Variational methods for the study of long-time behavior and traveling waves in parabolic gradient systems

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Abstract. As a model problem, consider the parabolic gradient system

$$u_t - u_{xx} = -\nabla V(u), \text{ in } \mathbb{R} \times (0, +\infty), \; V : \mathbb{R}^d \to \mathbb{R}, \; u : \mathbb{R} \times (0, +\infty) \to \mathbb{R}^d. \quad (0.1)$$

The system (0.1) appears in the modeling of some problems in biology (population dynamics) as well as in physics and chemistry (combustion, flame propagation). In several settings, it is known that the long-time dynamics obtained from initial conditions which invade a critical point of V are governed by the traveling wave solutions of (0.1), i. e., solutions of the type $u(x,t) = U(x - ct)$ for some profile U and speed c. The first proofs of this fact were provided in classical works by Fisher, Kolmogorov, Petrovski and Piskunov (1930s) and Fife and McLeod (1970s). These proofs are restricted to the scalar case $d = 1$ as they rely on the maximum principle. However, the works by Fife and McLeod in the 1970s identified a variational, gradient flow structure for (0.1) which suggested the possibility of a purely variational approach, not relying on the maximum principle and therefore able to handle the vectorial case $d \geq 2$. This program was initiated later, in the early 2000s, by Heinze, Muratov and Risler. It has lead to quite complete results in several vector-valued situations, but some challenging problems are still open. In this talk I will attempt to give a general introduction to this subject and then I will present my contributions to it, some of them obtained during my PhD studies (under the supervision of Fabrice Bethuel in Paris 6) and others later in collaboration with Emmanuel Risler (INSA de Lyon).