Periodic solutions of differential equations with weak singularities

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We look for positive $T$-periodic solutions of the model equation

$$x'' + a(t)x = \frac{b(t)}{x^\lambda} + c(t),$$

(1)

with $a, b, c \in L^1[0, T]$ and $\lambda > 0$. 
Summary of known results


\[ x'' = \frac{1}{x^\lambda} + c(t) \]  \hspace{1cm} (2)

If \( \lambda \geq 1 \) (strong force condition), \( c < 0 \) is a necessary and sufficient condition.

If \( 0 < \lambda < 1 \) (weak force condition):

\[ \exists c \text{ with } c < 0 \text{ such that } \nexists \text{ periodic sol.} \]

Since then, the strong force condition became standard in the related references:

- Topological degree
- Poincaré-Birkhoff Theorem

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Lazer-Solimini [Proc. A.M.S. 1987]

\[ x'' = \frac{1}{x^\lambda} + c(t) \]  \hspace{1cm} (2)

- If \( \lambda \geq 1 \) (strong force condition), \( \bar{c} < 0 \) is a necessary and sufficient condition.
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- If \( 0 < \lambda < 1 \) (weak force condition): \( \exists c \) with \( \bar{c} < 0 \) such that \( \not\exists \) periodic sol.
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Summary of known results


\[ x'' + k^2 x = \frac{b}{x^\lambda} + c(t) \]  \hspace{1cm} (3)

**Theorem**

For \( 0 < k^2 \leq \mu_1 := \left( \frac{\pi}{T} \right)^2 \) and \( \lambda, b > 0 \), eq. (3) has a \( T \)-periodic solution if

\[ c_* > - \left( \frac{\pi^2 - T^2 k^2}{T^2 \lambda b} \right)^{\frac{\lambda}{\lambda + 1}} (\lambda + 1) b \]  \hspace{1cm} (4)
Summary of known results


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For \(0 < k^2 \leq \mu_1 := \left( \frac{\pi}{T} \right)^2\) and \(\lambda, b > 0\), eq. (3) has a \(T\)-periodic solution if

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\[
k^2 = \mu_1 \implies c_* > 0
\]
At least for strong potentials, this result is optimal: 

**Counterexample** by D. Bonheure, C. Fabry, D. Smets [Discrete Contin. Dyn. Syst. (2002)]

\[ x'' + \mu_1 x = \frac{b}{x^3} + \epsilon \sin\left(\frac{2\pi}{T} t\right) \]

has no \( T \)-periodic solutions for \( \epsilon > 0 \) sufficiently small.
Summary of known results

- P.J.T. [J. Differential Equations (2003)]

**Theorem**

For $0 < k^2 < \mu_1 := \left(\frac{\pi}{T}\right)^2$ and $\lambda, b > 0$, eq.(3) has a $T$-periodic solution if

$$
\begin{align*}
    c^* &< 0, \\
    c^* &\leq \frac{c^*}{\cos^{\lambda}\left(\frac{kT}{2}\right)} + \frac{k}{T} \sin kT \left(\frac{b}{|c^*|}\right)^{\frac{1}{\lambda}}.
\end{align*}
$$

(5)

**Theorem**

Let be $k^2 = \mu_1$ and $\lambda, b > 0$. If

$$
\gamma(t) = \int_t^{t+T} c(s) \sin \left( \pi \frac{s - t}{T} \right) \, ds > 0, \quad \forall t,
$$

(6)

eq.(3) has a $T$-periodic solution.

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eq.(3) has a $T$-periodic solution.

**Note:** $\gamma(t)$ is the unique $T$-periodic solution of the linear equation $x'' + \mu_1 x = c(t)$. 
Work to be done

The results of Rachunková et al. and Bonheure-deCoster do not cover important cases

\[ x'' + a(t)x = b(t)x + c(t) \]

where the 'Brillouin equation'

The results of P.J.T. do not cover the "critical" value \( \mu_1 \).
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\[ x'' + a(t)x = \frac{b(t)}{x^\lambda} + c(t) \]
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\[ x'' + a(t)x = \frac{1}{x^\lambda} \]

“Brillouin equation”
The results of Rachunková et al. and Bonheure-deCoster do not cover important cases

\[ x'' + a(t)x = \frac{1}{x^\lambda} \]

The results of P.J.T. do not cover the “critical” value \( \mu_1 \).
Let us consider

\[ x'' + a(t)x = f(t, x) + c(t), \] 

(7)

with \( a, c \in L^1[0, T] \) and \( f \in Car([0, T] \times \mathbb{R}^+, \mathbb{R}) \).

**Standing Hypothesis:**

\((H1)\) The Hill's equation \( x'' + a(t)x = 0 \) is non-resonant and the corresponding Green's function \( G(t, s) \) is non-negative for every \((t, s) \in [0, T] \times [0, T]\).

**Note:**

If \( a(t) \equiv k_2 \), \((H1)\) if and only if \( 0 < k_2 \leq \mu_1 \).
Let us consider

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**STANDING HYPOTHESIS:**

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**Note:** If \( a(t) \equiv k^2 \), \((H1) \iff 0 < k^2 \leq \mu_1\)
Main result.

Define

\[ \gamma(t) = \int_0^T G(t, s)c(s)ds, \]

Theorem

Let us assume that there exist \( b \succ 0 \) and \( \lambda > 0 \) such that

\[ 0 \leq f(t, x) \leq \frac{b(t)}{x^\lambda}, \quad \text{for all } x > 0, \quad \text{for a.e. } t \]

If \( \gamma_* > 0 \), then there exists a \( T \)-periodic solution of (7).
Proof.

Schauder’s fixed point theorem to

\[ F[x](t) := \int_0^T G(t, s) [f(s, x(s)) + c(s)] \, ds = \]

\[ = \int_0^T G(t, s)f(s, x(s)) \, ds + \gamma(t) \]
Proof.

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Define

\[ K = \{ x \in C_T : r \leq x(t) \leq R \text{ for all } t \} \]

then

\[ \mathcal{F}(K) \subset K \]

by taking

\[ r := \gamma^*, \quad R = \frac{\beta^*}{\gamma_*^\lambda} + \gamma^*. \]
The case $\gamma^* = 0$.

**Theorem**

Let us assume $(H_1)$ and that there exist $b, \hat{b} \succ 0$ and $0 < \lambda < 1$ such that

$$0 \leq \hat{b}(t)x^\lambda \leq f(t, x) \leq b(t)x^\lambda,$$

for all $x > 0$, for a.e. $t$.

If $\gamma^* = 0$, then there exists a $T$-periodic solution of

$(7)$.

Sometimes, the presence of a weak nonlinearity is an ADVANTAGE.

Open problem for strong singularities!!
The case $\gamma_* = 0$.

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Sometimes, the presence of a weak nonlinearity is an ADVANTAGE.

Open problem for strong singularities!!
The particular case $c(t) \equiv 0$.

$$x'' + a(t)x = \frac{b(t)}{x^\lambda}$$
The particular case $c(t) \equiv 0$.

$x'' + a(t)x = \frac{b(t)}{x^\lambda}$

Define

$$\beta(t) = \int_0^T G(t, s)b(s)ds$$

**Theorem**

*If $b > 0$ and $0 < \lambda < 1$, then there exists a $T$-periodic solution such that*

$$\left(\frac{\beta_*}{\beta_*^\lambda}\right)^{\frac{1}{1-\lambda^2}} \leq x(t) \leq \left(\frac{\beta_*}{\beta_*^\lambda}\right)^{\frac{1}{1-\lambda^2}}$$
The particular case $c(t) \equiv 0$.

$$x'' + a(t)x = \frac{b(t)}{x^\lambda}$$

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**Optimal bounds:**
The particular case $c(t) \equiv 0$.

$$x'' + a(t)x = \frac{a(t)}{x^\lambda}$$

Define

$$\beta(t) = \int_0^T G(t, s)a(s)ds$$

**Theorem**

If $b > 0$ and $0 < \lambda < 1$, then there exists a $T$-periodic solution such that

$$\left(\frac{\beta_*}{\beta^*_\lambda}\right)^{\frac{1}{1-\lambda^2}} \leq x(t) \leq \left(\frac{\beta_*}{\beta^*_\lambda}\right)^{\frac{1}{1-\lambda^2}}$$

*Optimal bounds:* if $a(t) \equiv b(t)$,
The particular case $c(t) \equiv 0$.

$$x'' + a(t)x = \frac{a(t)}{x^\lambda}$$

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If $b > 0$ and $0 < \lambda < 1$, then there exists a $T$-periodic solution such that

$$\left( \frac{\beta_*}{\beta_*^\lambda} \right)^{\frac{1}{1-\lambda^2}} \leq x(t) \leq \left( \frac{\beta_*}{\beta_*^\lambda} \right)^{\frac{1}{1-\lambda^2}}$$

**Optimal bounds:** If $a(t) \equiv b(t)$, then $\beta_* = \beta^* = 1$ and we get the exact solution $x(t) = 1$. 

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The case \( \gamma^* \leq 0 \).

\[ x'' + a(t)x = \frac{b(t)}{x^\lambda} + c(t), \]
The case $\gamma^* \leq 0$.

\[ x'' + a(t)x = \frac{b(t)}{x^\lambda} + c(t), \]

**Theorem**

*Let us assume that $b \succ 0$ and $0 < \lambda < 1$. If $\gamma^* \leq 0$ and

\[ \gamma^* \geq \left[ \frac{\beta_*}{\beta^* \lambda^2} \right] \frac{1}{1-\lambda^2} \left( 1 - \frac{1}{\lambda^2} \right) \] (8)

then there exists a positive $T$-periodic solution.*
The case $\gamma^* \leq 0$.

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**Theorem**

Let us assume that $b \succ 0$ and $0 < \lambda < 1$. If $\gamma^* \leq 0$ and

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then there exists a positive $T$-periodic solution.

**Note:** The bound goes to $-\beta_*$ when $\lambda \to 0^+$.
Back to the equation with fixed coefficients.

\[ x'' + k^2 x = \frac{b}{x^\lambda} + c(t) \]
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**Corollary**

*Let us assume that* \( 0 < \lambda < 1 \) *and* \( 0 < k^2 \leq \mu_1 := \left( \frac{\pi}{T} \right)^2 \). *Then, there exists a positive* \( T \)-periodic solution if* \( c(t) < 0 \) *for a.e. t and*

\[ c^* \geq \left[ b k^{2\lambda} \lambda \frac{2\lambda^2}{1-\lambda} \right]^{\frac{1}{1+\lambda}} (\lambda^2 - 1). \] *(9)*

Note: Now, the bound goes to \( -b \) when \( \lambda \to 0^+ \).
Back to the equation with fixed coefficients.

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**Corollary**

Let us assume that \(0 < \lambda < 1\) and \(0 < k^2 \leq \mu_1 := \left(\frac{\pi}{T}\right)^2\). Then, there exists a positive \(T\)-periodic solution if \(c(t) < 0\) for a.e. \(t\) and

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**Note:** Now, the bound goes to \(-b\) when \(\lambda \to 0^+\).
Existence beyond $\mu_1$.

\[ x'' + k^2 x = \frac{b(t)}{x^\lambda} + c(t) \]
Existence beyond $\mu_1$.

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Existence beyond $\mu_1$.

$$x'' + k^2 x = \frac{b(t)}{x^\lambda} + \bar{c} + \tilde{c}(t)$$

Define the sequence

$$\mu_n = \left( \frac{n\pi}{T} \right)^2$$

$\mu_{2k+1} \equiv$ eigenvalues of the Dirichlet problem
$\mu_{2k} \equiv$ eigenvalues of the periodic problem
Existence beyond $\mu_1$.

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**Theorem**

*Let us assume that $k^2 \neq \mu_{2n}, n \in \mathbb{N}^*$. Then, for any $\tilde{c} \in L^1[0, T]$ there exists $C_0 > 0$ such that the eq. possesses a unique positive $T$-periodic for any $\bar{c} > C_0$.***
Existence beyond $\mu_1$.

$x'' + k^2 x = \frac{b(t)}{x^\lambda} + \bar{c} + \tilde{c}(t)$

Define the sequence

$$\mu_n = \left(\frac{n\pi}{T}\right)^2$$

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**Theorem**

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**Note:** No sign condition over $b$!!
Stability beyond $\mu_1$.

\[ x'' + k^2 x = \frac{b(t)}{x^\lambda} + \bar{c} + \tilde{c}(t) \]
Stability beyond $\mu_1$.

$$x'' + k^2 x = \frac{b(t)}{x^\lambda} + c + \tilde{c}(t)$$

**Theorem**

Let us assume that $k^2 \neq \left( \frac{n\pi}{mT} \right)^2$ for all $n, m \in \mathbb{N}^*$ with $1 \leq m \leq 4$ and $b(t) > 0$ for a.e. $t$. Then, for any $\tilde{c} \in L^1[0, T]$ there exists $C_1 > C_0 > 0$ such that for any $\tilde{c} > C_1$ the unique $T$-periodic solution is Lyapunov stable.