Multiple positive radial solutions for a Dirichlet problem involving the mean curvature operator in Minkowski space

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Abstract

We study the Dirichlet problem with mean curvature operator in Minkowski space

\[
\text{div}\left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}}\right) + \lambda \left[\mu(|x|) v^q\right] = 0 \quad \text{in } B(R), \quad v = 0 \quad \text{on } \partial B(R),
\]

where \(\lambda > 0\) is a parameter, \(q > 1\), \(R > 0\), \(\mu : [0, \infty) \to \mathbb{R}\) is continuous, strictly positive on \((0, \infty)\) and \(B(R) = \{x \in \mathbb{R}^N : |x| < R\}\). Using upper and lower solutions and Leray–Schauder degree type arguments, we prove that there exists \(\Lambda > 0\) such that the problem has zero, at least one or at least two positive radial solutions according to \(\lambda \in (0, \Lambda)\), \(\lambda = \Lambda\) or \(\lambda > \Lambda\). Moreover, \(\Lambda\) is strictly decreasing with respect to \(R\).

Keywords: Dirichlet problem; Positive radial solutions; Mean curvature operator; Minkowski space; Leray–Schauder degree; Upper and lower solutions

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1. Introduction

In this paper we present some non-existence, existence and multiplicity results for radial solutions of Dirichlet problems in a ball, associated to the mean curvature operator in the flat Minkowski space

\[ \mathbb{L}^{N+1} := \{(x, t): x \in \mathbb{R}^N, t \in \mathbb{R}\} \]

endowed with the Lorentzian metric

\[ \sum_{j=1}^{N} (dx_j)^2 - (dt)^2, \]

where \((x, t)\) are the canonical coordinates in \(\mathbb{R}^{N+1}\).

These problems are originated in the study – in differential geometry or special relativity, of maximal or constant mean curvature hypersurfaces, i.e., spacelike submanifolds of codimension one in \(\mathbb{L}^{N+1}\), having the property that their mean extrinsic curvature (trace of its second fundamental form) is respectively zero or constant (see e.g. [1,9,21]). More specifically, let \(M\) be a spacelike hypersurface of codimension one in \(\mathbb{L}^{N+1}\) and assume that \(M\) is the graph of a smooth function \(v: \Omega \to \mathbb{R}\) with \(\Omega\) a domain in \(\{(x, t): x \in \mathbb{R}^N, t = 0\} \simeq \mathbb{R}^N\). The spacelike condition implies \(|\nabla v| < 1\) and the mean curvature \(H\) satisfies the equation

\[ \text{div} \left( \frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \right) = NH(x, v) \text{ in } \Omega. \]

If \(H\) is bounded, then it has been shown in [3] that the above equation has at least one solution \(u \in C^1(\Omega) \cap W^{2,2}(\Omega)\) with \(u = 0\) on \(\partial \Omega\).

In this paper we consider the Dirichlet boundary value problem

\[ \text{div} \left( \frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \right) + \lambda \left[ \mu(|x|) v^q \right] = 0 \text{ in } B(R), \quad v = 0 \text{ on } \partial B(R), \quad (1) \]

where \(\lambda > 0\) is a parameter, \(q > 1\), \(R > 0\), \(\mu: [0, \infty) \to \mathbb{R}\) is continuous, strictly positive on \((0, \infty)\) and \(B(R) = \{x \in \mathbb{R}^N: |x| < R\}\).

Using a variational type argument, in [8] it is shown that if

\[ (q + 1)R^N < \lambda N \int_0^R r^{N-1} \mu(r)(R - r)^{q+1} \, dr, \]

then problem (1) has at least one positive classical radial solution. In particular, it is clear that the above condition is satisfied provided that \(\lambda\) is sufficiently large. On account of the main result of this paper (Theorem 1), this result becomes more precise. Namely, we prove (Corollary 1) that

- there exists \(\Lambda > 0\) such that (1) has zero, at least one or at least two positive classical radial solutions according to \(\lambda \in (0, \Lambda)\), \(\lambda = \Lambda\) or \(\lambda > \Lambda\). Moreover, \(\Lambda\) is strictly decreasing with respect to \(R\).
Up to our knowledge, such bifurcation scheme is completely new and has not been described before in related problems. If we compare with known results for classical elliptic equations with convex-concave nonlinearities (see for instance [2]), the bifurcation diagram is reversed in some sense. In particular, the non-existence of solutions for small values of the bifurcation parameter is a striking effect and a genuine consequence of the Minkowski mean curvature operator.

In the case $\mu = 1$, it is interesting to compare (1) with the analogous problem in the Euclidean context:

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}}\right) + \lambda v^q = 0 \quad \text{in } B(R), \quad v = 0 \quad \text{on } \partial B(R),$$

with $1 < q < \frac{N+2}{N-2}$. The assumption on $q$ is natural because, from [19] it follows that (2) has no nontrivial solutions if $q \geq \frac{N+2}{N-2}$. Notice also that, according to [13], all positive solutions of (2) have radial symmetry. Using critical point theory, in [11] it is proved that (2) has at least one positive radial solution for $\lambda$ sufficiently large. One the other hand, in [10] it is shown that if $\lambda = 1$ then there exists a non-negative number $R^*$ such that (2) has at least one positive radial solution for every $R > R^*$; this is done by means of a generalization of a Liouville type theorem concerning ground states due to Ni and Serrin. Also, notice that in [20] it has been shown that there exists $R_0 > 0$ such that (2) has no positive radial solution when $R < R_0$. The case $q = 1$ is considered in [17] for $\lambda$ in a left neighborhood of the principal eigenvalue of $-\Delta$ in $H^1_0$. In dimension one for $R = 1$, in [14] it is given a complete description of the exact number of positive solutions of (2).

For $\mu(r) \equiv r^m$, the analogous semilinear problem in which the mean curvature operator is replaced by the Laplacian is

$$\Delta v + |x|^m v^q = 0 \quad \text{in } B(1), \quad v = 0 \quad \text{on } \partial B(1),$$

and we point out that, as shown in [18], the above problem has a positive radial solution provided that $1 < q < \frac{N+2m+2}{N-2}$ and $N \geq 3, m > 0$.

Setting, as usual, $r = |x|$ and $v(x) = u(r)$, we reduce the Dirichlet problem (1) to the mixed boundary value problem

$$\left( r^{N-1} \frac{u'}{\sqrt{1-u'^2}} \right)' + r^{N-1} \left[ \lambda \mu(r) u^q \right] = 0, \quad u'(0) = 0 = u(R).$$

The rest of the paper is organized as follows. In Section 2 we associate to a larger class of problems of type (3) a fixed point operator and we prove a lower and upper solution result (Proposition 1). A Cauchy problem associated to the differential equation in (3) is studied in Section 3. The main result of this section (Proposition 2) will be employed to prove the monotonicity of $\Lambda$ with respect to $R$. By means of a degree computation inspired in the proof of the cone compression–expansion theorem by Krasnosel’skiĭ (see [15]), in Section 4 we show that the Leray–Schauder index in zero of the fixed point operator introduced in Section 2 is 1. Section 5 is devoted to the proof of the main result.

For other results concerning the Neumann problem associated to prescribed mean curvature operator in Minkowski space we refer the reader to [5–7,16].
2. A fixed point operator, lower and upper solutions and degree

In this section we consider problems of the type
\[(r^{N-1} \phi(u'))' + r^{N-1} g(r, u) = 0, \quad u'(0) = 0 = u(R),\] (4)
where \(N \geq 2\) is an integer, \(R > 0\) and the following main hypotheses hold true:

\((H_\phi)\) \(\phi: (-a, a) \rightarrow \mathbb{R} (0 < a < \infty)\) is an odd, increasing homeomorphism;
\((H_g)\) \(g: [0, R] \times \mathbb{R} \rightarrow \mathbb{R}\) is a continuous function.

In the sequel, the space \(C := C[0, R]\) will be endowed with the usual sup-norm \(\| \cdot \|_\infty\) and \(C^1 := C^1[0, R]\) will be considered with the norm \(\|u\| = \|u\|_\infty + \|u'\|_\infty\). Also, we shall use the closed subspace of \(C^1\) defined by
\[C^1_M = \{ u \in C^1: u'(0) = 0 = u(R) \}.\]

For \(u_0 \in C^1_M\), we set \(B(u_0, \rho) := \{ u \in C^1_M: \|u\| < \rho \} (\rho > 0)\) and, for shortness, we shall write \(B_\rho\) instead \(B(0, \rho)\).

Recall, by a solution of (4) we mean a function \(u \in C^1\) with \(\|u'\|_\infty < a\), such that \(r^{N-1} \phi(u') \in C^1\) and (4) is satisfied.

Setting
\[\sigma(r) := 1/r^{N-1} \quad (r > 0),\]
we introduce the linear operators
\[S: C \rightarrow C, \quad Su(r) = \sigma(r) \int_0^r t^{N-1} u(t) \, dt \quad (r \in (0, R]), \quad Su(0) = 0;\]
\[K: C \rightarrow C^1, \quad Ku(r) = \int_r^R u(t) \, dt \quad (r \in [0, R]).\]

It is easy to see that \(K\) is bounded and standard arguments, invoking the Arzela–Ascoli theorem, show that \(S\) is compact. This implies that the nonlinear operator \(K \circ \phi^{-1} \circ S: C \rightarrow C^1\) is compact. On the other hand, an easy computation shows that, for a given function \(h \in C\), the mixed problem
\[\left(r^{N-1} \phi(u')\right)' + r^{N-1} h(r) = 0, \quad u'(0) = 0 = u(R),\]
has an unique solution \(u\) given by
\[u = K \circ \phi^{-1} \circ S \circ h.\]
Next, let $N_g$ be the Nemytskii operator associated to $g$, i.e.,

$$N_g : C \rightarrow C, \quad N_g(u) = g(\cdot, u(\cdot)).$$

Noticing that $N_g$ is continuous and takes bounded sets into bounded sets, we have the following fixed point reformulation of problem (4).

**Lemma 1.** A function $u \in C^1_M$ is a solution of (4) if and only if it is a fixed point of the compact nonlinear operator

$$\mathcal{N}_g : C^1_M \rightarrow C^1_M, \quad \mathcal{N}_g = K \circ \phi^{-1} \circ S \circ N_g.$$

Moreover, any fixed point $u$ of $\mathcal{N}_g$ satisfies

$$\|u'\|_{\infty} < a, \quad \|u\|_{\infty} < aR$$

and

$$d_{LS}[I - \mathcal{N}_g, B_{\rho}, 0] = 1 \quad \text{for all } \rho \geq a(R + 1).$$

**Proof.** Inequalities in (5) follow immediately from the fact that the range of $\phi^{-1}$ is $(-a, a)$. Next, consider the compact homotopy

$$\mathcal{H} : [0, 1] \times C^1_M \rightarrow C^1_M, \quad \mathcal{H}(\tau, \cdot) = \tau \mathcal{N}_g(\cdot).$$

One has that

$$\mathcal{H}([0, 1] \times C^1_M) \subset B_{a(R+1)},$$

which together with the invariance under homotopy of the Leray–Schauder degree, imply that

$$d_{LS}[I - \mathcal{H}(0, \cdot), B_{\rho}, 0] = d_{LS}[I - \mathcal{H}(1, \cdot), B_{\rho}, 0],$$

for all $\rho \geq a(R + 1)$. The result follows from $\mathcal{H}(0, \cdot) = 0, \mathcal{H}(1, \cdot) = \mathcal{N}_g$ and $d_{LS}[I, B_{\rho}, 0] = 1$. \hfill $\square$

A lower solution of (4) is a function $\alpha \in C^1$ such that $\|\alpha'\|_{\infty} < a$, $r^{N-1}\phi(\alpha') \in C^1$ and

$$(r^{N-1}\phi(\alpha'(r)))' + r^{N-1}g(r, \alpha(r)) \geq 0 \quad (r \in [0, R]), \quad \alpha(R) \leq 0.$$ 

Similarly, an upper solution of (4) is defined by reversing the above inequalities.

**Proposition 1.** If (4) has a lower solution $\alpha$ and an upper solution $\beta$ such that $\alpha(r) \leq \beta(r)$ for all $r \in [0, R]$, then (4) has a solution $u$ such that $\alpha(r) \leq u(r) \leq \beta(r)$ for all $r \in [0, R]$. 

Proof. Let \( \gamma : [0, R] \times \mathbb{R} \to \mathbb{R} \) be the continuous function defined by

\[
\gamma(r, u) = \begin{cases} 
\alpha(r), & \text{if } u < \alpha(r), \\
u, & \text{if } \alpha(r) \leq u \leq \beta(r), \\
\beta(r), & \text{if } u > \beta(r), 
\end{cases}
\]

and define \( G : [0, R] \times \mathbb{R} \to \mathbb{R} \) by \( G(r, u) = g(r, \gamma(r, u)) \). We consider the modified problem

\[
\left( r^{N-1} \phi(u') \right)' + r^{N-1} \left[ G(r, u) - u + \gamma(r, u) \right] = 0, \quad u'(0) = 0 = u(R). \tag{6}
\]

It follows from [4] that problem (6) has at least one solution.

We show that if \( u \) is a solution of (6), then \( \alpha(r) \leq u(r) \leq \beta(r) \) for all \( r \in [0, R] \). This will conclude the proof.

Suppose that there exists some \( r_0 \in [0, R] \) such that

\[
\max_{[0, R]} (\alpha - u) = \alpha(r_0) - u(r_0) > 0.
\]

If \( r_0 \in (0, R) \) then \( \alpha'(r_0) = u'(r_0) \) and there is a sequence \( \{r_k\} \) in \((0, r_0)\) converging to \( r_0 \) such that \( \alpha'(r_k) - u'(r_k) \geq 0 \). As \( \phi \) is an increasing homeomorphism, this implies

\[
r_k^{N-1} \phi(\alpha'(r_k)) - r_0^{N-1} \phi(\alpha'(r_0)) \geq r_k^{N-1} \phi(u'(r_k)) - r_0^{N-1} \phi(u'(r_0)),
\]

implying that

\[
\left( r^{N-1} \phi(\alpha'(r)) \right)'_{r=r_0} \leq \left( r^{N-1} \phi(u'(r)) \right)'_{r=r_0}.
\]

Hence, because \( \alpha \) is a lower solution of (4), we obtain

\[
\left( r^{N-1} \phi(\alpha'(r)) \right)'_{r=r_0} \leq \left( r^{N-1} \phi(\alpha'(r)) \right)'_{r=r_0} \leq \left( r^{N-1} \phi(u'(r)) \right)'_{r=r_0}.
\]

a contradiction. If \( r_0 = R \) then \( \alpha(R) - u(R) > 0 \). But \( u(R) = 0 \) and \( \alpha(R) \leq 0 \), obtaining again a contradiction. Finally, if \( r_0 = 0 \) then there exists \( r_1 \in (0, R] \) such that \( \alpha(r) - u(r) > 0 \) for all \( r \in [0, r_1] \) and \( \alpha'(r_1) - u'(r_1) \leq 0 \). It follows that

\[
r_1^{N-1} \phi(\alpha'(r_1)) \leq r_1^{N-1} \phi(u'(r_1)).
\]

On the other hand, integrating (6) from 0 to \( r_1 \) and using that \( \alpha \) is a lower solution of (4) we obtain
\[ \begin{align*}
    r_1^{N-1} \phi(u'(r_1)) &= \int_0^{r_1} r^{N-1} \left[ -g(r, \alpha(r)) + u(r) - \alpha(r) \right] dr \\
    &< \int_0^{r_1} (r^{N-1} \phi(\alpha'(r)))' dr \\
    &= r_1^{N-1} \phi(u'(r_1)),
\end{align*} \]
a contradiction. Consequently, \( \alpha(r) \leq u(r) \) for all \( r \in [0, R] \). Analogously, it follows that \( u(r) \leq \beta(r) \) for all \( r \in [0, R] \). The proof is completed. \( \square \)

**Lemma 2.** Assume that (4) has a lower solution \( \alpha \) and an upper solution \( \beta \) such that \( \alpha(r) \leq \beta(r) \) for all \( r \in [0, R] \), and let \( \Omega_{\alpha, \beta} := \{ u \in C_1^1 : \alpha \leq u \leq \beta \} \). Assume also that problem (4) has an unique solution \( u_0 \) in \( \Omega_{\alpha, \beta} \) and there exists \( \rho_0 > 0 \) such that \( B(u_0, \rho_0) \subset \Omega_{\alpha, \beta} \). Then,
\[ d_{LS}[I - N_{g}, B(u_0, \rho), 0] = 1 \quad \text{for all} \quad 0 < \rho \leq \rho_0, \]
where \( N_{g} \) is the fixed point operator associated to (4).

**Proof.** Let \( N_{\gamma} \) be the fixed point operator associated to the modified problem (6). From the proof of Proposition 1 it follows that any fixed point \( u \) of \( N_{\gamma} \) is contained in \( \Omega_{\alpha, \beta} \) and \( u \) is also a fixed point of \( N_{g} \). It follows that \( u_0 \) is the unique fixed point of \( N_{\gamma} \). Then, from Lemma 1 and the excision property of the Leray–Schauder degree one has that
\[ d_{LS}[I - N_{\gamma}, B(u_0, \rho), 0] = 1 \quad \text{for all} \quad \rho > 0. \]
The result follows from the fact that
\[ N_{\gamma}(u) = N_{g}(u) \quad \text{for all} \quad u \in \overline{B}(u_0, \rho_0). \] \( \square \)

**3. A Cauchy problem**

In this section we consider the Cauchy problem
\[ \begin{align*}
    (r^{N-1} \phi(u'(r)))' + r^{N-1} \left[ \lambda \mu(r)p(u(r)) \right] &= 0 \quad (r \in [0, R]), \\
    u(0) &= \xi, \quad u'(0) = 0,
\end{align*} \]
where \( \lambda, \xi > 0 \) and

- \( \mu : [0, R] \to \mathbb{R} \) is continuous;
- \( p : \mathbb{R} \to \mathbb{R} \) is Lipschitz continuous on bounded sets.

We denote \( \mu_M := \max_{[0,R]} |\mu| \). In the proof of the next result we use some ideas from the last section in [12].
Proposition 2. Assume \((H_\phi)\) and that \(\phi\) is of class \(C^1\), \(\phi' > 0\). Then, problem (7) has an unique solution \(u(\lambda, \xi; \cdot)\) and the mapping \((\lambda, \xi) \mapsto u(\lambda, \xi; \cdot)\) is continuous from \((0, \infty) \times (0, \infty)\) to \(C^1\).

Proof. We divide the proof in three steps.

1. Existence. Consider the nonlinear compact operator

\[
C : C \to C, \quad Cu(r) \equiv \xi - \int_0^r \phi^{-1}\left(\frac{1}{tN-1} \int_0^t s^{N-1}\left[\lambda\mu(s)p\left(u(s)\right)\right]ds\right)dt.
\]

One has that \(u \in C\) is solution of (7) if and only if \(u = Cu\). Using that \(\|Cu\|_\infty < \xi + aR\) for all \(u \in C\), it follows from Schauder’s fixed point theorem that \(C\) has at least one fixed point \(u\) which is a solution of (7). Notice that

\[
\|u\|_\infty < \xi + aR. \tag{8}
\]

2. Uniqueness. Let \(u\) and \(v\) be solutions of (7) and

\[
\omega = \phi(u') - \phi(v'), \quad \psi = \lambda\mu[p(v) - p(u)].
\]

It follows that, for all \(r \in [0, R]\), one has

\[
|\omega(r)| = \left|1_{rN-1} \int_0^r t^{N-1}\psi(t)dt\right| \leq \frac{R}{N}\sup_{[0,r]}|\psi|.
\]

On the other hand, from (8) we have

\[
|\psi(r)| \leq M|u(r) - v(r)| \quad (r \in [0, R]),
\]

where \(M = \lambda L\mu_M\) and \(L\) is the Lipschitz constant of \(p\) corresponding to the interval \([- (\xi + aR), \xi + aR]\). Hence, using that \(u(0) = v(0)\), we infer that for all \(r \in [0, R]\),

\[
|\psi(r)| \leq M \int_0^r |u'(t) - v'(t)|dt \leq \frac{M}{m} \int_0^r |\omega(t)| dt,
\]

where \(m\) is the minimum of \(\phi'\) on the interval \([0, \max\{\|u'\|_\infty, \|v'\|_\infty\}]\). It follows that

\[
|\omega(r)| \leq \frac{MR}{mN} \int_0^r |\omega(t)| dt \quad (r \in [0, R]),
\]

which together with Gronwall’s inequality imply \(\omega = 0\), hence \(u = v\).
3. Continuous dependence on \((\lambda, \xi)\). Let \(u(\lambda, \xi; \cdot)\) be the unique solution of (7). For \(l, h \in \mathbb{R}\) sufficiently small, we set

\[
u := u(\lambda, \xi; \cdot), \quad v := u(\lambda + l, \xi + h; \cdot).
\]

From (8) we may assume that

\[
\|v\|_{\infty} < \xi + 1 + aR.
\]

This and

\[
-v'(r) = \phi^{-1}\left(\frac{1}{r^{N-1}} \int_0^r s^{N-1} \left[(\lambda + l)\mu(s)p(v(s))\right] ds\right)
\]

imply that there exists \(\delta > 0\), which is independent on \(l\) and \(h\), such that

\[
\|v'\|_{\infty} \leq \delta < a.
\]

Let \(\omega, \psi\) be as in Step 2. Using (9), for all \(r \in [0, R]\), one has

\[
|\omega(r)| = \left|\frac{1}{r^{N-1}} \int_0^r t^{N-1} \left[\psi(t) - l\mu(t)p(v(t))\right] dt\right| \leq \frac{R}{N} \left[\sup_{[0,r]} |\psi| + |l|c\right],
\]

where \(c = \mu_M \max_{[-(\xi+1+aR), \xi+1+aR]} |p|\). On the other hand, arguing as above we infer that for all \(r \in [0, R]\),

\[
|\psi(r)| \leq \frac{M}{k} \int_0^r |\omega(t)| dt + M|h|,
\]

where \(M = \lambda L\mu_M\) and \(L\) is the Lipschitz constant of \(p\) corresponding to the interval \([-\xi + 1 + aR, \xi + 1 + aR]\), and \(k\) is the minimum of \(\phi'\) on the interval \([0, \delta]\). It follows

\[
|\omega(r)| \leq \frac{cR|l| + MR|h|}{N} + \frac{MR}{kN} \int_0^r |\omega(t)| dt \quad (r \in [0, R]),
\]

which together with Gronwall’s inequality imply that

\[
|\omega(r)| \leq \left(\frac{cR|l| + MR|h|}{N}\right) \exp\left(\frac{MR^2}{kN}\right) \quad (r \in [0, R]).
\]

So, \(\|u' - v'\|_{\infty} \to 0\) as \(l, h \to 0\), implying also that \(\|u - v\|_{\infty} \to 0\).
4. Non-negative nonlinearities, positive solutions and degree around zero

Here, we consider mixed boundary value problems of the type

\[
(r^{N-1} \phi(u'))' + r^{N-1} f(r,u) = 0, \quad u'(0) = 0 = u(R),
\]

where \( N \geq 2 \) is an integer, \( R > 0 \) under hypotheses \((H_\phi)\) and

\((H_f)\quad f : [0, R] \times [0, \infty) \to [0, \infty) \) is continuous and \( f(r, s) > 0 \) for all \((r, s) \in (0, R] \times (0, \infty)\).

We need the following elementary result, which is proved in \([8]\).

**Lemma 3.** Assume \((H_\phi), \ (H_f)\) and let \( u \) be a nontrivial solution of

\[
(r^{N-1} \phi(u'))' + r^{N-1} f(r, |u|) = 0, \quad u'(0) = 0 = u(R).
\]

Then \( u > 0 \) on \([0, R)\) and \( u \) is strictly decreasing.

Notice that, by virtue of Lemma 3, \( u \) is a nontrivial solution of the mixed boundary value problem (11) if and only if \( u \) is a positive solution of (10). In this case, \( u \) is strictly decreasing.

Let \( \mathcal{N}_f \) be the fixed point operator associated to (11). In the next lemma we assume that \( f \) is sublinear with respect to \( \phi \) at zero.

**Lemma 4.** Assume \((H_\phi), \ (H_f)\),

\[
\lim_{s \to 0^+} \frac{f(r, s)}{\phi(s)} = 0 \quad \text{uniformly for } r \in [0, R]
\]

and

\[
\liminf_{s \to 0} \frac{\phi(\sigma s)}{\phi(s)} > 0 \quad \text{for all } \sigma > 0.
\]

Then there exists \( \rho_0 > 0 \) such that

\[
d_{LS}[I - \mathcal{N}_f, B_\rho, 0] = 1 \quad \text{for all } 0 < \rho \leq \rho_0.
\]

**Proof.** Using (13) we can find \( \varepsilon > 0 \) such that

\[
R \varepsilon / N < \liminf_{s \to 0} \frac{\phi(s/R)}{\phi(s)}.
\]

From (12) it follows that there exists \( s_\varepsilon > 0 \) such that

\[
f(r, s) \leq \varepsilon \phi(s) \quad \text{for all } (r, s) \in [0, R] \times [0, s_\varepsilon].
\]
Let us consider the compact homotopy
\[ H : [0, 1] \times C^1_M \to C^1_M, \quad H(\tau, u) = \tau N_f(u). \]
We will show that there exists \( \rho_0 > 0 \) such that
\[ u \neq H(\tau, u) \quad \text{for all} \quad (\tau, u) \in [0, 1] \times (\overline{B}_{\rho_0} \setminus \{0\}). \quad (16) \]

By contradiction, assume that one has
\[ u_k = \tau_k N_f(u_k) \]
with \( \tau_k \in [0, 1], u_k \in C^1_M \setminus \{0\} \) for all \( k \in \mathbb{N} \) and \( \|u_k\| \to 0 \). Using Lemma 3 it follows that \( u_k \) are strictly decreasing functions which are also strictly positive on \([0, R]\). Passing if necessary to a subsequence, we may assume that \( \|u_k\| \leq s_\delta \) for all \( k \in \mathbb{N} \), and then using (15) it follows
\[ f(r, u_k(r)) \leq \varepsilon \phi(\|u_k\|) \quad \text{for all} \quad r \in [0, R], \ k \in \mathbb{N}. \]
This implies that, for any \( k \in \mathbb{N} \),
\[ \|u_k\| \leq \int_0^R \phi^{-1}\left( \frac{\varepsilon \phi(\|u_k\|)}{\varepsilon R} \right) dr dt \]
\[ \leq R \phi^{-1}\left( \frac{\varepsilon R}{\phi(\|u_k\|)} \right). \]
It follows
\[ \frac{\phi\left(\frac{1}{R} \|u_k\|\right)}{\phi(\|u_k\|)} \leq \frac{\varepsilon R}{\phi(\|u_k\|)} \quad (k \in \mathbb{N}), \]
which together with \( \|u_k\| \to 0 \) contradict (14). Hence, (16) holds true. So, for any \( \rho \in (0, \rho_0] \) one has
\[ d_{LS}[I - H(1, \cdot), B_\rho, 0] = d_{LS}[I - H(0, \cdot), B_\rho, 0], \]
implying that
\[ d_{LS}[I - N_f, B_\rho, 0] = d_{LS}[I, B_\rho, 0] = 1, \]
and the proof is complete. \( \square \)
5. Main result

Now, we come to study the one-parameter problem (3) under the hypothesis

\((H)\) \(N \geq 2\) is an integer, \(R > 0\), \(q > 1\) and \(\mu : [0, \infty) \to \mathbb{R}\) is continuous, \(\mu(r) > 0\) for all \(r > 0\).

As the results in the previous sections apply with \(\phi(s) = s \sqrt{1 - s^2}\) \((s \in (-1, 1))\), note that \(u \in C^1\) is a positive solution of (3) if and only if \(u\) is a nontrivial solution of

\[
\left( r^{N-1} \frac{u'}{\sqrt{1-u'^2}} \right)' + r^{N-1} \left[ \lambda \mu(r) |u|^q \right] = 0, \quad u'(0) = 0 = u(R); \tag{17}
\]

in this case, \(u\) is strictly decreasing.

The main result of the paper is the following one. Notice that \(\mu_M = \max_{[0,R]} \mu\).

**Theorem 1.** Under hypothesis \((H)\), there exists \(\Lambda > 2N/(\mu_M R^{q+1})\) such that problem (3) has zero, at least one or at least two positive solutions according to \(\lambda \in (0, \Lambda), \lambda = \Lambda\) or \(\lambda > \Lambda\). Moreover, \(\Lambda\) is strictly decreasing with respect to \(R\).

**Proof.** We denote

\[ S_j := \{ \lambda > 0 : \text{(3) has at least } j \text{ positive solutions} \} \]

\[ = \{ \lambda > 0 : \text{(17) has at least } j \text{ non-trivial solutions} \} \quad (j = 1, 2) \]

and divide the proof in three steps.

1. **Finding \(\Lambda\).** Let \(\lambda > 0\) and \(u\) be a positive solution of (3). Integrating (3) on \([0, r]\), it follows

\[
-r^{N-1} \frac{u'(r)}{\sqrt{1-u'^2(r)}} = \lambda \int_0^r t^{N-1} \mu(t) u^q(t) dt \quad \text{for all } r \in [0, R].
\]

Using that \(u\) is strictly decreasing on \([0, R]\), we deduce that, for all \(r \in [0, R]\), one has

\[
-r^{N-1} u'(r) \leq -r^{N-1} \frac{u'(r)}{\sqrt{1-u'^2(r)}} \leq \lambda u^q(0) \mu_M R^N / N
\]

and integrating over \([0, R]\), we obtain

\[
u(0) \leq \lambda u^q(0) \mu_M R^2 / (2N). \tag{18}
\]
This, together with $0 < u(0) < R$ (see (5)) and $q > 1$ imply
\[ \lambda > 2N/(\mu MR^{q+1}). \]

From [8, Corollary 2] we know that (3) has a least one positive solution for $\lambda > 0$, sufficiently large. In particular, $S_1 \neq \emptyset$ and we can define
\[ \Lambda = \Lambda(R) := \inf S_1. \]

Clearly, we have $\Lambda \geq 2N/(\mu MR^q + 1)$. We claim that $\Lambda \in S_1$. Indeed, let $\{\lambda_k\} \subset S_1$ be a sequence converging to $\Lambda$, and $u_k \in C^1_M$ be positive on $[0, R)$ such that
\[ u_k = K \circ \phi^{-1} \circ S \circ (\lambda_k \mu u_k^q). \]

Then, from (5) and the Arzelà–Ascoli theorem, we infer that there exists $u \in C$ such that, passing eventually to a subsequence, $\{u_k\}$ converges to $u$ in $C$. So, it follows that $u \geq 0$ and
\[ u = K \circ \phi^{-1} \circ S \circ (\Lambda \mu u^q). \]

Using (18) we deduce that there is a constant $c_1 > 0$ such that $u_k(0) > c_1$, for all $k \in \mathbb{N}$. This ensures that $u(0) \geq c_1$, hence $u > 0$ on $[0, R)$ (by Lemma 3) and the claim is proved. Also, it is clear that $\Lambda > 2N/(\mu MR^q + 1)$.

Next, let $\lambda_0 > \Lambda$ be arbitrarily chosen. We shall apply Proposition 1 to show that $\lambda_0 \in S_1$. In this view, let $u_1$ be a positive solution for (3) corresponding to $\lambda = \Lambda$. It is easy to see that $u_1$ is a lower solution for (17) with $\lambda = \lambda_0$. To construct an upper solution, let $H > 0$, $\tilde{R} > R$ and consider the mixed problem
\[ \left( r^{N-1} \frac{u'}{\sqrt{1-u'^2}} \right)' + r^{N-1} H = 0, \quad u'(0) = 0 = u(\tilde{R}). \quad (19) \]

Then, by a simple integration, one has that the unique (positive) solution of (19) is given by
\[ u(r) = \frac{N}{H} \left[ \sqrt{1 + \frac{H^2}{N^2} \tilde{R}^2} - \sqrt{1 + \frac{H^2}{N^2} r^2} \right] \quad (r \in [0, \tilde{R}]). \]

For fixed $\lambda_2 > \lambda_0$, let $u_2$ be the solution of (19) corresponding to $H = \lambda_2 \mu M \tilde{R}^q$. Using that $u_2(R) > 0$ and
\[ \lambda_0 \mu(r) u_2^q(r) \leq \lambda_2 \mu M \tilde{R}^q \quad (r \in [0, R]), \]

it follows that $u_2$ is an upper solution for (17) with $\lambda = \lambda_0$. Since
\[ u_2(R) = \frac{N}{\sqrt{\left( \frac{1}{(\lambda_2 \mu M)^2 \tilde{R}^{2q}} + \frac{\tilde{R}^2}{N^2} - \left( \frac{1}{(\lambda_2 \mu M)^2 \tilde{R}^{2q}} + \frac{R^2}{N^2} \right) }}, \]
we can find $\tilde{R}$ sufficiently large, such that $u_1(0) < u_2(R)$. Then, taking into account that $u_1, u_2$ are strictly decreasing, we infer that $u_1 < u_2$ on $[0, R]$. By virtue of Proposition 1, we get $\lambda_0 \in S_1$. Therefore, we have

$$S_1 = [\Lambda, \infty).$$

2. Multiplicity. We use some ideas from the proof of Theorem 3.10 in [2]. Let $\lambda_0 > \Lambda$. We shall apply Lemmas 1, 2, 4 to show that $\lambda_0 \in S_2$. With this aim, let $u_1, u_2$ be constructed as in Step 1 and $u_0$ be a solution of (17) with $\lambda = \lambda_0$ such that $u_1 \leq u_0 \leq u_2$, i.e., $u_0 \in \Omega_{u_1,u_2}$ (see Lemma 2).

First, we claim that there exists $\varepsilon > 0$ with $B(u_0, \varepsilon) \subset \Omega_{u_1,u_2}$. Notice that, for all $r \in [0, R]$, one has

$$u_2(r) = \int_{r}^{\tilde{R}} \phi^{-1}\left(\int_{t}^{\tilde{R}} s^{N-1} \left[\lambda_{2}\mu M \tilde{R}^q\right] ds\right) dt,$$

implying that

$$u_2(r) > \int_{r}^{R} \phi^{-1}\left(\int_{t}^{R} s^{N-1} \left[\lambda_{2}\mu(s)u_2^q(s)\right] ds\right) dt \\
\geq \int_{r}^{R} \phi^{-1}\left(\int_{t}^{R} s^{N-1} \left[\lambda_0\mu(s)u_0^q(s)\right] ds\right) dt \\
= u_0(r),$$

so, there exists $\varepsilon_2 > 0$ such that $v \leq u_2$ for all $v \in B(u_0, \varepsilon_2)$. Similar arguments show that $u_1 < u_0$ on $[0, R/2]$. Thus, we can find $\varepsilon_1' > 0$ so that

$$v \in C^1_M \quad \text{and} \quad \|v - u_0\|_{\infty} \leq \varepsilon_1' \quad \Rightarrow \quad v \geq u_1 \quad \text{on} \quad [0, R/2]. \tag{20}$$

On the other hand, we have

$$-u_0' = \phi^{-1} \circ S \circ [\lambda_0\mu u_0^q] \quad \text{and} \quad -u_1' = \phi^{-1} \circ S \circ [A\mu u_1^q],$$

yielding $u_0' < u_1'$ on $[R/2, R]$. So, we can find $\varepsilon_1 \in (0, \varepsilon_1')$ sufficiently small, such that $v' < u_1'$ on $[R/2, R]$ whenever $v \in B(u_0, \varepsilon_1)$. Then, using $u_0(R) = 0 = v(R)$, we deduce that $v > u_1$ on $[R/2, R]$, for all $v \in B(u_0, \varepsilon_1)$. Now, on account of (20), the claim follows with any $\varepsilon \in (0, \min\{\varepsilon_1, \varepsilon_2\})$.

Next, if (17) has a second solution contained in $\Omega_{u_1,u_2}$, this solution is nontrivial and the proof of the multiplicity part is completed. If not, using Lemma 2 we deduce that

$$d_{LS}[I - N_{\tilde{\lambda}_0}, B(u_0, \rho), 0] = 1 \quad \text{for all} \quad 0 < \rho \leq \varepsilon,$$
where $\mathcal{N}_{\lambda_0}$ is the fixed point operator associated to (17) with $\lambda = \lambda_0$. On the other hand, from Lemma 1 one has

$$d_{LS}[I - \mathcal{N}_{\lambda_0}, B_\rho, 0] = 1 \quad \text{for all } \rho \geq R + 1,$$

and from Lemma 4 we have

$$d_{LS}[I - \mathcal{N}_{\lambda_0}, B_\rho, 0] = 1 \quad \text{for all } \rho \text{ sufficiently small}.$$

Now, consider $\rho_1, \rho_2 > 0$ sufficiently small and $\rho_3 \geq R + 1$ such that $B(u_0, \rho_1) \cap B_{\rho_2} = \emptyset$ and $B(u_0, \rho_1) \cup B_{\rho_2} \subset B_{\rho_3}$. Then, from the additivity-excision property of the Leray–Schauder degree it follows that

$$d_{LS}[I - \mathcal{N}_{\lambda_0}, B_{\rho_3} \setminus \overline{B(u_0, \rho_1) \cup \overline{B_{\rho_2}}}, 0] = -1,$$

which, together with the existence property of the Leray–Schauder degree, imply that $\mathcal{N}_{\lambda_0}$ has a fixed point $\tilde{u}_0 \in B_{\rho_3} \setminus \overline{B(u_0, \rho_1) \cup \overline{B_{\rho_2}}}$. We infer that (3) has a second positive solution.

3. Monotonicity of $\Lambda$. Let $u_0$ be a nontrivial solution of (17) with $\lambda = \lambda_0 := \Lambda(R_0)$ and $R = R_0$. We fix $R > R_0$. Then, setting $\xi_0 = u_0(0)$, from Proposition 2 with $p(s) = |s|^q$, one has that $u(\lambda_0, \xi_0; \cdot)|_{[0, R_0]} = u_0$. Since $u(\lambda_0, \xi_0; \cdot)$ is strictly decreasing on $[0, R]$ (this is easily seen) and $u(\lambda_0, \xi_0; R_0) = 0$, it follows that $u(\lambda_0, \xi_0; R) < 0$. Using again Proposition 2, we infer that there exists $\varepsilon > 0$ such that $u(\lambda, \xi_0; R) < 0$ for all $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$; in particular, $u(\lambda, \xi_0; \cdot)$ is a lower solution of (17). Arguing exactly as in Step 1, we can show that (17) has an upper solution $\beta_\lambda$ such that $u(\lambda, \xi_0, \cdot) \leq \beta_\lambda$ on $[0, R]$. Then, applying Proposition 1 we deduce that (17) has at least one nonzero solution which is also a strictly positive solution of (3). Consequently, $\Lambda(R_0) > \Lambda(R)$ and the proof is complete.

Corollary 1. Under hypothesis (H), there exists $\Lambda > 2N/(\mu M R^{q+1})$ such that problem (1) has zero, at least one or at least two positive classical radial solutions according to $\lambda \in (0, \Lambda)$, $\lambda = \Lambda$ or $\lambda > \Lambda$. Also, $\Lambda$ is strictly decreasing with respect to $R$.

Example 1. If $N \geq 2$ is an integer and $q > 1$, $m \geq 0$, $R > 0$ are real numbers, then there exists $\Lambda > 2N/R^{m+q+1}$ such that the problem

$$\text{div} \left( \frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \right) + \lambda |x|^m v^q = 0 \quad \text{in } \mathcal{B}(R), \quad v = 0 \quad \text{on } \partial \mathcal{B}(R),$$

has zero, at least one or at least two positive classical radial solutions according to $\lambda \in (0, \Lambda)$, $\lambda = \Lambda$ or $\lambda > \Lambda$. In addition, $\Lambda$ is strictly decreasing with respect to $R$.

Remark 1. The reader will emphasize that, excepting the part concerning the monotonicity of $\Lambda$ as function of $R$, the statements of Theorem 1 and Corollary 1 still remain true if the continuous weight function $\mu$ is defined only on $[0, R]$ instead of $[0, \infty)$ and positive on $(0, R]$.
References