Recent advances on mathematical models involving singular nonlinearities

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Model I:
A mass-spring model of electrostatically actuated micro-electro-mechanical system
The model

Figure: Mass-spring model of electrostatically actuated MEMS
The model

\[ my'' + cy' + ky = \frac{\varepsilon_0 A}{2} \frac{V^2(t)}{(d - y)^2}, \quad (1) \]

with \( V(t) = v_{dc} + v_{ac} \cos(\omega t) \) (AC-DC voltage)
The model

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Static pull-in

For a DC voltage $V(t) = v_{dc} > 0$, equilibria are the roots of $y(d - y^2) = \varepsilon_0 A v_{dc}^2 k$, giving rise to a saddle-node bifurcation.

Figure: Saddle-node bifurcation at $d_0 = \frac{2}{3}(\varepsilon_0 A v_{dc}^2 k)^{-1/3}$
For a DC voltage $V(t) = v_{dc} > 0$, equilibria are the roots of

$$y(d - y^2) = \frac{\varepsilon_0 Av_{dc}^2}{2k},$$

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**Figure:** Saddle-node bifurcation at $d_0 = \frac{3}{2} \left( \frac{\varepsilon_0 A v_{dc}^2}{k} \right)^{1/3}$. 
Dynamic pull-in

For an AC-DC voltage \( V(t) = v_{dc} + v_{ac}\cos(\omega t) \),

Dynamic pull-in \(\equiv\) non-autonomous saddle-node bifurcation
Let $V(t)$ be a continuous, positive, $T$-periodic function with $T = \frac{2\pi}{\omega}$. By convenience, we call $V_m = \min_{[0,T]} V(t)$, $V_M = \max_{[0,T]} V(t)$.

Theorem

There exists $d_0 > 0$ such that

1. If $d < d_0$, (1) has no $T$-periodic solutions.
2. If $d = d_0$, (1) has at least one $T$-periodic solution.
3. If $d > d_0$, (1) has at least two $T$-periodic solutions.

Besides, $d_0$ admits the following quantitative estimate

$$\frac{3}{2} \left( \frac{\varepsilon_0 AV_m^2}{k} \right)^{1/3} \leq d_0 \leq \frac{3}{2} \left( \frac{\varepsilon_0 AV_M^2}{k} \right)^{1/3}.$$  \hspace{1cm} (2)
Main result

Let $V(t)$ be a continuous, positive, $T$-periodic function with $T = \frac{2\pi}{\omega}$. By convenience, we call $V_m = \min_{[0,T]} V(t)$, $V_M = \max_{[0,T]} V(t)$.

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The model as a singular equation

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The change of variables

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The change of variables

\[ u = d - y \]

leads to

\[ u'' + c u' + ku + \frac{a(t)}{u^2} = s, \quad (3) \]

with \( c, k > 0, s := kd/m \) and \( a(t) := \frac{\varepsilon_0 A}{2m} V^2(t) \).
Sketch of the proof

A $T$-periodic solution of

$$u'' + cu' + ku + \frac{a(t)}{u^2} = s$$

is a fixed point of the functional

$$\Phi[u] := L^{-1} \left[ s - (k + 1)u + \frac{a(t)}{u^2} \right]$$

where $Lu := u'' + cu' - u$. 

$\Phi$ is a compact operator on the Banach space of the $T$-periodic continuous functions and the Leray-Schauder degree $\deg_{LS}(I - \Phi, \Omega)$ is well-defined whenever $\Phi$ has no fixed points in the boundary of $\Omega$. 
Existence of the unstable branch: lower and upper solution method
Sketch of the proof

- Existence of the unstable branch: lower and upper solution method
- Multiplicity (second branch): excision of the degree
Theorem

Assume that

\[ 4k < \frac{\varepsilon_0 AV_m^2}{2} \left( \frac{\omega c V_m^2}{\pi kd V_M^2} \right)^3 + \omega^2 + \frac{c^2}{m}. \]  

(4)

Then, if \( d > d_0 \), there exist exactly two \( T \)-periodic solutions, one asymptotically stable and another unstable.
Stability

Theorem

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Then, if \( d > d_0 \), there exist exactly two \( T \)-periodic solutions, one asymptotically stable and another unstable.

For the physical parameters: \( m = 3.5 \times 10^{-11} \text{ Kg} \), \( k = 0.17 \text{ N/m} \), \( c = 1.78 \times 10^{-6} \text{ Kg/s} \), \( A = 1.6 \times 10^{-9} \text{ m}^2 \), \( \varepsilon_0 = 8.85 \times 10^{-12} \text{ F/m} \). If \( V(t) = 10 + 2 \cos(\omega t) \text{ V} \), then the bifurcation value is bounded by \( 2.62033 \mu m < d_0 < 3.4336 \mu m \). If \( d > d_0 \) and \( \omega \geq 0.76772 s^{-1} \) then there are exactly two periodic solutions, one asymptotically stable and the other unstable.
Open problem I

To identify the dynamic pull-in (non-autonomous saddle-node bifurcation) when $V(t)$ change its sign.
Model II:
Motion of fluid particles induced by a prescribed vortex path in a circular domain
A fixed point vortex on the unbounded plane:

\[
\dot{\zeta} = \frac{\Gamma}{2\pi i} \left( \frac{1}{\zeta - z} \right)
\]

where the complex variable \(\zeta\) represents the evolution on the position of a particle transport induced by the flux generated by a fixed vortex placed at \(z\).
A fixed point vortex on the unbounded plane:

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where the complex variable \( \zeta \) represents the evolution on the position of a particle transport induced by the flux generated by a fixed vortex placed at \( z \).

This is a planar system with hamiltonian structure, where the stream function

\[ \Psi(\zeta) = \frac{\Gamma}{2\pi} \ln |\zeta - z| \]

plays the role of the hamiltonian.
Influence of a circular domain of radius $R$:

$$\frac{\dot{\zeta}}{2\pi i} = \frac{\Gamma}{2\pi i} \left( \frac{1}{\zeta - z} - \frac{1}{\zeta - \frac{R^2}{|z|^2} z} \right).$$

The first term models the action of the vortex whereas the second term corresponds to the wall influence on the flow.
The model

Influence of a circular domain of radius $R$:

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\frac{\dot{\zeta}}{\zeta} = \frac{\Gamma}{2\pi i} \left( \frac{1}{\zeta - z} - \frac{1}{\zeta - \frac{R^2}{|z|^2} z} \right).
$$

The first term models the action of the vortex whereas the second term corresponds to the wall influence on the flow.

Now the hamiltonian is

$$
\Psi(\zeta) = \frac{\Gamma}{2\pi} \ln \left| \frac{\zeta - z}{\bar{z}\zeta - R^2} \right|
$$
The model

**Figure:** Stream lines of a fixed vortex located at \((1, 0)\) in the circular domain of radius \(R = 2\).
The model

If the vortex is moving following a prescribed path $z(t)$:

$$
\dot{\zeta} = \frac{\Gamma}{2\pi i} \left( \frac{1}{\zeta - z(t)} - \frac{1}{\zeta - \frac{R^2}{|z(t)|^2} z(t)} \right). \tag{5}
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The model

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\dot{\zeta} = \frac{\Gamma}{2\pi i} \left( \frac{1}{\zeta - z(t)} - \frac{1}{\zeta - \frac{R^2}{|z(t)|^2} z(t)} \right).
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(5)

the hamiltonian

$$
\psi(t, \zeta) = \frac{\Gamma}{2\pi} \ln \left| \frac{\zeta - z(t)}{z(t)\zeta - R^2} \right|
$$

is no more a conserved quantity.
The main result

**Theorem 1**

Let \( z : \mathbb{R} \to \mathbb{C} \) be a \( T \)-periodic function of class \( C^1 \), such that \( |z(t)| < R \) for all \( t \). Then, for every integer \( k \geq 1 \), system (5) has infinitely many \( kT \)-periodic solutions lying in the disk \( B_R(0) \). More precisely, for every integer \( k \geq 1 \), there exists an integer \( j_k^* \) such that, for every integer \( j \geq j_k^* \), system (5) has two \( kT \)-periodic solutions \( \zeta_{k,j}^{(1)}(t), \zeta_{k,j}^{(2)}(t) \) such that, for \( i = 1, 2 \),

\[
\| \zeta_{k,j}^{(i)} \|_\infty \leq R \quad \text{and} \quad \text{rot}_{kT}(\zeta_{k,j}^{(i)}) = j. \quad (6)
\]

Moreover, for every \( k \geq 1 \), \( j \geq j_k^* \) and \( i = 1, 2 \),

\[
\lim_{j \to +\infty} \left| \zeta_{k,j}^{(i)}(t) - z(t) \right| = 0, \quad \text{uniformly in } t \in [0, kT]. \quad (7)
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The main result

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- A. Boscaggin, P.J. Torres, Periodic motions of fluid particles induced by a prescribed vortex path in a circular domain, Physica D 261 (2013) 81-84
Open problem II

Existence of periodic solutions with rotation number equal to zero.
Open problem II

Existence of periodic solutions with rotation number equal to zero.

Figure: Stream lines induced by a vortex path $z(t) = \exp(it)$ in the circular domain of radius $R = 2$. 
Model III:
Water transport across a cell membrane with fluctuating environmental conditions
The model

\begin{align*}
\dot{w}_1 &= \frac{x_{np}}{w_1} + \sum_{j=2}^{n} \frac{w_j}{w_1} - \sum_{i=1}^{n} M_i(t), \\
\dot{w}_k &= b_k \left( M_k(t) - \frac{w_k}{w_1} \right), \quad k = 2, \ldots, n.
\end{align*}

(8)
The model

\[
\dot{w}_1 = \frac{x_{np}}{w_1} + \sum_{j=2}^{n} \frac{w_j}{w_1} - \sum_{i=1}^{n} M_i(t), \quad (8)
\]

\[
\dot{w}_k = b_k \left( M_k(t) - \frac{w_k}{w_1} \right), \quad k = 2, \ldots, n.
\]

\(w_1(t)\equiv\) intracellular water volume
\(w_k(t), k = 2, \ldots, n\equiv\) amount of permeating intracellular solute species
\(x_{np} \geq 0\equiv\) amount of non-permeating intracellular solute species (salts)
\(M_1: \mathbb{R} \to [0, +\infty)\equiv\) extracellular concentration of non–permeating solute
\(M_k: \mathbb{R} \to [0, +\infty) \quad k = 2, \ldots, n\equiv\) extracellular concentrations of permeating solute species
The model

\[ \dot{w}_1 = \frac{x_{np}}{w_1} + \sum_{j=2}^{n} \frac{w_j}{w_1} - \sum_{i=1}^{n} M_i(t), \]

\[ \dot{w}_k = b_k \left( M_k(t) - \frac{w_k}{w_1} \right), \quad k = 2, \ldots, n. \]  

(8)

\( w_1(t) \equiv \) intracellular water volume

\( w_k(t), \quad k = 2, \ldots, n \equiv \) amount of permeating intracellular solute species

\( x_{np} \geq 0 \equiv \) amount of non-permeating intracellular solute species (salts)

\( M_1 : \mathbb{R} \to [0, +\infty) \equiv \) extracellular concentration of non–permeating solute

\( M_k : \mathbb{R} \to [0, +\infty) \quad k = 2, \ldots, n \equiv \) extracellular concentrations of permeating solute species

Main results

Theorem

Assume that $x_{np} > 0$. Then, system (8) has a $T$-periodic solution if and only if $\bar{M}_1 > 0$. 

P.J. Torres, Periodic oscillations of a model for membrane permeability with fluctuating environmental conditions, to appear in Journal of Mathematical Biology
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Theorem

Assume that $x_{np} = 0$, $M_1(t) \equiv 0$ and

(H$_1$) there exists $b > 0$ such that $b = b_k$ for every $k = 2, \ldots, n$.

Then, system (8) has infinitely many $T$-periodic solutions.
Main results

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$(H_1)$ there exists $b > 0$ such that $b = b_k$ for every $k = 2, \ldots, n$.

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$x_{np} > 0$: conditions for uniqueness and asymptotic stability
Open problem III

\[ x_{np} > 0: \text{conditions for uniqueness and asymptotic stability} \]

\[ x_{np} = 0: \text{existence without condition (H1)} \]
THANK YOU FOR YOUR ATTENTION!!