Resultados óptimos de existencia y unicidad de solución para ecuaciones casilineales singulares

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Outline

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1. Background on quasilinear problems
2. New contributions
3. Highlights of the proofs
4. Work in progress

$$\begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{u^\theta} + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

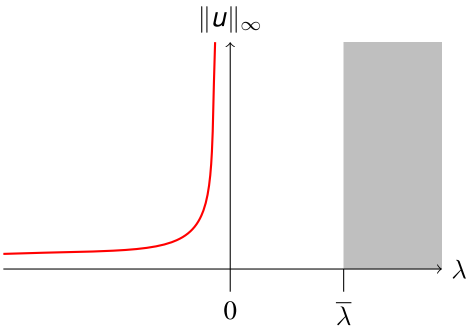
- $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ bounded smooth domain
- $\lambda \in \mathbb{R}$
- $1 < q \leq 2$
- $\theta \geq 0$
- $0 \leq \mu \in L^\infty(\Omega)$
- $0 \lneq f \in L^\infty(\Omega)$

The solutions will be understood in the **weak** sense, and will be **bounded**.

Nonsingular problem ($\theta = 0$)

$$\begin{cases} -\Delta u = \lambda u + \mu |\nabla u|^2 + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

- $\lambda < 0$: [Boccardo, Murat, Puel. 80's and 90's], [Barles, Murat. 1995].



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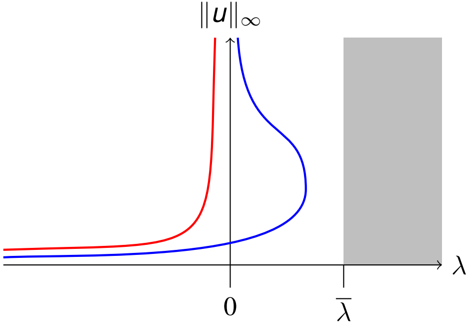
- $\lambda < 0$: [Boccardo, Murat, Puel. 80's and 90's], [Barles, Murat. 1995].
- $\lambda = 0$: [Ferone, Murat. 2000], [Abdellaoui, Dall'Aglio, Peral. 2006], [Porretta. 2010].
- $\lambda > 0$: [Arcoya, De Coster, Jeanjean, Tanaka. 2015].

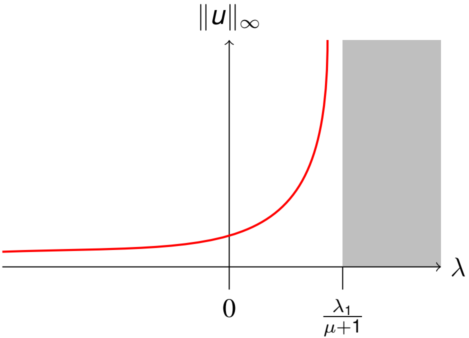


Problem with singularity ($\theta = 1$)

$$\begin{cases} -\Delta u = \lambda u + \mu \dfrac{|\nabla u|^2}{u} + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

- $\lambda < 0$: [Giachetti, Murat. 2009].
- $\lambda = 0$: [Arcoya, Boccardo, Leonori, Porretta. 2010].
- $\lambda > 0$: [Arcoya, Moreno-Mérida, 2017].

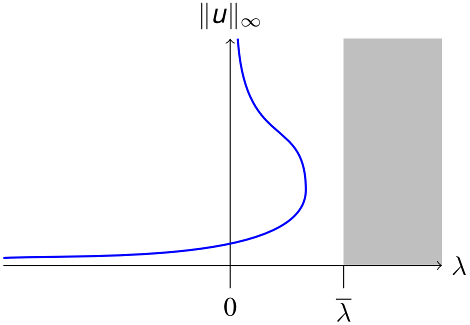
Is this optimal result due only to the presence of a singularity at $u = 0$?



Another singular problem ($0 < \theta < 1$)

$$\begin{cases} -\Delta u = \lambda u + \mu \dfrac{|\nabla u|^2}{u^\theta} + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

- $\lambda > 0$: [Carmona, Leonori, L.M., Martínez-Aparicio. 2017].



The key point is not the singularity itself, but the fact that the homogeneous equation

$$\begin{cases} -\Delta u = \lambda u + \mu \dfrac{|\nabla u|^2}{u} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

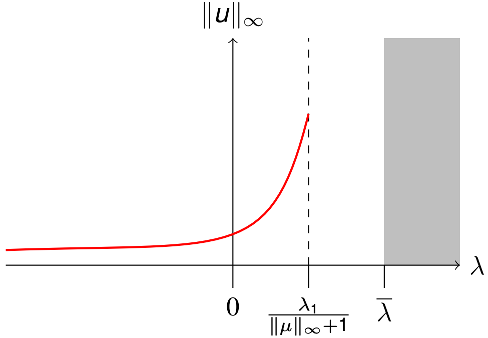
has the following property:

If u is a solution, then tu is also a solution for all $t > 0$.

Hence, a similar optimal existence result should be expected if $\mu = \mu(x)$. However...

$$\begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^2}{u} + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

- $\lambda > 0$: [Arcoya, Moreno-Mérida. 2017].



... $\frac{\lambda_1}{\|\mu\|_\infty+1}$ may not be optimal. Can we find an optimal value for λ?

Outline

J. Carmona, T. Leonori, S.L.M., P.J. Martínez-Aparicio, *Quasilinear elliptic problems with singular and homogeneous lower order terms. Submitted.*

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{u^{q-1}} + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Our goals

1. Determine the set $L = \{\lambda \in \mathbb{R} : \text{ there exists a solution to } (P_\lambda)\}$.
2. Study uniqueness of solution to (P_λ) for all $\lambda \in L$.
3. Analyze other qualitative properties of the solutions: continuity with respect to λ and bifurcation from infinity.

Hypotheses

$$0 \leq \mu \in L^\infty(\Omega), \; 0 \lneqq f \in L^\infty(\Omega),$$

either $1 < q < 2$, or $q = 2$ and $\|\mu\|_\infty < 1$.

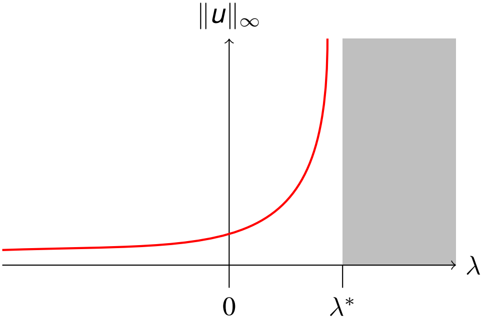
Theorem (J. Carmona, T. Leonori, S.L.M., P.J. Martínez-Aparicio)

Under the previous hypotheses, there exists $\lambda^* = \lambda^*(\mu, q, \Omega) \in (0, \lambda_1]$ such that the following holds for problem (P_λ):

$f \gneqq 0 \implies$ *Either $L = (-\infty, \lambda^*)$ or $L = (-\infty, \lambda^*]$. Uniqueness of solution to (P_λ) for all $\lambda \leq 0$.*

$inf_\Omega(f) > 0 \implies$ *$L = (-\infty, \lambda^*)$. Uniqueness of solution to (P_λ) for all $\lambda \in L$. The set $\Sigma = \{(\lambda, u_\lambda) : u_\lambda$ solves $(P_\lambda)\}$ is a continuum in $L \times C(\overline{\Omega})$ that bifurcates from infinity to the left of λ^*.*

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{u^{q-1}} + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$



$$\lambda^* = \sup \{\lambda \in \mathbb{R} | \exists v \text{ supersolution to } (E_\lambda) \text{ with } inf_\Omega(v) > 0\}$$

$$(E_\lambda) \qquad \begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{u^{q-1}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

H. Berestycki, L. Nirenberg, S. R. S. Varadhan, *The principal eigenvalue and maximum principle for second-order elliptic operators in general domains.* *Comm. Pure Appl. Math.* **47** (1994), no. 1, 47-92.

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Here, λ^* is well defined and $\lambda^* \in (0, \lambda_1]$. Moreover, we prove the following

Theorem (J. Carmona, T. Leonori, S.L.M., P.J. Martínez-Aparicio)

There exists a solution to (E_λ) if and only if $\lambda = \lambda^*$.

$\lambda^* = \sup\{\lambda \in \mathbb{R} | \exists v$ supersolution to (E_λ) with $inf_\Omega(v) > 0\}$

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Theorem (J. Carmona, T. Leonori, S.L.M., P.J. Martínez-Aparicio)

There exists a solution to (E_λ) *if and only if* $\lambda = \lambda^*$.

$\lambda < \lambda^* \implies$ nonexistence to $(E_\lambda) \implies$ existence to (P_λ)

Outline

Existence to (P_λ) for $\lambda < \lambda^*$

$$(Q_n) \qquad \begin{cases} -\Delta u_n = \lambda u_n + \mu(x)\dfrac{|\nabla u_n|^q}{\left|u_n + \frac{1}{n}\right|^{q-1}} + f(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Three steps:

1. $\exists u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ solution to (Q_n) $\forall n \in \mathbb{N}$.
   Proof: Sub and supersolutions method.
2. $\{u_n\}$ is bounded in $L^\infty(\Omega)$.
   Proof: By contradiction, using the nonexistence to (E_λ) for $\lambda < \lambda^*$.
3. $u_n \to u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ solution to (P_λ).
   Proof: $L^\infty(\Omega)$ estimate and positive local lower bound (p.l.l.b.).

Existence to (P_λ) for $\lambda < \lambda^*$

$$(Q_n) \qquad \begin{cases} -\Delta u_n = \lambda u_n + \mu(x) \dfrac{|\nabla u_n|^q}{\left|u_n + \frac{1}{n}\right|^{q-1}} + f(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

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- If $\|u_n\|_\infty \to \infty \implies z_n := \dfrac{u_n}{\|u_n\|_\infty}$ is bounded in $L^\infty(\Omega)$ and satisfies

$$\begin{cases} -\Delta z_n = \lambda z_n + \mu(x)\dfrac{|\nabla z_n|^q}{\left|z_n + \frac{1}{n\|u_n\|_\infty}\right|^{q-1}} + \dfrac{f(x)}{\|u_n\|_\infty} & \text{in } \Omega, \\ z_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Existence to (P_λ) for $\lambda < \lambda^*$

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- Moreover, using carefully the maximum principle we prove the p.l.l.b.:

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0 : \quad z_n \geq c_\omega \text{ in } \omega,\ \forall n.$$

Existence to (P_λ) for $\lambda < \lambda^*$

$$(Q_n) \qquad \begin{cases} -\Delta u_n = \lambda u_n + \mu(x)\dfrac{|\nabla u_n|^q}{\left|u_n + \frac{1}{n}\right|^{q-1}} + f(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Using the a priori estimates we can pass to the limit and find a solution to

$$\begin{cases} -\Delta z = \lambda z + \mu(x)\dfrac{|\nabla z|^q}{z^{q-1}} & \text{in } \Omega, \\ z > 0 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

CONTRADICTION since $\lambda < \lambda^*$. □

Nonexistence to (E_λ) for $\lambda < \lambda^*$

$$\lambda^* = \sup\{\lambda \in \mathbb{R} | \exists v \text{ supersolution to } (E_\lambda) \text{ with } inf_\Omega(v) > 0\}$$

Theorem (J. Carmona, T. Leonori, S.L.M., P.J. Martínez-Aparicio)

If u, v are a subsolution and a supersolution to (E_λ) respectively, and $inf_\Omega(v) > 0$, then $u \leq v$ in Ω.

1. Assume that u is a solution to (E_λ) with $\lambda < \lambda^*$.
2. By homogeneity of (E_λ), the same holds for tu for all $t > 0$.
3. There exists a supersolution v to (E_λ) with $inf_\Omega(v) > 0$.
4. The Comparison Principle implies that $tu \leq v$ in Ω.
5. CONTRADICTION by taking t large. □

Optimality of λ^*: nonexistence for $\lambda > \lambda^*$ and $\lambda = \lambda^*$

$$\lambda^* = \sup\{\lambda \in \mathbb{R} | \exists v \text{ supersolution to } (E_\lambda) \text{ with } inf_\Omega(v) > 0\}$$

- If u is a solution to (P_λ) with $\lambda > \lambda^*$, then there exist $\gamma, \varepsilon > 0$ such that

$$v = \varepsilon(\varphi_1^\gamma + 1) + u^\gamma$$

is a supersolution to $(E_{\bar\lambda})$ with $inf_\Omega(v) > 0$, for $\lambda^* < \bar\lambda < \lambda \implies$ CONTRADICTION □

What if $\lambda = \lambda^*$?

- If $inf_\Omega(f) > 0$ and u is a solution to (P_{λ^*}), then there exists $c > 0$ such that $v = u + c$ is a supersolution to (E_{λ^*+c}) with $inf_\Omega(v) > 0 \implies$ CONTRADICTION □

Comments on uniqueness and bifurcation

Theorem (J. Carmona, T. Leonori, S.L.M., P.J. Martínez-Aparicio)

If u, v are a subsolution and a supersolution to (P_λ) respectively, and $\inf_\Omega(f) > 0$, then $u \leq v$ in Ω.

Corollary

If $\inf_\Omega(f) > 0$, then problem (P_λ) admits at most one solution.

Proposition

If $\inf_\Omega(f) > 0$, then the set $\Sigma = \{(\lambda, u_\lambda) : u_\lambda \text{ solves } (P_\lambda)\}$ is a continuum in $L \times C(\overline{\Omega})$ that bifurcates from infinity to the left of λ^*.

Outline

An observation

$$(Q_n) \qquad \begin{cases} -\Delta u_n = \lambda u_n + \mu(x) \dfrac{|\nabla u_n|^q}{\left|u_n + \frac{1}{n}\right|^{q-1}} + f(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

In order to pass to the limit in (Q_n) we used the p.l.l.b.:

$$\forall \omega \subset\subset \Omega, \ \exists c_\omega > 0 : \quad u_n \geq c_\omega \text{ in } \omega, \ \forall n.$$

Thus, the solution is avoided locally. However,

$$\text{if } \left\{ \frac{|\nabla u_n|^q}{\left|u_n + \frac{1}{n}\right|^{q-1}} \right\} \text{ is bounded in } L^1(\Omega),$$

(which happens when $1 < q < 2$) we can pass to the limit without using the p.l.l.b.

An observation

$$(Q_n) \qquad \begin{cases} -\Delta u_n = \lambda u_n + \mu(x)\dfrac{|\nabla u_n|^q}{\left|u_n + \frac{1}{n}\right|^{q-1}} + f(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

For fixed $\phi \in C_c^1(\Omega)$, $\delta > 0$, we split

$$\int_\Omega \mu(x)\frac{|\nabla u_n|^q}{\left|u_n + \frac{1}{n}\right|^{q-1}}\phi = \int_{\{|u_n|\geq\delta\}} \mu(x)\frac{|\nabla u_n|^q}{\left|u_n + \frac{1}{n}\right|^{q-1}}\phi + \int_{\{|u_n|<\delta\}} \mu(x)\frac{|\nabla u_n|^q}{\left|u_n + \frac{1}{n}\right|^{q-1}}\phi$$

An observation

$$(Q_n) \qquad \begin{cases} -\Delta u_n = \lambda u_n + \mu(x)\dfrac{|\nabla u_n|^q}{\left|u_n + \frac{1}{n}\right|^{q-1}} + f(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, we may consider problems for which one does not expect to have a positive local lower bound. For instance, if f changes sign.

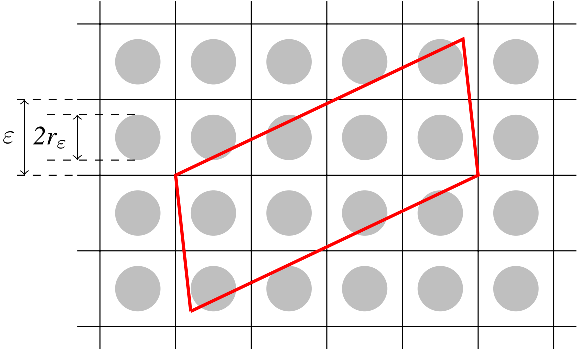
Theorem

If $f \not\equiv 0$ *(but may change sign),* $1 < q < 2$ *and* $\lambda < \lambda^*$*, there exists a solution to*

$$\begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{|u|^{q-1}} + f(x) & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega. \end{cases}$$

Another example without p.l.l.b.: *Homogenization*

- Let Ω^ε be a domain consisting of removing many small spherical holes from Ω uniformly distributed.



Another example without p.l.l.b.: *Homogenization*

- Let Ω^ε be a domain consisting of removing many small spherical holes from Ω uniformly distributed.
- Let $u^\varepsilon : \Omega^\varepsilon \to \mathbb{R}$ be a solution to
$$\begin{cases} -\Delta u^\varepsilon = \lambda u^\varepsilon + \mu(x)\dfrac{|\nabla u^\varepsilon|^q}{|u^\varepsilon|^{q-1}} + f(x) & \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon. \end{cases}$$
- Define $u^\varepsilon = 0$ in $\Omega \setminus \Omega^\varepsilon$. $\implies$ There is not a p.l.l.b. for $\{u^\varepsilon\}$

Theorem

If $1 < q < 2$ and $\lambda < \lambda^*$, there exists a constant $\sigma > 0$ such that u^ε weakly converges in $H_0^1(\Omega)$ to a solution to the problem
$$\begin{cases} -\Delta u = (\lambda - \sigma)u + \mu(x)\dfrac{|\nabla u|^q}{|u|^{q-1}} + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Thanks for your attention!

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