A variational approach for the Neumann problem in some FLRW spacetimes

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Abstract

In this paper we study, using critical point theory for strongly indefinite functionals, the Neumann problem associated to some prescribed mean curvature problems in a FLRW spacetime with one spatial dimension. We assume that the warping function is even and positive and the prescribed mean curvature function is odd and sublinear. Then, we show that our problem has infinitely many solutions. The keypoint is that our problem has a Hamiltonian formulation. The main tool is an abstract result of Clark type for strongly indefinite functionals.

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1 Introduction

In this paper we initiate the study by variational methods of the Neumann problem associated to the prescribed mean curvature equation in a certain family of Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetimes. The FLRW

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metric models a spatially homogeneous and isotropic Universe and has the expression

$$ds^2 = -dt^2 + f(t)dx^2,$$

where $f(t)$ is a positive function of time called the scale factor or warping function in the related literature (see for example [15]).

The mathematical problem under consideration in this paper is described as follows. Let $f \in C^1([-\varepsilon, \varepsilon])$ be a positive function. For $R > \rho \geq 0$ and a given continuous function $g : [\rho, R] \times \mathbb{R} \to \mathbb{R}$, we look for solutions of the following ODE with Neumann conditions

$$\begin{gathered} -\left(\frac{q'}{\sqrt{f(q)^2 - q'^2}}\right)' - \frac{f(q)f'(q)}{\sqrt{f(q)^2 - q'^2}} = f(q)g(r,q) \\ |q'| < f(q) \quad \text{in} \quad [\rho, R], \\ q'(\rho) = 0 = q'(R). \end{gathered} \tag{1}$$

A solution of the above Neumann problem is a function $q \in C^1([\rho, R])$ such that $||q||_\infty < \varepsilon$, $|q'| < f(q)$ in $[\rho, R]$, $\frac{q'}{\sqrt{f(q)^2 - q'^2}} \in C^1([\rho, R])$ and the above problem holds true.

The study of Neumann problems in FLRW spacetimes started very recently with the paper [13] (see also [16]). The main result of these works is an existence result in a ball centered in the origin and a small radius, proved by using Leray-Schauder degree. In the present paper, following [12], we prove a multiplicity result assuming that the prescribed mean curvature function is odd and sublinear and the warping function is even. To the best of our knowledge, it is the first time that variational methods are applied to this problem.

Consider $G : [\rho, R] \times \mathbb{R} \to \mathbb{R}$ given by

$$G(r, q) = \int_0^q f(t)g(r, t)dt.$$

The main result of this paper is the following

Theorem 1 *Assume that $g(r, \cdot)$ is odd for all $r \in [\rho, R]$, f is even with*

$$f(0) = \max_{[-\varepsilon,\varepsilon]} f, \quad f(0) - f(q) \leq dq^2, \tag{2}$$

for all $q \in [-\varepsilon, \varepsilon]$ for some $d > 0$. If

$$\lim_{q \to 0} \frac{G(r, q)}{q^2} = +\infty \quad \textit{uniformly in} \;\; r \in [\rho, R], \tag{3}$$

then, (1) has infinitely many solutions q_k with $||q_k||_\infty \to 0$ as $k \to \infty$.

The following example illustrates the applicability of the result.

Example 1 Consider the warping function (see [13, 15])

$$f(t) = \beta(\cos \alpha t)^{2/3} \qquad (\alpha, \beta > 0)$$

or the Minkowski case (see [4, 8, 9])

$$f(t) = 1.$$

Assume that $1 < s < 2 < t$, $\lambda > 0$ and $\mu \in \mathbb{R}$. Then, the Neumann problem (1) with

$$g(u) = \lambda |u|^{s-2}u + \mu |u|^{t-2}u$$

has infinitely many solutions q_k with $||q_k||_\infty \to 0$ as $k \to \infty$. See [1, 6] for the analogous result for the Laplacian operator.

For the proof, a key observation is that the above Neumann problem is equivalent to the Hamiltonian system

$$p' = -H_q(r,p,q), \quad q' = H_p(r,p,q), \quad p(\rho) = 0 = p(R),$$

where the Hamiltonian function H is given by

$$H(r,p,q) = f(q)\sqrt{1+p^2} - f(0) + G(r,q),$$

and

$$p = \frac{q'}{\sqrt{f(q)^2 - q'^2}}.$$

Once the hamiltonian setting has been identified, our main tool is a modification of a Clark's type result for strongly indefinite functionals given in Theorem 1.2 from [12]. Using this abstract result, it is shown in [12] that a general Hamiltonian system has infinitely many periodic solutions if the Hamiltonian function is even in the spatial variable and sublinear around zero. We observe that our H defined above does not satisfy this condition due to the presence of the term $f(q)\sqrt{1+p^2} - f(0)$. An analogous situation occurs in [11] in the case of first and second order superquadratic Hamiltonian systems. To prove both cases simultaneously, the main idea in [11] is to use an auxiliar operator B of the form

$$B(p,q) = (\mu^s p, \mu^t q),$$

together with a minimax theorem including B which is a version of Benci - Rabinowitz minimax theorem for strongly indefinite functionals (see [7, 14]). Notice that a different situation occurs for first and second order forced superlinear autonomous systems. It is proved separately in [2, 3] that this type of systems have infinitely many periodic solutions, and there is no proof unifying both cases. In this paper we use like in [11] an auxiliar operator L_μ and a critical point theorem including this operator which is a version of Theorem 1.2 from [12]. Then, we will apply this abstract result to the action functional associated to the above Hamiltonian system.

Problem (1) corresponds to the prescribed mean curvature equation with Neumann conditions in a FLRW spacetime with n=1 spatial dimension. The consideration of the Neumann problem in the ball for a general FLRW spacetime with arbitrary dimension $n \geq 1$ is an interesting open problem. The use of a Theorem of Clark type imposes necessarily the even symmetry in the functional, which implies serious restrictions on the warping and the prescribed curvature functions. As a counterpart, for $n = 1$ we find an infinite number of solutions, a result that seems quite new in this context. At this moment, we are not able to identify a variational structure for the case $n \geq 2$, which is another intriguing open question.

This paper is organized as follows. In Section 2, we prove the abstract version of Theorem 1.2 from [12]. In Section 3, we show that our problem has a Hamiltonian formulation and we introduce the functional setting. Section 4 is devoted to the proof of the main result.

2 An abstract result

Let E be a Hilbert space such that $E = \oplus_{m\in\mathbb{Z}} E_m$ where $\dim E_m = 1$ for all $m \in \mathbb{Z}$. We denote $E^{\pm} = \oplus_{m\geq 1} E_{\pm m}$. For any positive integer k let $L_k : E \to E$ be a linear bijective operator such that $(||L_k||)$ is a bounded sequence and

$$L_k\Big(\bigoplus_{m=-n}^{n} E_m\Big) = \bigoplus_{m=-n}^{n} E_m \quad \text{for all } n \geq 1.$$

On the other hand, let

$$X_n = \bigoplus_{m\geq -n} E_m \quad (n \geq 1).$$

For $I \in C^1(E, \mathbb{R})$, we recall that I satisfies (PS) condition if any sequence $(u_k) \subset E$ for which $(I(u_k))$ is bounded and $I'(u_k) \to 0$ as $k \to \infty$, possesses a convergent subsequence. Also, I is said to satisfy the $(PS)^*$ condition with respect to the sequence of subspaces (X_n) if for any subsequence (n_j) of (n), any sequence (u_{n_j}) such that $u_{n_j} \in X_{n_j}$ for all j, $(I(u_{n_j}))$ is bounded and $||(I|_{X_{n_j}})'(u_{n_j})|| \to 0$ as $j \to \infty$, contains a subsequence converging to a critical point of I.

In the next lemma, we denote $S_\rho = \{u \in E : ||u|| = \rho\}$.

Lemma 1 *Let $I \in C^1(E, \mathbb{R})$ be even with $I(0) = 0$, $I|_{E^+}$ satisfies (PS) condition and I satisfies $(PS)^*$ condition with respect to the sequence of subspaces (X_n). Suppose that*

(i) the functional $I|_{E^+}$ is bounded from below,

(ii) there exists $\rho_k, \varepsilon_k > 0$ with $\rho_k \to 0$ such that, for all $k \geq k_0$

$$\sup_{L_k(S_{\rho_k} \cap (\oplus_{m\leq k} E_m))} I \leq -\varepsilon_k,$$

for some positive integer k_0.

Then I possesses a sequence of critical points (u_k) such that $||u_k|| \to 0$ as $k \to \infty$.

Proof. We will show, following [12], that at least one of the following propositions holds.

(i) There exists a sequence of critical points (u_k) such that $I(u_k) < 0$ for all k and $||u_k|| \to 0$ as $k \to \infty$.

(ii) There exists $r > 0$ such that for any $0 < a \leq r$ there exists a critical point $u \in S_a$ with $I(u) = 0$.

Assume by contradiction that both (i) and (ii) are false. It follows that there exists $0 < r_1 < r_0$ such that if one of the following propositions holds true

- $||u|| \leq r_0$ and $I(u) < 0$,
- $||u|| = r_1$ and $I(u) = 0$,

then, $I'(u) \neq 0$. Using that $I(0) = 0$, we may assume that

$$I(u) > -1 \quad \text{for all } ||u|| \leq r_0. \tag{4}$$

Let us denote $I_A^c = \{u \in E : u \in A, I(u) \leq c\}$. Then, using the $(PS)^*$ condition it follows that there exists a, b with $0 < a < r_1 < b \leq r_0$, $\nu_1 > 0$ and a positive integer n_1, such that for any $n \geq n_1$ one has that

$$||(I|_{X_n})'(u)|| \geq \nu_1 \quad \text{for all } u \in I^0_{X_n} \text{ with } a \leq ||u|| \leq b. \tag{5}$$

Now, for $n \geq n_1$, consider $K^n = \{u \in X_n : (I|_{X_n})'(u) = 0\}$ and let $W_n : X_n \setminus K^n \to X_n$ be an odd pseudogradient vector field of $I|_{X_n}$. For a fixed $u \in X_n \setminus K^n$ there exists a unique maximal solution $\eta_n(\cdot, u) : [0, T_n(u)[\to X_n \setminus K^n$ of the following Cauchy problem in $X_n \setminus K^n$

$$\frac{d}{dt}\eta(t, u) = -W_n(\eta(t, u)) \ (t \geq 0), \quad \eta(0, u) = u.$$

We also define $\eta_n(t, u) = u$ for $t \geq 0$ and $T_n(u) = \infty$ if $u \in K^n$. Then, using (5) we deduce that if $u \in I^0_{X_n \setminus K^n}$ and $0 \leq t_1 < t_2 < T_n(u)$ are such that $||\eta_n(t_1, u)|| = a$, $||\eta_n(t_2, u)|| = b$ and $a < ||\eta_n(t, u)|| < b$ for all $t \in]t_1, t_2[$, then

$$I(\eta_n(t_2, u)) \leq -\mu_0 := -\frac{(b-a)\nu_1}{2}. \tag{6}$$

Next, using that I is even, $I(0) = 0$, $I|_{E^+}$ is bounded from below and satisfies (PS) it follows that

$$k_0 := \gamma(I^{-\mu_0}_{E^+}) < \infty, \tag{7}$$

where γ is the generalized genus (see [5]). Then, using that $I^{-\mu_0}_{E^+}$ is closed in $I^{-\mu_0}_{X_n}$ and (7), one has that

$$\gamma(I^{-\mu_0}_{X_n}) \leq \gamma(X_n \setminus E^+) + \gamma(I^{-\mu_0}_{E^+}) = n + 1 + k_0. \tag{8}$$

Let $k_1 > k_0$ be such that $\rho_k ||L_k|| < a$ for all $k \geq k_1$. Fix $k \geq k_1$. Then, the choice of r_0 and $(PS)^*$ imply that there exists $\nu_2 > 0$ and $n_2 \geq n_1 + k + 1$ such that for any $n \geq n_2$ one has that

$$||(I|_{X_n})'(u)|| \geq \nu_2 \quad \text{for all } u \in I_{X_n}^{-\varepsilon_k},\ ||u|| \leq r_0. \tag{9}$$

We denote

$$A_k = L_k \left(S_{\rho_k} \cap \bigoplus_{j=-n}^{k} E_j \right)$$

and note that $A_k \subset \oplus_{j=-n}^{n} E_j \subset X_n$.

One has that for all $u \in A_k$ there exists $s_n(u) \in]0, T_n(u)[$ such that $||\eta_n(s_n(u), u)|| > b$. Suppose by contradiction that there exists $u \in A_k$ such that $||\eta_n(t,u)|| \leq b \leq r_0$ for all $t \in]0, T_n(u)[$. From (4) it follows that

$$-1 < I(\eta_n(t,u)) \leq I(u) \leq -\varepsilon_k \quad \text{for all } t \in]0, T_n(u)[.$$

Hence, for all $t \in]0, T_n(u)[$, using the properties of W_n,

$$\begin{aligned} 1 \geq 1 - \varepsilon_k &\geq I(u) - I(\eta_n(t,u)) \\ &\geq \int_0^t ||(I|_{X_n})'(\eta_n(s,u))||^2 ds \\ &\geq \nu_2^2 t, \end{aligned}$$

which implies that $T_n(u) \leq \nu_2^{-2}$. It follows that

$$||\eta_n(t_1, u) - \eta_n(t_2, u)|| \leq 2(t_2 - t_1)^{1/2} \quad \text{for all } 0 < t_1 < t_2 < T_n(u),$$

and there exists the limit $u^* = \lim_{t \to T_n(u)-} \eta_n(t,u)$. One has that $u^* \in I_{X_n \cap B_b}^{-\varepsilon_k}$, which implies that $u^* \in X_n \backslash K_n$. This gives a contradiction with the maximality of the interval $[0, T_n(u)[$.

Now, consider $u \in A_k$. Using that $||u|| < a$ and $||\eta_n(s_n(u), u)|| > b$, it follows that there exists $0 < t_1 < t_2 < s_n(u)$ such that $||\eta_n(t_1, u)|| = a$, $||\eta_n(t_2, u)|| = b$ and $a < ||\eta_n(t,u)|| < b$ for all $t \in]t_1, t_2[$. Notice that $u \in I_{B_a}^{-\varepsilon_k}$, which implies that $u \in I_{X_n \backslash K^n}^0$. Hence, using (6) we deduce that

$$I(\eta_n(s_n(u), u)) < I(\eta_n(t_2, u)) \leq -\mu_0.$$

Then, following [12], for all $n \geq n_2$ there exists an odd continuous function

$$h_n : A_k \to I_{X_n}^{-\mu_0}.$$

Finally, using (8), one has that

$$\begin{aligned} n + 1 + k &= \gamma \left(S_{\rho_k} \cap \bigoplus_{j=-n}^{k} E_j \right) \\ &= \gamma(A_k) \\ &\leq \gamma(h_n(A_k)) \\ &\leq \gamma(I_{X_n}^{-\mu_0}) \\ &\leq n + 1 + k_0, \end{aligned}$$

which contradicts the fact that $k > k_0$. The proof is completed. ∎

3 Hamiltonian formulation and the functional setting

3.1 The Hamiltonian

The following result shows that the above Neumann nonlinear differential equation has a Hamiltonian structure. We have the following key

Lemma 2 *Consider the Hamiltonian function* $H : [\rho, R] \times \mathbb{R} \times]-\varepsilon, \varepsilon[\to \mathbb{R}$ *given by*

$$H(r, p, q) = f(q)\sqrt{1 + p^2} - f(0) + G(r, q),$$

and $p, q \in C^1([\rho, R])$ *with* $||q||_\infty < \varepsilon$. *The following statements are equivalent:*
(i) q *is a solution of (1) and, for all* $r \in [\rho, R]$,

$$p(r) = \frac{q'(r)}{\sqrt{f(q(r))^2 - q'^2(r)}}. \tag{10}$$

(ii) (p, q) *is a solution of the Hamiltonian system*

$$p'(r) = -H_q(r, p(r), q(r)), \quad q'(r) = H_p(r, p(r), q(r)) \quad (r \in [\rho, R]) \tag{11}$$
$$p(\rho) = 0 = p(R).$$

Proof. We will prove that (ii) implies (i). The reversed implication is analogous and easier. One has that

$$p'(r) = -f'(q(r))\sqrt{1 + p^2(r)} - f(q(r))g(r, q(r)) \quad (r \in [\rho, R]),$$

and

$$q'(r) = \frac{f(q(r))p(r)}{\sqrt{1 + p^2(r)}} \quad (r \in [\rho, R]).$$

Hence

$$q'(\rho) = 0 = q'(R).$$

Then, using the second equation and that f is positive, it follows that

$$|q'(r)| < f(q(r)) \quad \text{for all} \quad r \in [\rho, R],$$

and (10) holds true. This, together with the second equation imply that, for all $r \in [\rho, R]$,

$$\sqrt{1 + p^2(r)} = \frac{f(q(r))}{\sqrt{f(q(r))^2 - q'^2(r)}}.$$

Then, the first equation gives the conclusion. ∎

3.2 The functional

Consider $\omega = \frac{\pi}{R-\rho}$ and, for any positive integer m,

$$e_m(r) = \sin(m\omega(r-\rho)) \quad (r \in [\rho, R])$$

Let $H_0^{1/2}$ be the set of functions $p \in L^2(\rho, R)$ having Fourier expansion

$$p = \sum_{m \geq 1} p_m e_m$$

such that

$$\sum_{m \geq 1} m p_m^2 < \infty.$$

Taking $\varphi \in H_0^{1/2}$ with $\varphi = \sum_{m \geq 1} \varphi_m e_m$, we define the scalar product

$$(p|\varphi)_{H_0^{1/2}} = \frac{\omega\pi}{2} \sum_{m \geq 1} m p_m \varphi_m.$$

It is well known that $H_0^{1/2}$ is a Hilbert space together with the above scalar product. Notice that $H_0^{1/2}$ is compactly embedded into $L^s(0, \pi)$ for all $1 \leq s < \infty$. In particular there is $l_s > 0$ such that

$$||p||_{L^s} \leq l_s ||p||_{H_0^{1/2}} \quad \text{for all } p \in H_0^{1/2}. \tag{12}$$

We need also a second fractional Sobolev space. For any nonnegative integer m, let

$$f_m(r) = \cos(m\omega(r-\rho)) \quad (r \in [\rho, R])$$

and $H^{1/2}$ be the set of functions $q \in L^2(\rho, R)$ having Fourier expansion

$$q = \sum_{m \geq 0} q_m f_m$$

such that

$$\sum_{m \geq 1} m q_m^2 < \infty.$$

Taking $\psi \in H^{1/2}$ with $\psi = \sum_{m \geq 0} \psi_m f_m$, we define the scalar product

$$(q|\psi)_{H^{1/2}} = (R-\rho) q_0 \psi_0 + \frac{\omega\pi}{2} \sum_{m \geq 1} m q_m \psi_m.$$

Like before, $H^{1/2}$ is compactly embedded into $L^s(0, \pi)$ for all $1 \leq s < \infty$. In particular there is $l_s > 0$ such that

$$||q||_{L^s} \leq l_s ||q||_{H^{1/2}} \quad \text{for all } q \in H^{1/2}. \tag{13}$$

Finally, consider the Hilbert space $E = H_0^{1/2} \times H^{1/2}$ endowed with usual scalar product

$$(z|w) = (p|\varphi)_{H_0^{1/2}} + (q|\psi)_{H^{1/2}}$$

for any $z = (p, q)$ and $w = (\varphi, \psi)$ in E.

In the sequel we will write $(\cdot|\cdot)$ and $||\cdot||$ for any scalar product and norm in the spaces defined above.

We introduce now a modified Hamiltonian function $\hat{H}$. Consider $\tilde{H} : [\rho, R] \times \mathbb{R}^2 \to \mathbb{R}$ smooth, such that $H = \tilde{H}$ on $[\rho, R] \times [-1, 1] \times [\frac{-\varepsilon}{2}, \frac{\varepsilon}{2}]$, $\tilde{H} = 0$ on $[\rho, R] \times (\mathbb{R}^2 \setminus [-2, 2] \times [-2, 2])$. The Hamiltonian $\tilde{H}$ is defined as follows. Let $\tilde{f} \in \mathcal{C}^\infty(\mathbb{R})$ be an even smooth positive function such that $\tilde{f} = f$ on $[\frac{-\varepsilon}{2}, \frac{\varepsilon}{2}]$, $\tilde{f} = \tilde{c} > 0$ constant on $\mathbb{R}\backslash] - \varepsilon, \varepsilon[$, $\tilde{f} \geq \tilde{c}$ on $\mathbb{R}$, and $\beta \in \mathcal{D}(\mathbb{R})$ be an even smooth positive function such that $\beta = 1$ on $[-1, 1]$, *supp* $\beta \subset] - 2, 2[$, β nonincreasing on $[1, 2]$. Consider $\tilde{H}$ given by

$$\tilde{H}(r, p, q) = \beta(p)[\tilde{f}(q)\sqrt{1 + p^2} - f(0)] + \beta(q)G(r, q),$$

for all $(r, p, q) \in [\rho, R] \times \mathbb{R}^2$. Now, we consider $\alpha \in \mathcal{D}(\mathbb{R}^2)$ an even smooth positive function such that *supp* $\alpha \subset [-3, 3] \times [-3, 3]$, $\alpha \leq 1$, $\alpha = 1$ on $[-2, 2] \times [-2, 2]$, then the hamiltonian $\hat{H}$ is defined by

$$\hat{H}(r, z) = \alpha(z)\tilde{H}(r, z) + (1 - \alpha(z))\hat{c}|z|^2$$

for all $(r, z) \in [\rho, R] \times \mathbb{R}^2$, where $(2l_2\omega)^{-1} > \hat{c} > 0$ is a small fixed constant. The reason for introducing $\hat{H}$ was that we could not verify $(PS)^*$ condition without the Hamiltonian growing at a prescribed rate at infinity,

Next, consider the functional

$$\mathcal{J} : H_0^{1/2} \to \mathbb{R} : \; p \mapsto \int_\rho^R \sqrt{1 + p^2} dr.$$

Then, it is easy to prove that $\mathcal{J} \in C^1(H_0^{1/2}, \mathbb{R})$ and

$$\mathcal{J}'(p)(\varphi) = \int_\rho^R \frac{p\varphi}{\sqrt{1 + p^2}} dr \quad \text{for all } p, \varphi \in H_0^{1/2}.$$

It follows that the following result holds true.

Lemma 3 *Assume that $\hat{H}$ satisfies the above conditions and consider the functional*

$$\hat{\mathcal{J}} : E \to \mathbb{R} : z \mapsto \int_\rho^R \hat{H}(r, z) dr.$$

One has that $\hat{\mathcal{J}} \in C^1(E, \mathbb{R})$ and

$$\hat{\mathcal{J}}'(z)(w) = \int_\rho^R \hat{H}_z(r, z) \cdot w dr,$$

for all $z, w \in E$.

Consider now the Hamiltonian system associated to $\hat{H}$

$$p'(r) = -\hat{H}_q(r, p(r), q(r)), \quad q'(r) = \hat{H}_p(r, p(r), q(r)) \quad (r \in [\rho, R]) \tag{14}$$
$$p(\rho) = 0 = p(R).$$

We define the action functional associated to (14) as follows. Consider the continuous symmetric bilinear form

$$B : E \times E \to \mathbb{R} : (z, w) \mapsto -\frac{\pi}{2} \sum_{m \geq 1} m(p_m \psi_m + \varphi_m q_m),$$

where

$$z = (\sum_{m \geq 1} p_m e_m, \sum_{m \geq 0} q_m f_m)$$

and

$$w = (\sum_{m \geq 1} \varphi_m e_m, \sum_{m \geq 0} \psi_m f_m).$$

For $z, w \in H_0^1 \times H^1$ with $z = (p, q)$ and $w = (\varphi, \psi)$ one has that

$$B[z, w] = \int_\rho^R (p\psi' + \varphi q')dr.$$

Consider now the quadratic form associated to B,

$$A : E \to \mathbb{R} : z \mapsto \frac{1}{2} B[z, z].$$

One has that, for $z \in H_0^1 \times H^1$ with $z = (p, q)$

$$A(z) = \int_\rho^R pq' dr.$$

Next, consider

$$E^\pm = \{(p, q) \in E : p = \sum_{m \geq 1} p_m e_m, q = \sum_{m \geq 1} (\mp p_m) f_m\},$$
$$E_0 = \{(0, q_0) : q_0 \in \mathbb{R}\},$$

and notice that $E = E^- \oplus E_0 \oplus E^+$. It follows that for

$$z = (\sum_{m \geq 1} p_m e_m, \sum_{m \geq 0} q_m f_m) \in E,$$

one has $z = z^- + z^0 + z^+$, where $z^0 = (0, q_0)$ and

$$z^\pm = \left(\sum_{m \geq 1} \left(\frac{p_m \mp q_m}{2} \right) e_m, \sum_{m \geq 1} \left(\frac{q_m \mp p_m}{2} \right) f_m \right).$$

This implies that

$$A(z) = (2\omega)^{-1}(||z^+||^2 - ||z^-||^2) \quad \text{for all } z \in E.$$

Using Lemma 3 and a classical strategy (see [14]), it is not difficult to show the following

Lemma 4 *Consider the action functional*

$$I : E \to \mathbb{R} : z \mapsto A(z) - \int_\rho^R \hat{H}(r, z)dr.$$

Then $I \in C^1(E, \mathbb{R})$ *with*

$$I'(z)(w) = B[z, w] - \int_\rho^R \hat{H}_z(r, z) \cdot w dr \quad (z, w \in E).$$

Moreover, if $z \in E$ *with* $I'(z) = 0$*, then* $z = (p, q)$ *is a solution of (14).*

3.3 (PS) and $(PS)^*$

Consider, for any positive integer m,

$$E_{\pm m} = \{(p, q) \in E : p = p_m e_m, q = (\mp p_m) f_m\},$$

and remark that

$$\bigoplus_{m=-k}^{k} E_m = \{(p, q) \in E : p = \sum_{m=1}^{k} p_m e_m, q = \sum_{m=0}^{k} q_m f_m\}.$$

Lemma 5 *One has that* $I|_{E^+}$ *satisifes* (PS) *condition and* I *has* $(PS)^*$ *condition with respect to the sequence of subspaces*

$$X_n = \bigoplus_{m \geq -n} E_m \quad (n \geq 1).$$

Proof. We will prove that I has $(PS)^*$ condition with respect to (X_n). Consider (n_j) a subsequence of (n) and $z_j \in X_{n_j}$ be such that $(I(z_j))$ be bounded and $||(I|_{X_{n_j}})'(z_j)|| \to 0$ as $j \to \infty$. Note that for all $j \geq 1$,

$$B[z_j, z_j^+] = \omega^{-1}||z_j^+||^2, \qquad \int_\rho^R z_j \cdot z_j^+ dr = \int_\rho^R |z_j^+|^2 dr.$$

It follows that

$$(I|_{X_{n_j}})'(z_j)[z_j^+] \geq \omega^{-1}||z_j^+||^2 - c_1 \int_\rho^R |z_j^+| dr - 2\hat{c}l_2||z_j^+||^2,$$

which implies that $(||z_j^+||)$ is bounded. Similarly, one has that $(||z_j^-||)$ is bounded. In particular $(||z_j^+ + z_j^-||_{L^2})$ is bounded. On the other hand, using moreover that $(I(z_j))$ is bounded, we deduce that $||z_j||_{L^2}$ is bounded. It follows that $(|z_j^0|)$ is bounded and $||z_j||$ is bounded. Hence, passing to a subsequence, one has that $z_j \rightharpoonup z$ weakly in E and $z_j \to z$ in L^2 and a.e. on $[\rho, R]$. We can assume also that $z_j^0 \to z^0$. One has that

$$\begin{aligned}(I'(z_j) - I'(z))[z_j^+ - z^+] &= \omega^{-1}||z_j^+ - z^+||^2 \\ &+ \int_\rho^R (\hat{H}_z(r,z) - \hat{H}_z(r,z_j)) \cdot [z_j^+ - z^+]dr,\end{aligned}$$

which implies that $||z_j^+ - z^+|| \to 0$ as $j \to \infty$. Now, let P_j be the orthogonal projection onto X_{n_j}, and note that $||z^- - P_j z^-|| \to 0$ as $j \to \infty$. On the other hand, like above, one has that $||z_j^- - P_j z^-|| \to 0$ as $j \to \infty$, and $||z_j^- - z^-|| \to 0$. Hence, we have that $||z - z_j|| \to 0$ and z is a critical point of I. The fact that $I|_{E^+}$ has (PS) follows in the same way and it is more easy. ■

4 Proof of the main result

One has that

$$\sqrt{1+p^2} - 1 \geq \frac{1}{4\sqrt{2}} p^2 \qquad ((r,p) \in [0,R] \times [-1,1]).$$

This, together with (2) imply that for all $(r,p,q) \in [\rho, R] \times [-1,1] \times [\frac{-\varepsilon}{2}, \frac{\varepsilon}{2}]$,

$$\begin{aligned}\beta(p)[\tilde{f}(q)\sqrt{1+p^2} - f(0)] = f(q)[\sqrt{1+p^2} - 1] \\ -[f(0) - f(q)] \\ \geq \tilde{c}\frac{1}{4\sqrt{2}}p^2 - dq^2.\end{aligned}$$

It follows that for all $(r,p,q) \in [\rho, R] \times [-1,1] \times [-1,1]$,

$$\hat{H}(r,p,q) \geq \tilde{c}\frac{1}{4\sqrt{2}}p^2 - \tilde{d}q^2 + G(r,q), \tag{15}$$

for some $\tilde{d} > 0$. Then, from (3) and (15), it follows that there exist $c_1 > 0$ such that for any $\lambda > 0$ there exists $c_2 = c_2(\lambda)$ with

$$\hat{H}(r,p,q) \geq c_1 p^2 + \lambda q^2 - c_2(\lambda) q^4 \quad \text{for all} \quad (r,p,q) \in [\rho, R] \times \mathbb{R}^2. \tag{16}$$

Consider $\mu > 0$ and the linear operator

$$L_\mu : E \to E : (p,q) \mapsto (p, \mu q).$$

One has that $||L_\mu|| \leq 1+\mu$ and

$$L_\mu(\bigoplus_{m=-j}^{j} E_m) = \bigoplus_{m=-j}^{j} E_m \quad \text{for all } j \geq 1.$$

Let $z = L_\mu(p,q)$, where $(p,q) = (p^-,q^-) + (0,q_0) + (p^+,q^+)$ and

$$(p^-,q^-) = (\sum_{m\geq 1} a_m e_m, \sum_{m\geq 1} a_m f_m),$$

$$(p^+,q^+) = (\sum_{m=1}^{k} b_m e_m, \sum_{m=1}^{k} (-b_m) f_m),$$

with k a positive fixed integer. It follows that $z^0 = (0, \mu q_0)$, and if we denote $z^\pm = (\tilde{p}^\pm, \tilde{q}^\pm)$, then

$$\tilde{p}^\pm = \sum_{m=1}^{k} \frac{(a_m+b_m) \mp \mu(a_m-b_m)}{2} e_m + \sum_{m\geq k+1} \frac{(1\mp\mu)a_m}{2} e_m.$$

This implies that

$$A(z) = \frac{\mu\pi}{2}\left(\sum_{m=1}^{k} m b_m^2 - \sum_{m\geq 1} m a_m^2\right).$$

On the other hand

$$||p||_{L^2}^2 = \frac{R-\rho}{2}\left(\sum_{m=1}^{k}(a_m+b_m)^2 + \sum_{m\geq k+1} a_m^2\right),$$

$$||q||_{L^2}^2 = \frac{R-\rho}{2}\left(\sum_{m=1}^{k}(a_m-b_m)^2 + \sum_{m\geq k+1} a_m^2\right) + (R-\rho)q_0^2,$$

and

$$||(p,q)||^2 = (R-\rho)q_0^2 + \omega\pi\left(\sum_{m=1}^{k} m b_m^2 + \sum_{m\geq 1} m a_m^2\right).$$

Then, from (16) it follows that

$$I(z) \leq A(z) - c_1||p||_{L^2}^2 - \lambda\mu^2||q||_{L^2}^2 + c_2\mu^4||q||_{L^4}^4$$

$$\leq \frac{\mu\pi}{2}\left(\sum_{m=1}^{k} m b_m^2 - \sum_{m\geq 1} m a_m^2\right) - (c_1+\lambda\mu^2)(R-\rho)\left(\sum_{m=1}^{k} b_m^2 + \sum_{m\geq 1} a_m^2\right)$$

$$+(\mu^2\lambda - c_1)(R-\rho)\sum_{m=1}^{k} a_m b_m - \lambda\mu^2(R-\rho)q_0^2 + c_2\mu^4||q||_{L^4}^4.$$

Now, consider

$$\mu_k = \frac{2c_1(R-\rho)}{k\pi}, \qquad \lambda_k = c_1/\mu^2.$$

Then, using that

$$k\sum_{m=1}^{k} b_m^2 \geq \sum_{m=1}^{k} m b_m^2,$$

it follows that

$$\frac{\mu_k \pi}{2}\sum_{m=1}^{k} m b_m^2 - (c_1 + \lambda_k \mu^2)(R-\rho)\sum_{m=1}^{k} b_m^2 \leq -\frac{\mu_k\pi}{2}\sum_{m=1}^{k} m b_m^2.$$

Hence, there exists $k_0 > 0$ such that for all $k \geq k_0$, one has that

$$\begin{aligned} I(z) &\leq -\frac{\mu_k\pi}{2}\left(\sum_{m=1}^{k} m b_m^2 + \sum_{m\geq 1} m a_m^2\right) - c_1(R-\rho)q_0^2 + c_2\mu_k^4 l_4 ||q||_{H^{1/2}}^4 \\ &\leq -\frac{\mu_k}{2\omega}||(p,q)||^2 + c_2\mu_k^4 l_4 ||(p,q)||^4. \end{aligned}$$

Consider $\rho_k < 1/k$ such that

$$-(2\omega)^{-1} + c_2 l_4 \mu_k^3 \rho_k^2 < 0.$$

Then, it follows that I satisfies (ii) from Lemma 1.

Next, one has that there exists $c_3 > 0$ such that

$$\hat{H}(r,z) \leq \hat{c}|z|^2 + c_3 \qquad \text{for all } (r,z) \in [\rho, R] \times \mathbb{R}^2.$$

This implies that for any $z \in E^+$ one has that

$$\begin{aligned} I(z) &\geq (2\omega)^{-1}||z^+||^2 - \hat{c}||z||_{L^2}^2 - c_3(R-\rho) \\ &\geq ((2\omega)^{-1} - \hat{c}l_2)||z^+||^2 - c_3(R-\rho), \end{aligned}$$

which together with $(2l_2\omega)^{-1} > \hat{c} > 0$ imply that I satisfies (i) from Lemma 1. Then, using Lemma 5 we deduce that I satisfies the assumptions of Lemma 1 and I has a sequence (z_k) of critical points such that $||z_k|| \to 0$ as $k \to \infty$. Passing to a subsequence one has that $||z_k||_{L^2} \to 0$ and $z_k \to 0$ a.e. on $[\rho, R]$ as $k \to \infty$. On the other hand from Lemma 4 one has that $z_k = (p_k, q_k)$ is a solution of the Hamiltonian system (14). Then, integrating in (14) and using the convergence in L^2 and a.e. on $[\rho, R]$ it follows that $||z_k||_\infty \to 0$ as $k \to \infty$. This implies that $z_k = (p_k, q_k)$ is a solution of (11), and using Lemma 2 we deduce that q_k is a solution of (1). Moreover, $||q_k||_\infty \to 0$ as $k \to \infty$.

The proof of Theorem 1 is complete.

References

[1] A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, J. Funct. Anal. 122 (1994), 519-543.

[2] A. Bahri, H. Berestycki, Forced vibrations of superquadratic Hamiltonian systems, Acta Math. 152 (1984), 143-197.

[3] A. Bahri, H. Berestycki, Existence of forced oscillations for some nonlinear differential equations, Commun. Pure Appl. Math. 37 (1984), 403-442.

[4] R. Bartnik, L. Simon, Spacelike hypersurfaces with prescribed boundary values and mean curvature, Comm. Math. Phys. 87 (1982-83), 131-152.

[5] T. Bartsch, Z. Liu, Location and critical groups of critical points in Banach spaces with an application to nonlinear eigenvalue problems, Adv. Differential Eq. 9 (2004), 645-676.

[6] T. Bartsch, M. Willem, On an elliptic equation with concave and convex nonlinearities, Proc. Amer. Math. Soc. 123 (1995), 3555-3561.

[7] V. Benci, P.H. Rabinowitz, Critical point theorems for indefinite functionals, Invent. Math. 52 (1979), 241-273.

[8] C. Bereanu, P. Jebelean, J. Mawhin, Variational methods for nonlinear perturbations of singular ϕ-Laplacians, Rend. Lincei Mat. Appl. 22 (2011), 89-111.

[9] C. Bereanu, P. Jebelean, J. Mawhin, Radial solutions of Neumann problems involving mean extrinsic curvature and periodic nonlinearities, Calc. Var. PDE 46 (2013), 113-122.

[10] C. Bereanu, P. Jebelean, P.J. Torres, Positive radial solutions for Dirichlet problems with mean curvature operators in Minkowski space, J. Funct. Anal. 264 (2013), 270-287.

[11] P.L. Felmer, Periodic solutions of "superquadratic" Hamiltonian systems, J. Differential Eq. 102 (1993), 188-207.

[12] Z. Liu, Z.-Q. Wang, On Clark's theorem and its applications to partially sublinear problems, Ann. I.H. Poincare - AN 32 (2015), 1015-1037.

[13] J. Mawhin, P.J. Torres, Prescribed mean curvature graphs with Neumann boundary conditions in some FLRW spacetimes, J. Differential Eq. 261 (2016), 7145-7156.

[14] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Reg. Conf. Ser. Math., vol.65, AMS, Providence, 1986.

[15] B.S. Ryden, Introduction to Cosmology, Pearson, 2014.

[16] P.J. Torres, The prescribed mean curvature problem with Neumann boundary conditions in FLRW spacetimes, Rend. Istit. Mat. Univ. Trieste 49 (2017), 1-7.