flow matrix for the problem a^{gh} has the following form

$$OD^{gh} = \begin{array}{c} \\ \\ g \\ \\ \\ \end{array} \overset{\displaystyle h}{\begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix}}.$$

First, we note that, for any other municipality $k \in S \setminus \{i,j\}$, $\omega_{hf} = \omega_{fg} = 0$ for all $s_h, s_g \in S_q$ and all $s_f \in S \setminus S_k$. By applying *null municipality*, we find that $R_k(a^{gh}) = 0$. In addition, municipalities i and j are symmetric in the problem a^{gh}, and thus by *symmetry*, it must be true that $R_i(a^{gh}) = R_j(a^{gh}) = \frac{C}{2}$.

Let $a = (M,S,OD,C) \in \mathbb{A}$ be a problem without any restriction. We can additively split the problem a into other $\Omega(OD)$ problems a^{gh} such that

$$OD = \sum_{\{g,h\} \subseteq S} OD^{gh} \quad \text{and} \quad C^{gh} = \frac{C}{\Omega(OD)}.$$

We already know that for each of those problems, $a^{gh} = (M, S, OD^{gh}, C^{gh})$. *Weighted additivity* implies that, for each $i \in M$,

$$\frac{\Omega(OD)}{C} R_i(a) = \sum_{\{g,h\} \subseteq S} \frac{\Omega(OD^{gh})}{C^{gh}} R_i(a^{gh}),$$

or equivalently,

$$R_i(a) = \frac{C}{\Omega(OD)} \sum_{\{g,h\} \subseteq S} \frac{\Omega(OD^{gh})}{C^{gh}} R_i(a^{gh}).$$

We note the following

$$R_i(a^{gh}) = \begin{cases} 0 & \text{if } s_g, s_h \notin S_i \\ \frac{C^{gh}}{2} & \text{otherwise} \end{cases}.$$

Therefore,

$$\begin{aligned} R_i(a) = & \frac{C}{\Omega(OD)} \sum_{\{g,h\} \subseteq S} \frac{\Omega(OD^{gh})}{C^{gh}} R_i(a^{gh}) \\ = & \frac{C}{\Omega(OD)} \left[\sum_{s_g \in S_i,\, s_h \in S} \frac{\Omega(OD^{gh})}{C^{gh}} \frac{C^{gh}}{2} + \sum_{s_g \in S,\, s_h \in S_i} \frac{\Omega(OD^{gh})}{C^{gh}} \frac{C^{gh}}{2} \right] \\ = & \frac{C}{\Omega(OD)} \left[\frac{1}{2} \Omega_i^+(OD) + \frac{1}{2} \Omega_i^-(OD) \right] \\ = & \frac{C}{2\Omega(OD)} \cdot \Omega_i(OD) \\ = & SP_i(a). \end{aligned}$$

□

The next theorem states that if we require symmetry, bilateral ratio consistency, and weighted additivity, then the cost is split equally among all the municipalities regardless of the number of stations and their users.

Theorem 2. *A rule satisfies symmetry, bilateral ratio consistency, and weighted additivity if and only if it is the uniform rule.*

Proof. First, we check that the uniform rule satisfies the three properties of the statement.

(a) Symmetry. Let $\{i, j\} \subseteq M$ such that $\Omega_{ii}(OD) = \Omega_{jj}(OD)$, and $\Omega_{ik}(OD) = \Omega_{jk}(OD)$ and $\Omega_{ki}(OD) = \Omega_{kj}(OD)$ for all $k \in M \setminus \{i, j\}$. Then, we holds that $\Omega_i(OD) = \Omega_j(OD)$. By definition of the uniform rule, we immediately find that $U_i(a) = U_j(a)$.

(b) Bilateral ratio consistency. Let us consider the problems where $a = (M, S, OD, C)$ and $a_{\{i,j\}} = \big(\{i,j\}, S_i \cup S_j, OD_{\{i,j\}}, C\big)$, and if we apply the uniform rule, we obtain the following.

$$\frac{U_i(a)}{U_j(a)} = \frac{\frac{C}{m}}{\frac{C}{m}} = 1 = \frac{\frac{C}{2}}{\frac{C}{2}} = \frac{U_i(a_{\{i,j\}})}{U_j(a_{\{i,j\}})}.$$

(c) Weighted additivity. Now, for each $i \in M$,

$$\begin{aligned}\sum_{t=1}^{T} \frac{\Omega(OD_t)}{C_t} U_i\left(M, S, OD_t, C_t\right) &= \sum_{t=1}^{T} \frac{\Omega(OD_t)}{C_t} \frac{C_t}{m} \\ &= \frac{1}{m} \sum_{t=1}^{T} \Omega(OD_t) \\ &= \frac{C}{m} \frac{\Omega(OD)}{C} \\ &= \frac{\Omega(OD)}{C} \cdot U_i\left(M, S, OD, C\right).\end{aligned}$$

Let us consider the converse. Let R be a rule that is symmetric, bilateral ratio consistent, and weighted additive. We show that $R = U$. Let $a^{gh} = (M, S, OD^{gh}, C) \in \mathbb{A}$ be a problem such that all of the entries in the flow matrix are null, except for the entry gh that is equal to one ($\omega_{ef} = 0$ for all $(e, f) \neq (g, h)$, and $\omega_{gh} = 1$). In this case, there are two municipalities (not necessarily different) $i, j \in M$ to which these stations belong, i.e., $s_g \in S_i$ and $s_h \in S_j$. The flow matrix for problem a^{gh} has the following form

$$OD^{gh} = \begin{array}{c} \\ \\ g \\ \\ \\ \end{array}\begin{array}{c} \quad\quad\quad h \\ \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix}\end{array}.$$

We note that for any other pair of municipalities $\{k, l\} \neq \{i, j\}$, the reduced flow matrix $OD^{gh}_{\{k,l\}}$ is null (all of the entries are equal to zero). Therefore, k and l are symmetric in the reduced problem $\left(\{k, l\}, S_k \cup S_l, OD^{gh}_{\{k,l\}}, C\right)$, and thus due to the *symmetry*, it must be true that

$$R_k\left(\{k, l\}, S_k \cup S_l, OD^{gh}_{\{k,l\}}, C\right) = R_l\left(\{k, l\}, S_k \cup S_l, OD^{gh}_{\{k,l\}}, C\right) = \frac{C}{2}.$$

By applying the *bilateral ratio consistency*, we also know that

$$\frac{R_k\left(a^{gh}\right)}{R_l\left(a^{gh}\right)} = \frac{R_k\left(\{k, l\}, S_k \cup S_l, OD^{gh}_{\{k,l\}}, C\right)}{R_l\left(\{k, l\}, S_k \cup S_l, OD^{gh}_{\{k,l\}}, C\right)} = 1 \quad \Leftrightarrow \quad R_k\left(a^{gh}\right) = R_l\left(a^{gh}\right).$$

This fact combined with the requirement that $\sum_{k=1}^{m} R_k\left(a^{gh}\right) = C^{gh}$ imply that $R_k\left(a^{gh}\right) = \frac{C^{gh}}{m}$ for all $k \in M$.

Let $a = (M, S, OD, C) \in \mathbb{A}$ be a problem without any restriction. We can additively split problem a into other $\Omega(OD)$ problems a^{gh} such that

$$OD = \sum_{\{g,h\}\subseteq S} OD^{gh} \quad \text{and} \quad C^{gh} = \frac{C}{\Omega(OD)}.$$

We already know that for each of those problems, $a^{gh} = (M, S, OD^{gh}, C^{gh})$, the *weighted additivity* implies that for each $i \in M$:

$$\frac{\Omega(OD)}{C} R_i(a) = \sum_{\{g,h\}\subseteq S} \frac{\Omega(OD^{gh})}{C^{gh}} R_i(a^{gh}),$$

or equivalently,

$$R_i(a) = \frac{C}{\Omega(OD)} \sum_{\{g,h\}\subseteq S} \frac{\Omega(OD^{gh})}{C^{gh}} R_i(a^{gh}).$$

We already know that $R_i(a^{gh}) = \frac{C^{gh}}{m}$ for any $i \in M$, so we have the following.

$$\begin{aligned} R_i(a) = & \frac{C}{\Omega(OD)} \sum_{\{g,h\}\subseteq S} \frac{\Omega(OD^{gh})}{C^{gh}} R_i(a^{gh}) \\ = & \frac{C}{\Omega(OD)} \sum_{\{g,h\}\subseteq S} \frac{\Omega(OD^{gh})}{C^{gh}} \frac{C^{gh}}{m} \\ = & \frac{C}{m} \\ = & U_i(a). \end{aligned}$$

□

Finally, the next theorem provides an axiomatic characterization of the track-based proportional rule.

Theorem 3. *The unique rule that satisfies adjacent symmetry, weak null municipality, trip decomposition, and weighted additivity is the track-based proportional rule.*

Proof. First, we prove that the track-based proportional rule satisfies the four properties in the statement above.

(a) Adjacent symmetry. Let $\{i, j\} \subseteq M$, such that $g \in S_i, h \in S_j$, $|g-h| = 1$, and $\omega_{gh} + \omega_{hg} = \Omega(OD)$. Then

$$TP_i(a) = \frac{C}{\Omega(OD)} \frac{\omega_{gh} + \omega_{hg}}{2} = \frac{C}{2} = TP_j(a).$$

(b) Weak null municipality. Let us assume that for $i \in M$, the condition that $\omega_{gh} = \omega_{hg} = 0$ for all $j \leq i$, for all $s_g \in S_j$, and all $s_h \in S$ holds. Then, this condition implies that for all $s_g, s_h \in S$, and $s_f \in S_i$, such that $g \leq f \leq h, g \neq h$, $\omega_{gh} = \omega_{hg} = 0$. Therefore, by the definition of the track-based proportional rule, $R_i(a) = 0$. The proof is completely analogous for the other condition.

(c) Trip decomposition. Let $a = (M, S, OD, C), a' = (M, S, OD', C) \in \mathbb{A}$, such that:

(a) $h - g > 1$

(b) $\omega'_{g(g+1)} = \omega_{g(g+1)} + \frac{\omega_{gh}}{|h-g|}; \omega'_{(g+1)(g+2)} = \omega_{(g+1)(g+2)} + \frac{\omega_{gh}}{|h-g|}, \ldots, \omega'_{(h-1)h} = \omega_{(h-1)h} + \frac{\omega_{gh}}{|h-g|}$; and $\omega'_{gh} = 0$;

(c) $\omega'_{ef} = \omega_{ef}$, if $(ef) \neq (gh)$,

then for each $i \in M$, $TP_i(a')$ is given by

$$TP_i(a') = \frac{C}{\Omega(OD')} \cdot \sum_{s_f \in S_i} \sum_{\substack{s_d, s_e \in S, s_d \neq s_e \\ f \in [e,d] \text{ or } f \in [d,e]}} \frac{\omega'_{de}}{(2 - \lceil \frac{|f-d| \cdot |e-f|}{(e-d)^2} \rceil)|e-d|},$$

We distinguish two situations, as follows:

(a) $\forall s_f \in S_i, f \notin [g, h]$. In this situation, we have:

$$\omega'_{de} = \omega_{de} \text{ and } \omega'_{ed} = \omega_{ed}, \forall s_d, s_e \in S, \text{ such that } f \in [e, d].$$

Therefore, $TP_i(a') = TP_i(a)$.

(b) There exists $s_f \in S_i$, such that $f \in [g, h]$. In this situation only the trips in ascending and consecutive order from g to h are different from a to a'. Therefore, we will now focus on these trips. We distinguish three cases, as follows:

i. If we assume that $g < f = g + k < h, k \in \mathbb{Z}_+$, then we only have to consider the trips from $s_{(g+k-1)}$ to $s_{(g+k)}$ and from $s_{(g+k)}$ to $s_{(g+k+1)}$. In this case, the corresponding two terms in $TP_i(a')$ are given by:

$$\frac{\omega_{(g+k-1)(g+k)}+\frac{\omega_{gh}}{|h-g|}}{(2-0)1}+\frac{\omega_{(g+k)(g+k+1)}+\frac{\omega_{gh}}{|h-g|}}{(2-0)1}=$$

$$\frac{\omega_{(g+k-1)(g+k)}}{2}+\frac{\omega_{(g+k)(g+k+1)}}{2}+\frac{\omega_{gh}}{|h-g|}.$$

The first two terms correspond to the same terms in problem a and the third term corresponds to the term associated with station s_f in the trip from s_g to s_h in problem a.

ii. Let us consider that $f=g$. In this case, we only have to consider the trip from s_g to s_{g+1}. The corresponding term in $TP_i(a')$ is given by

$$\frac{\omega_{g(g+1)}+\frac{\omega_{gh}}{|h-g|}}{(2-0)1}=\frac{\omega_{g(g+1)}}{2}+\frac{\omega_{gh}}{2|h-g|}.$$

The first term corresponds to the same term in problem a and the second term corresponds to the term associated with station s_f in the trip from s_g to s_h in problem a.

iii. Let us consider $f=h$. This case is analogous to Case ii.

Therefore, we can conclude that $TP_i(a')=TP_i(a)$.

(d) Weighted additivity. First, it is easy to prove that $\omega_{gh}=\sum_{t=1}^{T}\omega_{gh}^{t}$. Now, for each $i\in M$, we have the following:

$$\sum_{t=1}^{T}\frac{\Omega(OD_t)}{C_t}TP_i(a_t)=$$

$$\sum_{t=1}^{T}\frac{\Omega(OD_t)}{C_t}\left(\frac{C_t}{\Omega(OD_t)}\cdot\sum_{s_f\in S_i}\sum_{\substack{s_g,s_h\in S,s_g\neq s_h\\ f\in[g,h]\text{ or }f\in[h,g]}}\frac{\omega_{gh}^{t}}{(2-\lceil\frac{|f-g|\cdot|h-f|}{(h-g)^2}\rceil)|h-g|}\right)=$$

$$\sum_{t=1}^{T}\sum_{s_f\in S_i}\sum_{\substack{s_g,s_h\in S,s_g\neq s_h\\ f\in[g,h]\text{ or }f\in[h,g]}}\frac{\omega_{gh}^{t}}{(2-\lceil\frac{|f-g|\cdot|h-f|}{(h-g)^2}\rceil)|h-g|}=$$

$$\sum_{s_f\in S_i}\sum_{\substack{s_g,s_h\in S,s_g\neq s_h\\ f\in[g,h]\text{ or }f\in[h,g]}}\frac{\sum_{t=1}^{T}\omega_{gh}^{t}}{(2-\lceil\frac{|f-g|\cdot|h-f|}{(h-g)^2}\rceil)|h-g|}=$$

$$\sum_{s_f\in S_i}\sum_{\substack{s_g,s_h\in S,s_g\neq s_h\\ f\in[g,h]\text{ or }f\in[h,g]}}\frac{\omega_{gh}}{(2-\lceil\frac{|f-g|\cdot|h-f|}{(h-g)^2}\rceil)|h-g|}=\frac{\Omega(OD)}{C}TP_i(a).$$

Conversely, let R be a rule that satisfies adjacent symmetry, weak null municipality, trip decomposition, and weighted additivity. We now show that $R=TP$. Let $a^{gh}=(M,S,OD^{gh},C)\in\mathbb{A}$

be a problem such that all of the entries in the flow matrix are null, except for the gh entry, which is equal to w_{gh} ($\omega_{ef} = 0$ for all $(e, f) \neq (g, h)$). In this case, there are two municipalities (not necessarily different) $i, j \in M$ to which those stations belong, i.e., $s_g \in S_i$ and $s_h \in S_j$. The flow matrix for the problem a^{gh} has the following form

$$OD^{gh} = \begin{array}{c} \\ \\ \\ g \\ \\ \\ \end{array} \overset{\displaystyle h}{\begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & \dots & \omega_{gh} & \dots & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix}}.$$

Now, ω_{gh} in OD^{gh} can be distributed equally among the consecutive trips from s_g to s_h. The latter matrix can be written as the sum of matrices such that the only non-null entry corresponds to two consecutive stations. Therefore, we consider $a'^{gh} = (M, S, OD'^{gh}, C) \in \mathbb{A}$, such that $|g - h| = 1$.

R satisfies *adjacent symmetry* and *weak null municipality*, so $R_i(a'^{gh}) = R_j(a'^{gh}) = \frac{C}{2}$, which matches with the track-based proportional rule.

Let $a^{gh} = (M, S, OD^{gh}, C) \in \mathbb{A}$. Let us consider that $g < h$, and the other case is completely analogous. Let $a'^{gh} = (M, S, OD'^{gh}, C) \in \mathbb{A}$, such that $\omega'_{(g+i)(g+i+1)} = \frac{\omega_{gh}}{|h-g|}$, for all $i = 0, 1, 2, \dots, h - g - 1$ and the remaining entries are zero.

Now, R satisfies *trip decomposition*, so we find that $R(a^{gh}) = R(a'^{gh})$. We can additively split the problem a'^{gh} into $|h - g|$ problems $a''^{(g+i)(g+i+1)} = (M, S, OD''^{(g+i)(g+i+1)}, C^{(g+i)(g+i+1)}) \in \mathbb{A}, i = 0, 1, 2, \dots, h - g - 1$, such that $\omega''_{(g+i)(g+i+1)} = \frac{\omega_{gh}}{|h-g|}$ and the remaining entries are zero, and

$$C^{(g+i)(g+i+1)} = \frac{C}{|h - g|}.$$

Therefore, we have:

$$OD'^{gh} = \sum_{i=0}^{h-g-1} OD''^{(g+i)(g+i+1)} \quad \text{and} \quad C = \sum_{i=0}^{h-g-1} C^{(g+i)(g+i+1)}.$$

R satisfies *weighted additivity*, for each $i \in M$, so the following holds:

$$\frac{\Omega(OD'^{gh})}{C}R_i(a'^{gh}) = \sum_{i=0}^{h-g-1} \frac{\Omega(OD''^{(g+i)(g+i+1)})}{C^{(g+i)(g+i+1)}}R_i(a''^{(g+i)(g+i+1)}),$$

or equivalently,

$$R_i(a'^{gh}) = \frac{C}{\Omega(OD'^{gh})} \sum_{i=0}^{h-g-1} \frac{\Omega(OD''^{(g+i)(g+i+1)})}{C^{(g+i)(g+i+1)}}R_i(a''^{(g+i)(g+i+1)}).$$

We have proved that R and TP coincide for problems with only traffic of passengers between two consecutive stations, so we find that:

$$R_i(a'^{gh}) = \frac{C}{\Omega(OD'^{gh})} \sum_{i=0}^{h-g-1} \frac{\Omega(OD''^{(g+i)(g+i+1)})}{C^{(g+i)(g+i+1)}}TP_i(a''^{(g+i)(g+i+1)}).$$

Now, the track-based proportional rule satisfies *weighted additivity*, so the following holds

$$R_i(a'^{gh}) = \frac{C}{\Omega(OD'^{gh})} \cdot \frac{\Omega(OD'^{gh})}{C}TP_i(a'^{gh}) = TP_i(a'^{gh}).$$

Therefore, $R_i(a^{gh}) = TP_i(a^{gh})$, for all $i \in M$.

Finally, let $a = (M, S, OD, C) \in \mathbb{A}$ be a problem without any restriction. We can additively split the problem a into other $\Omega(OD)$ problems a^{gh} such that

$$OD = \sum_{\{g,h\}\subseteq S} OD^{gh} \quad \text{and} \quad C^{gh} = \frac{C \cdot \omega_{gh}}{\Omega(OD)}.$$

We already know that for each of those problems, $a^{gh} = (M, S, OD^{gh}, C^{gh})$, the *weighted additivity* implies that, for each $i \in M$,

$$\frac{\Omega(OD)}{C}R_i(a) = \sum_{\{g,h\}\subseteq S} \frac{\Omega(OD^{gh})}{C^{gh}}R_i(a^{gh}),$$

or equivalently,

$$R_i(a) = \frac{C}{\Omega(OD)} \sum_{\{g,h\}\subseteq S} \frac{\Omega(OD^{gh})}{C^{gh}}R_i(a^{gh}).$$

Therefore, $R(a^{gh}) = TP(a^{gh})$ and the track-based proportional rule satisfies *weighted additivity*, so for each $i \in M$, we obtain:

$$R_i(a) = \frac{C}{\Omega(OD)} \sum_{\{g,h\}\subseteq S} \frac{\Omega(OD^{gh})}{C^{gh}}TP_i(a^{gh}) = TP_i(a).$$

□

5 Logical independence of the properties

In this section we show that all of the properties used in the characterization of each solution are necessary.

Proposition 2. *Null municipality, symmetry and weighted additivity are necessary in the characterization of the station-based proportional rule.*

Proof. (a) The uniform rule satisfies symmetry and weighted additivity but it does not satisfy null municipality by definition, because all municipalities are allocated with part of the fixed cost independetly of the traffic in the transport system.

(b) Let $a = (M, S, OD, C) \in \mathbb{A}$ be a problem and for each $i \in M$ we define the following rule:

$$R_i(a) = \begin{cases} 0 & \text{if } \Omega_i(OD) = 0 \\ \frac{C}{|K|} & \text{otherwise} \end{cases},$$

where $K = \{i \in M : \Omega_i(OD) \neq 0\}$.

By definition, this rule satisfies null municipality and symmetry, but not weighted additivity. Indeed, let us consider the case of a trolley line that passes across three municipalities $M = \{1, 2, 3\}$ with three stations $S = \{s_1, s_2, s_3\}$, which are distributed as follows: $S_1 = \{s_1\}, S_2 = \{s_2\}$ and $S_3 = \{s_3\}$. The fixed cost is $C = 6$ and the OD matrix is given by

$$OD = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

and

i	$\Omega_i^+(OD)$	$\Omega_i^-(OD)$	$\Omega_i(OD)$
1	2	2	4
2	2	2	4
3	2	2	4

Therefore, all of the municipalities are symmetric so they must pay the same $\frac{C}{3}$, and thus:

$$R(a) = (2, 2, 2) \,.$$

Now we divide the cost C into $C_1 + C_2 = 2 + 4 = 6$ and the OD matrix into $OD_1 + OD_2$ in the following manner:

$$OD = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

For example, for Municipality 1, we obtain:

$$\frac{6}{6} \cdot 2 \neq \frac{2}{2} \cdot 0 + \frac{4}{4} \cdot \frac{4}{3}.$$

Therefore, this rule does not satisfy weighted additivity.

(c) Let $a = (M, S, OD, C) \in \mathbb{A}$ be a problem. For each $i \in M$, we define the following rule:

$$R_i(a) = \begin{cases} \frac{i}{3} \cdot C & \text{if } |M| = 2 \text{ and } |S| = 2 \\ SP_i(a) & \text{otherwise.} \end{cases}$$

For $|M| = 2$ and $|S| = 2$, it is clear that this rule satisfies weighted additivity because it does not depend on the OD matrix. Furthermore, in this case, null municipality is meaningless because it implies that there is no traffic at all in the network. Finally, in this case, this rule does not satisfy symmetry because the allocation of the fixed cost depends on the names of the agents. For the remaining cases, this rule satisfies null municipality, symmetry and weighted additivity. Therefore, this rule satisfies null municipality, weighted additivity but no symmetry.

□

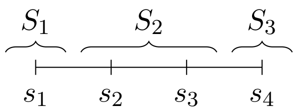
Proposition 3. *Symmetry, bilateral ratio consistency, and weighted additivity are necessary to characterize the uniform rule.*

Proof. (a) Let us consider the station-based uniform rule given by:

$$SU_i(a) = \frac{C}{n} \cdot |S_i|.$$

It is clear that this rule does not satisfy symmetry because it depends only on the number of stations in each municipality but not on the traffic through the network. Similar to the uniform rule, we can prove that the station-based uniform rule satisfies bilateral ratio consistency and weighted additivity.

(b) The station-based proportional rule satisfies symmetry and weighted additivity but not bilateral ratio consistency. We consider the case of a trolley line that passes across three municipalities $M = \{1, 2, 3\}$ with four stations $S = \{s_1, s_2, s_3, s_4\}$, which are distributed as follows:



The fixed cost is $C = 4$ and the OD matrix is

$$OD = \begin{pmatrix} 0 & 2 & 3 & 1 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

In this case, $SP(a) = \left(1, \frac{19}{8}, \frac{5}{8}\right)$ Now, if we suppose that Municipality 3 leaves the consortium, then the new (reduced) problem is: $a_{\{1,2\}} = \big(\{1,2\}, S_1 \cup S_2, OD_{\{1,2\}}, C\big)$, where

$$OD_{\{1,2\}} = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix},$$

For this reduced problem, we obtain:

$$SP(a_{\{1,2\}}) = \left(\frac{14}{11}, \frac{30}{11}\right).$$

Now we have:

$$\frac{1}{\frac{19}{8}} \neq \frac{\frac{14}{11}}{\frac{30}{11}}.$$

Therefore, the station-based proportional rule does not satisfy bilateral ratio consistency.

(c) Example of a rule that satisfies symmetry, bilateral ratio consistency but does not satisfy weighted additivity. Let $a = (M, S, OD, C) \in \mathbb{A}$ be a problem. We define the following rule for each $i \in M$:

$$R_i(a) = \begin{cases} \frac{C}{\sum_{j\in M}\Omega_{jj}(OD)} \cdot \Omega_{ii}(OD) & \text{if } \Omega_{kk}(OD) \neq 0 \text{ for all } k \in M \\ U_i(a) & \text{otherwise.} \end{cases}$$

By definition, this rule satisfies symmetry and bilateral ratio consistency. However, it does not satisfy weighted additivity because we can transform a problem with $\Omega_{kk}(OD) \neq 0$ for all $k \in M$ into problems where this condition does not hold, and we can then apply the uniform rule instead of the proportional distribution to the inner traffic in each municipality.

□

Proposition 4. *Adjacent symmetry, weak null municipality, trip decomposition, and weighted additivity are necessary to characterize the track-based proportional rule.*

Proof. (a) By the definition of the uniform rule, it is straightforward to check whether it satisfies adjacent symmetry, trip decomposition, and weighted additivity but not weak null municipality.

(b) Before giving a rule that satisfies weak null municipality, trip decomposition, and weighted additivity but not adjacent symmetry, we introduce the following

$$\omega_{[g,g+1]} = \sum_{\substack{k,h \\ k\leq g<g+1\leq h}} \frac{\omega_{kh}+\omega_{hk}}{|h-k|},$$

where $\omega_{[g,g+1]}$ is the number of passengers between two consecutive stations when all passengers are distributed equally among all tracks that they use in their trips. Now, we define the following rule:

$$R_i(a) = \frac{C}{\Omega(OD)} \sum_{s_g\in S_i} \left(\frac{g}{2g-1}\omega_{[g-1,g]} + \frac{g}{2g+1}\omega_{[g,g+1]} \right), \text{ for all } i \in M,$$

where $\omega_{[0,1]} = \omega_{[n,n+1]} = 0$.

By definition, this rule satisfies weak null municipality and trip decomposition. Analogous to the track-based proportional rule, we can prove that this rule satisfies weighted additivity. However, this rule does not satisfies adjacent symmetry because it depends on the name of the stations.

(c) It is easy to ckeck that the station-based proportional rule satisfies weak null municipality, adjacent symmetry, and weighted additivity. However, it does not satisfy trip decomposition as shown by the following example. Let us consider a problem with two municipalities and three stations, $S_1 = \{s_1\}$ and $S_2 = \{s_2, s_3\}$, where the fixed cost that needs to be distributed is 1 and the OD matrix is given by

$$OD = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The station-based proportional rule is $SP(a) = \left(\frac{1}{3}, \frac{2}{3}\right)$. Now, if we distribute the passengers such that there are only trips between consecutive stations, we obtain the following OD' matrix:

$$OD' = \begin{pmatrix} 0 & 1\frac{1}{2} & 0 \\ 1\frac{1}{2} & 0 & 1\frac{1}{2} \\ 0 & 1\frac{1}{2} & 0 \end{pmatrix},$$

and the station-based proportional rule is $SP(a') = \left(\frac{1}{4}, \frac{3}{4}\right)$. Therefore, the station-based proportional rule does not satisfies trip decomposition.

(d) Given an OD matrix, we define the following $[OD]$ matrix:

$$[\omega_{gh}] = \begin{cases} 0 & \text{if } |g-h| > 1 \\ \omega_{[g,h]} & \text{otherwise,} \end{cases}$$

where $\omega_{[g,h]}$ is defined as in (b).

Now, we define the following rule for each $i \in M$:

$$R_i(a) = \begin{cases} 0 & \text{if } \Omega_i([OD]) = 0 \\ \frac{C}{|K|} & \text{otherwise} \end{cases}$$

where $K = \{i \in M : \Omega_i([OD]) \neq 0\}$.

By definition, we can prove that this rule satisfies weak null municipality, adjacent symmetry, and trip decomposition but not weighted additivity.

□

Finally, we provide a table to summarize the properties that each characterized rule satisfies, where we highlight the properties used in the corresponding characterization.

Properties	Uniform	Station-based proportional	Track-based proportional
Null municipality	No	**Yes**	No
Weak null municipality	No	Yes	**Yes**
Symmetry	**Yes**	**Yes**	No
Adjacent symmetry	Yes	Yes	**Yes**
Bilateral ratio consistency	**Yes**	No	No
Trip decomposition	Yes	No	**Yes**
Weighted additivity	**Yes**	**Yes**	**Yes**
Additivity	Yes	No	No

6 Final comments

In this study, we proposed a model for addressing the problem of dividing the fixed cost of a tram line among the municipalities along that line. The allotment rule depends on the set of municipalities, the stations in each municipality, the cost that needs to be distributed, and the flow matrix. We showed that if we require that this rule satisfies the basic notions of fairness and stability, then we obtain unique solutions. In particular, we showed that null municipality (a municipality without users is exempted from payment), symmetry (symmetric municipalities contribute equally), and weighted additivity (the allocation is immune to splitting the problem) lead to the station-based proportional rule, which divides the cost in proportion to the number of passengers that use the stations in each city. Similarly, we characterized the track-based proportional rule in terms of the adjacent symmetry (symmetry with respect to adjacent stations), weak null municipality, trip decomposition (the rule is not altered by splitting a long trip into small trips), and weighted additivity. Finally, we proved that if we require the rule to satisfy symmetry, bilateral ratio consistency (stability with respect to changes in the set of municipalities), and weighted additivity, then we must allocate the cost uniformly among the cities.

As shown in many previous studies of cost sharing (e.g., Sánchez-Soriano et al. (2002), Ni & Wang (2007), and Kuipers et al. (2013)), we can naturally define a cost game as follows. The set of players is the set of municipalities M and the value function for each coalition $S \subset M$ is given by the following.

$$c(S) = \begin{cases} 0 & \text{if } S = \phi \\ C & \text{if } S \neq \phi \end{cases}.$$

The function c is constant due to the nature of the problem considered, so regardless of the number or identity of the municipalities involved in the coalition, the cost they incur is always C, e.g., the salary of the CEO must be paid irrespective of the use or the structure of the network. If we apply the basic solution concepts from previous studies of TU games to our problem, we obtain completely unsatisfactory allocations. The Shapley value and the nucleolus are the uniform split $\left(x = \left(\frac{C}{m}, \ldots, \frac{C}{m}\right)\right)$,[1] and the core is the whole set of feasible allocations $(\{x : \sum_{i=1}^{m} x_i = C\})$.

As shown by Shapley (1953), the unique solution of the previously described cost game that satisfies symmetry, null player, and additivity is the Shapley value, which matches with the uniform rule in our problem. Interestingly, Theorem 1 proves that if these same principles are translated into our model, then we characterize the station-based proportional rule instead. The uniform rule (or equivalently, the Shapley value for the cost game) can also be characterized using alternative properties.

[1] In fact, any symmetric solution for the cost leads to the uniform allocation.

References

Alcalde-Unzu, J., Gómez-Rúa, M., & Molis, E. (2015). Sharing the costs of cleaning a river: the upstream responsibility rule. *Games and Economic Behavior*, *90*, 134-50.

Algaba, E., Fragnelli, V., Llorca, N., & Sánchez-Soriano, J. (2017). Horizontal cooperation in a multimodal public transport system: the revenue allocation problem. *Mimeo*.

Bergantiños, G., Gómez-Rúa, M., Llorca, N., Pulido, M., & Sánchez-Soriano, J. (2014). A new rule for source connection problems. *European Journal of Operational Research*, *234*, 780-788.

Bergantiños, G., & Martínez, R. (2014). Cost allocation in asymmetric trees. *European Journal of Operational Research*, *237*, 975-987.

Gómez-Rúa, M. (2011). Sharing a polluted river through environmental taxes. *SERIEs*, *4*, 137-153.

Kuipers, J., Mosquera, M. A., & Zarzuelo, J. M. (2013). Sharing costs in highways: A game theoretic approach. *European Journal of Operational Research*, *228*, 158-168.

Littlechild, S. C., & Owen, G. (1973). A Simple Expression for the Shapley Value in a Special Case. *Management Science*, *20*, 370-372.

Maschler, M., Potters, J., & Reijnierse, H. (2010). The nucleolus of a standard tree game revisited: a study of its monotonicity and computational properties. *International Journal of Game Theory*, *39*, 89-104.

Ni, D., & Wang, Y. (2007). Sharing a polluted river. *Games and Economic Behavior*, *60*, 176-186.

Norde, H., Fragnelli, V., García-Jurado, I., Patrone, F., & Tijs, S. (2002). Balancedness of infrastructure cost games. *European Journal of Operational Research*, *136*, 635-654.

Sánchez-Soriano, J., Llorca, N., Meca, A., Molina, E., & Pulido, M. (2002). An integrated transport system for alacant's students. univercity. *Annals of Operations Research*, *109*, 41-60.

Shapley, L. S. (1953). A Value for N-person Games. *Annals of Mathematics Studies*, *28*, 307-317.

Sharkey, W. (1995). Network models in economics. In *Network routing* (Vol. 8, p. 713-765). Elsevier.

Slikker, M., & Nouweland, A. v. (2000). Network formation models with costs for stablishing links. *Review of Economic Design*, *5*, 333-362.

Sudhölter, P., & Zarzuelo, J. (2017). Characterizations of highway toll pricing methods. *European Journal of Operational Research*, *260*, 161-170.

Thomson, W. (2001). On the Axiomatic Method and its Recent Applications to Game Theory and Resource Allocation. *Social Choice and Welfare*, *18*, 327-386.

Thomson, W. (2014). Cost allocation and airport problems. *Rochester Center for Economics Research Working paper*, *34*.

van den Brink, R., & van der Laan, G. (2008). Comment on "sharing a polluted river". *Mimeo*.

van den Nouweland, A., Borm, P., van Golstein Brouwers, W., Bruinderink, R. G., & Tijs, S. (1996). A game theoretic approach to problems in telecommunication. *Management Science*, *42*, 294-303.

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