Orthogonality in the Hyperbolic Plane

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1 INTRODUCTION

Constructions which involve the orthogonality are necessary for the resolution of many problems of the plane hyperbolic geometry, such as the construction of the circumference, the horocycles, the hypercycles[4], translations according to a line, Saccheri[6] and Lambert[5] quadrilaterals and others, in the two models of Poincaré for the hyperbolic plane: the half{plane H^2 and the unit disc D^2[3].

These constructions can be classi¯ed in three types[2]:

1.- Determination of the orthogonal line to another one given through a point of it.

2.- Determination of the unique orthogonal line to a pair of ultraparallel lines.

3.- Determination of the orthogonal line to another one given through an outer point.

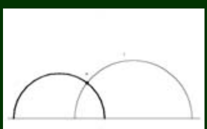
2 ORTHOGONAL LINE TO ANOTHER ONE GIVEN THROUGH A POINT OF IT

Given a line l determine the orthogonal line to l through a point P belonging to l.

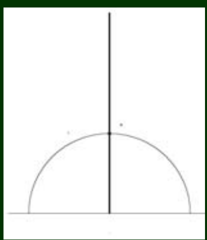
2.1 DETERMINATION IN H^2

Let α be the angle determined by the tangent euclidean line to l at the point P with the euclidean line $y = 0$. We can consider two cases.

1.- $\alpha \neq 0$



2.- $\alpha = 0$



2.2 DETERMINATION IN D^2

Let P be as point in l. By the Cayley transformation φ we construct the line $l' = \varphi^{-1}(l)$ and the point $P' = \varphi^{-1}(P)$ in H^2.

Now, in H^2, we construct the orthogonal line to l' through P'. Applying φ we obtain the corresponding orthogonal line in D^2.

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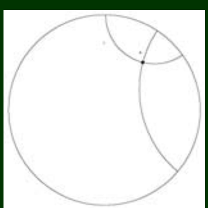
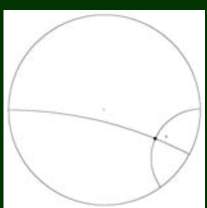
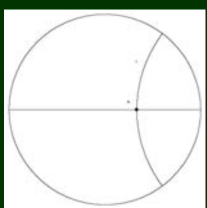


2.- $\alpha = 0$



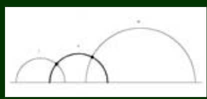


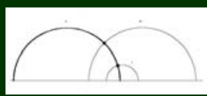
3 ORTHOGONAL LINE TO A PAIR OF ULTRAPARALLEL LINES

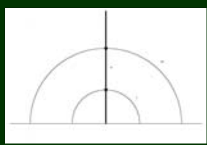
Given two ultraparallel lines l and m, determine the unique orthogonal line to both.

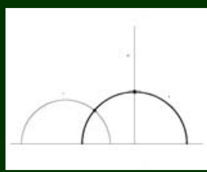
3.1 DETERMINATION IN H^2

From the euclidean point of view exist four possibilities of relative position for these two ultraparallel lines. From this, we can obtain the following situations:







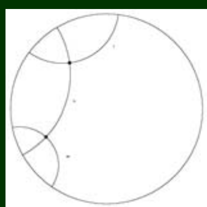


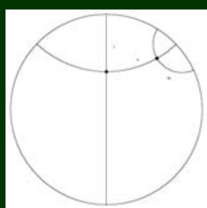
3.2 DETERMINATION IN D^2

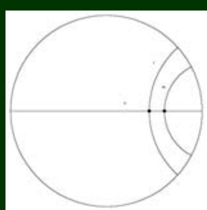
Let l and m two ultraparallel lines in D^2. By the Cayley transformation φ we construct the lines $l' = \varphi^{-1}(l)$ and $m' = \varphi^{-1}(m)$ in H^2.

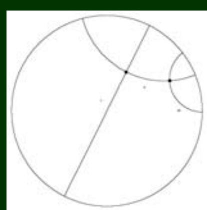
Now, in H^2, we construct the unique orthogonal line to l' and m'. Applying φ we obtain the corresponding orthogonal line in D^2.

We can obtain the following situations:







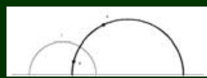


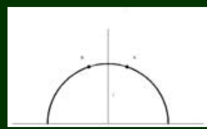
4 ORTHOGONAL LINE TO ANOTHER ONE GIVEN THRUGH AN OUTER POINT

Given a line l and an outer point of it P, determine the orthogonal line to l through P.

3.1 DETERMINATION IN H^2

Let Q be the refection of the point P respect to the line l. The sought orthogonal line is the line through P and Q.

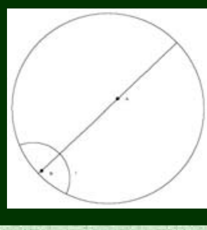


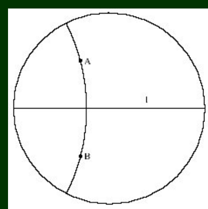


3.2 DETERMINATION IN D^2

Let l and P a line and an outer point in D^2, respectively. By the Cayley transformation φ we construct the line $l' = \varphi^{-1}(l)$ and $P' = \varphi^{-1}(P)$ in H^2.

Now, in H^2, we construct the orthogonal line to l' through P'. Applying φ we obtain the corresponding orthogonal line in D^2.





REMARK

We refer the reader to [3], which presents an electronic tool named Hyperbol ([1]) whose computational support is Mathematica software. This tool consists of modules that allow us to draw di®erent hyperbolic constructions in Poincare's models for the hyperbolic plane, usually denoted by H^2 and D^2. Such constructions include re°ections, rotations, translations, glide re°ections, and the orbits of a point. These isometries and geometric loci act on the hyperbolic plane; and if a euclidean element appears in some representation of this plane, we shall note it in an explicit way.

[1]Software available at http://www.ugr.es/local/ruiz/software.htm

REFERENCES

[1] H.S.M. Coxeter, Fundamentos de geometría, Limusa-Wiley, S.A. México, 1971.

[2] D. Gámez, Construcciones en Geometría Hiperbólica y Teselaciones mediante Grupos NEC Poligonales. Algoritmos de Automatización, PhD Thesis, Granada, 2001.

[3] D. Gámez, M. Pasadas, R. Pérez, C. Ruiz, Hyperbolic Plane Tesselations. Proceedings of the VI Journées Zaragoza-Pau de Mathematiques Appliquées et Statistique, pp. 257-264, Jaca, Spain, 1999.

[4] D. Gámez, M. Pasadas, R. Pérez, C. Ruiz, Orbits in the Hyperbolic Plane. Proccedings of the VII Journées Zaragoza-Pau de Mathematiques Appliquées et Statistique, Jaca, Spain, 2001.

[5] D. Gámez, M. Pasadas, R. Pérez, C. Ruiz, The Lambert quadrilateral and tesselations in the hyperbolic plane. Int. Math. J., Vol. 2, pp. 777-795, 2002.

[6] D. Gámez, M. Pasadas, R. Pérez, C. Ruiz, The Saccheri Quadrilateral, Translations and Tesselations in the Hyperbolic Plane (Submitted).