NEC Polygonal groups and tesselations

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NECS GROUPS AND FUNDAMENTAL REGIONS

Let X be the hyperbolic plane with the topology induced by the hyperbolic metric. The model we shall use is the open disc unit of Poincaré D^2. Let Γ be a subgroup of the group of hyperbolic isometries of X, $Iso(X)$. The action of Γ on X is the natural application $\Theta : \Gamma \times X \to X$, given by $\Theta(\gamma; z) = \gamma(z)$. For each $z \in X$ we denote the set $\Sigma = \{\gamma(z) : \gamma \in \Gamma\}$ as the orbit of z by the action of Γ upon X. The action of a subgroup Γ on X gives rise to the following relationship: $\forall z, w \in X;\ z \sim w,\ \exists \gamma \in \Gamma : w = \gamma(z)$.

This relationship $\sim$ is an equivalence relation on X, whose equivalence classes are the orbits of the elements of X by the action of Γ. The quotient set $X/\sim$ will be designated as $\Gamma \backslash X$ and is called the orbit space.

Definition 1 Let Γ be a subgroup of $Iso(X)$. Γ is said to be an NEC group, that is, a Non-Euclidean Crystallographic Group, if it is discrete (with compact open topology) and the orbit space is compact.

If Γ is an NEC group, then its rotations have multiple integer angles of $2\pi/n$, with $n \in \mathbb{N}$, and they do not contain limit rotations that is, rotations whose center lies on the infinity line.

The fundamental regions comprise the smallest tile that allow us to tessellate by means of the action of an NEC group. For this purpose we establish the following definition.

Definition 2 ([3]) Let $F \subset X$ be closed and Γ an NEC group. F is said to be a fundamental region for Γ if:

i) for each $z \in X$ there exists $\gamma \in \Gamma$ so that $\gamma(z) \in F$

ii) if $z \in F$ is such that $\gamma(z) \in F$, with $\gamma \neq 1$, then $z, \gamma(z)$ are in boundary of F

iii) $F = Closure(int(F))$.

It is important to point out that this definition is an improvement upon those used in [10] and [12], in the sense that we eliminate the cases of pathological regions having isolated points that correspond with other points of the fundamental region under consideration.

TESSELATIONS OF THE HYPERBOLIC PLANE

Definition 3 Let Γ be a NEC group. A Γ-tessellation of the hyperbolic plane X, with the fundamental region F is a set of regions $\{F_i\}_{i \in I}$ such that:

i) $\bigcup_{i \in I} F_i = X$

ii) $int(F_i) \cap int(F_j) = \emptyset$ for $i \neq j$

iii) $\forall i \in I; \exists \gamma \in \Gamma : F_i = \gamma(F)$.

In the study of subgroups of NEC groups, as well as int he tessellations of X, the following proposition is important:

Proposition 1 ([11]) Let Γ by an NEC group and Γ' a subgroup of Γ with the index p ($\Gamma : \Gamma' = p$). Let F^{Γ} by a fundamental region for Γ. Then Γ' has a fundamental region $F^{\Gamma'}$ resulting from the union of p congruent replicas of F^{Γ} and besides $|\Gamma| \, p = |\Gamma'|$.

Polygonal groups are those that are generated by reflections in the sides of a polygon. Certain hyperbolic polygons give rise to NEC groups, and the hyperbolic plane is tessellated with them.

Lemma 1 ([1]) Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be real numbers with $0 \le \alpha_i < \pi$ for each $i = 1, 2, \ldots, k$. Then, there exists a convex hyperbolic polygon F with angles $\alpha_1, \alpha_2, \ldots, \alpha_k$ if, and only if, $\sum_{i=1}^{k} \alpha_i < (k-2)\pi$

Theorem 1 ([10]) Let Γ be an NEC group. Then, there exists a convex polygon F that constitutes a fundamental region for Γ.

Theorem 2 Let F be a convex polygon of k sides and interior angles π/n_i, with $n_i \in \mathbb{N}; n_i \geq 2$, for each $i = 1, 2, \ldots, k$, with vertices P_i and sides determined by the segments $P_{i-1}P_i$ where $P_0 = P_k$, satisfying $\sum_{i=1}^{k} \pi/n_i < (k-2)\pi$. Let σ_i be the reflection on the line containing the segment $P_{i-1}P_i$ for each $i = 1, 2, \ldots, k$. Then the group Γ generated by the reflections $\sigma_1, \sigma_2, \ldots, \sigma_k$ (polygonal group) is an NEC group.

In addition, the family $\{\gamma(F) : \gamma \in \Gamma\}$ constitutes a Γ-tessellation of X.

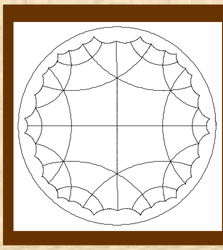
Example 1 Tessellations of the Poincaré disk constructed by M.C. Escher using polygonal groups: Cirkellimiet I (Circle limit I) and Cirkellimiet III (Circle limit III).

KALEIDOSCOPIC SACCHERI AND LAMBERT QUADRILATERALS

The Saccheri quadrilaterals have two consecutive right angles on a side named base, and two equal and acute angles opposite to the base.

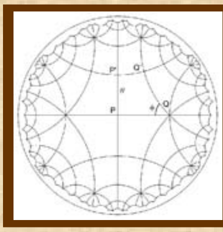
The Lambert quadrilateral is a quadrilateral with three right angles. The union of two Lambert quadrilaterals is a Saccheri quadrilateral: that is, it has two adjacent right angles and two equal acute angles. And, conversely, if in a Saccheri quadrilateral we trace the unique perpendicular line that is common to the base and the opposite side, we obtain two Lambert quadrilaterals, congruent by means of reflection with respect to the common perpendicular.

Theorem 3 ([7]) For every $R > 0$ and $n \in \mathbb{N}$, $n > 2$, there exists a unique Saccheri quadrilateral with the base R and acute angles $\varphi = \pi/n$, unique up to congruence, that tessellates the hyperbolic plane.

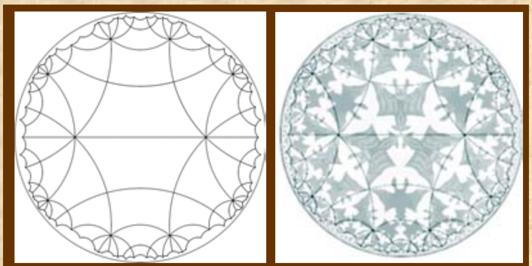


Example 2 As an example illustrating this situation, we show below tessellation of a Poincaré disc by means of a Saccheri quadrilateral for $\varphi = \frac{\pi}{3}$ and $R = 1$, created with the Hyperbol package for Mathematica software that has been developed by the authors.

Theorem 4 ([6]) For each $R > 0$ and $n \in \mathbb{N}$, $n > 2$, there exists a unique Lambert quadrilateral $L(P, Q, Q', P')$ with three right angles, side PP' or length R, and acute angle $\phi = \pi/n$, unique up to congruence, that tessellates the hyperbolic plane.



Example 3 As an example illustrating this situation, we show below two tessellations of the Poincaré disc, created also with the Hyperbol package, using a Lambert quadrilateral for $\phi = \pi/4$, and $R \simeq 0.881373$.



Example 4 We use the Saccheri quadrilateral that is obtained from the union of the Lambert quadrilateral, of the above example, and its reflection with respect to the line that contains the segment of length R. This Saccheri quadrilateral is subjacent in the work of Escher in the Cirkellimiet IV (Circle limit IV), known as Angels and Devils, as it is the union of three consecutive triangles, which contain the figures appearing in this creation.

TESSELLATIONS IN DE POINCARÉ HALF-PLANE

All the mathematical tools that we have described for D^2 can be transcribed in the model of the Poincaré half-plane denoted by H^2. Both models are geometrically equivalent and the interested reader cam find this equivalence in [3]. The Hyperbol package, mentioned above, also features drawing tools to work on the half-plane. Their usefulness is evident when we regard some of the works by M.C. Escher based upon this model. Two examples follows:

Example 5 Tessellation of the Poincaré half-plane, in the study work Houtsnede VI (Woodcut VI) by Escher, made with triangular groups ([4]).

Example 6 The work Schmetterlinge (Butterflies) by M.C. Escher ([14]) is based on an easily recognizable polygonal NEC group.

CONCLUSIONS

We refer the reader to [3], which presents an electronic tool named Hyperbol ([1]) whose computational support is Mathematica software. This tool consists of modules that allow us to draw different hyperbolic constructions in Poincare's models for the hyperbolic plane, usually denoted by H^2 and D^2. Such constructions include reflections, rotations, translations, glide reflections, and the orbits of a point. These isometries and geometric loci act on the hyperbolic plane; and if a euclidean element appears in some representation of this plane, we shall note it in an explicit way.

[1] Software available at http://www.ugr.es/local/ruiz/software.htm

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