

SOLUTIONS TO OVERDETERMINED ELLIPTIC PROBLEMS IN NONTRIVIAL EXTERIOR DOMAINS

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ABSTRACT. In this paper we construct nontrivial exterior domains $\Omega \subset \mathbb{R}^N$, for all $N \geq 2$, such that the problem

$$\begin{cases} -\Delta u + u - u^p = 0, & u > 0 & \text{in } \Omega, \\ u = 0 & & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = \text{cte} & & \text{on } \partial\Omega, \end{cases}$$

admits a positive bounded solution. This result gives a negative answer to the Berestycki-Caffarelli-Nirenberg conjecture on overdetermined elliptic problems in dimension 2, the only dimension in which the conjecture was still open. For higher dimensions, different counterexamples have been found in the literature; however, our example is the first one in the form of an exterior domain.

1. INTRODUCTION

This paper is concerned with the existence of solutions of a semilinear overdetermined elliptic problem in the form

$$(1.1) \quad \begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = c \neq 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^N$ is a regular domain, f is a Lipschitz function and ν stands for the exterior normal vector to $\partial\Omega$. Observe that the presence of two boundary conditions makes the problem overdetermined. Overdetermined boundary conditions arise naturally in free boundary problems, when the variational structure imposes suitable conditions on the separation interface, see for example [3].

In 1971 J. Serrin proved that if (1.1) is solvable for a bounded domain Ω , then Ω must be a ball ([24, 19]). This is also true if we replace the Laplacian operator by another

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strongly elliptic operator and if the function f depends also on the gradient of u . This result has many applications in Physics, and some of them are the following: 1) when a viscous incompressible fluid is moving in straight parallel streamlines through a pipe of given cross section, the tangential stress per unit area on the pipe wall is the same at all points if and only if the cross section is circular; 2) when a solid straight bar is subject to torsion, the magnitude of the resulting traction which occurs at the surface of the bar is independent of the position if and only if the bar has a circular cross section; 3) when a liquid is rising in a straight capillary tube, the liquid will rise to the same height at the tube wall if and only if the tube has circular section. Serrin's proof is based on the *Alexandrov reflection principle*, introduced in 1956 by Alexandrov in [2] to prove that the only compact, connected, embedded hypersurfaces in \mathbb{R}^N whose mean curvature is constant are the spheres. The reflection principle was used also in 1979 by Gidas, Ni and Nirenberg [13] to derive radial symmetry results for positive solutions of semilinear elliptic equations. After that paper the reflection principle has been named the *moving plane method*.

A natural dual version of the previous situation is to consider problem (1.1) in exterior domains, i.e., domains Ω given as the complement of a compact connected region $D \subset \mathbb{R}^N$. In Physics this situation corresponds to the case of very big domains (mathematically considered as unbounded) with a hole. We refer the reader to the survey [27] for more specific applications in Physics of elliptic overdetermined problems in exterior domains.

In the case of exterior domains, the following problem has been considered:

$$(1.2) \quad \begin{cases} \Delta v + g(v) = 0 & \text{in } \Omega, \\ v = a > 0 & \text{on } \partial\Omega, \\ \frac{\partial v}{\partial \nu} = c & \text{on } \partial\Omega, \\ 0 \leq v < a & \text{in } \Omega. \end{cases}$$

With the change of variables $u := a - v$ we have immediately a problem in the form (1.1) with the extra assumptions $u \leq a$. In this framework, the main research line has aimed to prove the counterpart of the Serrin's symmetry result, that is, to prove that Ω is the complement of a ball. For example under the assumptions that $g(t) \geq 0$ and that $t^{-\frac{n+2}{n-2}}g(t)$ is nonincreasing, Aftalion and Busca proved in [1] that if problem (1.2) has a solution then Ω is the complement of a ball. In [20] Reichel proved the same symmetry result but under different assumptions: he assumes that $g(t)$ is decreasing for small positive t and that $v \rightarrow 0$ at infinity. Finally, Sirakov ([28]) extends Reichel's result in several directions: to other domains (like the complement of a finite number of domains, with different constants a_i, c_i), to other elliptic operators, and moreover he shows that the condition $v < a$ can be replaced with the assumption $c_i \geq 0$.

As a consequence of [20, 28], we show in this paper that for some special functions f , the solution u of (1.1) must be radially symmetric and increasing (see subsection 2.1) and Ω is the complement of a ball. This includes, for instance, the case of the Allen-Cahn nonlinearity $f(u) = u - u^3$.

In the proofs of [1, 20, 28] the main tool used is the moving plane method from infinity, sometimes combined with the moving spheres method. Then their proofs show not only symmetry, but also monotonicity along the radius. At this point, we point out that there are radially symmetric solutions of problem (1.1) which are not monotone along the radius. For instance, there exists a non-monotone radial solution to the problem:

$$(1.3) \quad \begin{cases} \Delta u + u^p - u = 0 & \text{in } B_R^c, \\ u > 0 & \text{in } B_R^c, \\ u = 0 & \text{on } \partial B_R, \end{cases}$$

for any $p > 1$ and $R > 0$, where B_R is the ball of radius R and B_R^c is its complement (see for instance [10]). This equation receives the name of Nonlinear Schrödinger Equation and has been widely studied in the literature. Its solution increases in the radius up to a certain maximum, and then it decreases and converges to 0 at infinity. Therefore, the usage of the moving plane method from infinity is intrinsically restricted to some kind of nonlinearities and/or solutions u . The main goal of this paper is to prove that (1.1) is solvable for some exterior domain different from the complement of a ball. For that, we use a local bifurcation argument from the solutions of (1.3).

Before going further presenting our results, let us review the literature on overdetermined semilinear elliptic problems. In [6] Berestycki, Caffarelli and Nirenberg consider free boundary problems where the variational structure imposes overdetermined conditions on the boundary. The study of the regularity of the solutions by a blow-up technique led them to study problem (1.1) in epigraphs. Under some hypothesis on the nonlinearity f and on the behavior of the epigraph at infinity, they proved that the epigraph must be a half-space (these results were later extended by Farina and Valdinoci [11]). Motivated by this, and by the aforementioned results on exterior domains, they proposed in [6] the following conjecture:

BCN Conjecture: If $\mathbb{R}^N \setminus \overline{\Omega}$ is connected, then the existence of a bounded solution to problem (1.1) implies that Ω is either a ball, a half-space, a generalized cylinder $B^k \times \mathbb{R}^{N-k}$ (B^k is a ball in \mathbb{R}^k), or the complement of one of them.

This conjecture has been answered negatively for $N \geq 3$ in [26], where the third author finds a periodic perturbation of the straight cylinder $B^{N-1} \times \mathbb{R}$ that supports a periodic solution to problem (1.1) with $f(t) = \lambda t$.

In the last years, a parallelism between overdetermined elliptic problems and constant mean curvature surfaces, in the spirit of the correspondence of Alexandrof's and Serrin's

results, has been observed. Indeed, the counterexample to the BCN Conjecture built in [26] belongs to a smooth one-parameter family that can be seen as a counterpart of the family of *Delaunay surfaces*, see [25]. Such domains exist also in other homogeneous manifolds, as $\mathbb{S}^n \times \mathbb{R}$ or $\mathbb{H}^n \times \mathbb{R}$, as shown in [18] as the counterpart of other well known families of constant mean curvature surfaces. In [14] Hélein, Hauswirth and Pacard establish a kind of Weierstrass representation for overdetermined elliptic problems in dimension 2 with $f \equiv 0$ in analogy with minimal surfaces. Moreover, Traizet finds a one-to-one correspondence between solutions of problem (1.1) in dimension 2 with $f \equiv 0$ and a special class of minimal surfaces ([32]). Finally, in [9] Del Pino, Pacard and Wei consider problem (1.1) for functions f of Allen-Cahn type and they build new solutions in domains Ω with boundary close to a dilated embedded minimal surface in \mathbb{R}^3 with finite total curvature and nondegenerate, or to a dilated Delaunay surface.

If Ω is an epigraph, the problem is also related to the *De Giorgi's conjecture* (1978), that is still open in its full generality. This conjecture states that the entire solutions of the Allen-Cahn equation $\Delta u + u - u^3 = 0$ monotone in one direction must have level sets which are parallel hyperplanes if $N \leq 8$. The relationship between the De Giorgi's conjecture and overdetermined problems is not surprising if we recall that this conjecture is the counterpart of the *Bernstein's conjecture* on minimal surfaces (1914), that stated that all entire minimal graphs in \mathbb{R}^N should be hyperplanes, and which has been disproved by E. Bombieri, E. De Giorgi and E. Giusti for $N \geq 9$ ([7]). Starting from the Bombieri-De Giorgi-Giusti entire minimal graph, Del Pino, Kowalczyk and Wei build entire nontrivial monotone solutions to the Allen-Cahn equation if $N \geq 9$. In this spirit, Del Pino, Pacard and Wei has recently built nontrivial solutions for (1.1) for f of Allen-Cahn type in nontrivial epigraphs if $N \geq 9$, see [9]. In [34] Wang and Wei prove that this type of solutions do not exist if $N \leq 8$, a result that can be put in analogy with that of Savin for the De Giorgi conjecture ([23]). Finally, the notion of stability plays an important role in the De Giorgi conjecture, and also in overdetermined problems, see [33].

Coming back to the BCN Conjecture, we point out that all counterexamples mentioned above require $N \geq 3$, and we underline that all the examples of domains solving an overdetermined elliptic problem are linked to minimal or constant mean curvature surfaces.

In this paper we give a counterexample in the form of a exterior domain for any dimension $N \geq 2$. This gives a definitive negative answer to the conjecture, also in dimension two. Partial positive answers to the BCN conjecture in dimension 2 have been given in several works, see [11, 14, 21, 22, 32, 34]. In [21] the authors show that the conjecture holds in dimension 2 under the hypothesis that $\partial\Omega$ is unbounded. The counterexample we give in this paper shows that such hypothesis is actually sharp.

Finally, this is the first example of a domain solving an overdetermined elliptic problem that has no clear counterpart in the theory of minimal or constant mean curvature surfaces.

A first statement of our result (see Section 2 for a more detailed statement) is the following:

Theorem 1.1. *Let $N \in \mathbb{N}$, $N \geq 2$, $1 < p < \frac{N+2}{N-2}$ ($p > 1$ if $N = 2$). There exist smooth exterior domains Ω different from the complement of a ball such that the overdetermined problem*

$$(1.4) \quad \begin{cases} -\Delta u + u - u^p = 0, & u > 0 & \text{in } \Omega, \\ u = 0 & & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = cte & & \text{on } \partial\Omega, \end{cases}$$

admits a bounded solution.

Observe that, for any $R > 0$, the solutions to (1.3) form a trivial family of solutions of (1.4). In this paper we use a local bifurcation argument to show that, from this family of trivial solutions, there are nontrivial solutions in nontrivial domains bifurcating at some values of the radius. The proof uses a general bifurcation result in the spirit of Krasnoselskii. In order to do that, the nondegeneracy of the Dirichlet problem is essential, but in general this is false at least for some radii R . Under some symmetry assumptions, we find a spectral gap for the Dirichlet problem associated to (1.3), that is, we show that it is nondegenerate for $R \in (0, R_0)$, for some $R_0 > 0$. Another important issue of the proof is to show that bifurcation occurs exactly in that interval. This is made by studying the behavior of the first Steklov eigenvalue of the linearized operator.

The paper is organized as follows. In Section 2 we state a more precise version of Theorem 1.1, which is the main result of the paper. Moreover, we complement our analysis by showing the validity of the BCN conjecture for exterior domains and some special nonlinearities f . In Section 3 we give some preliminary results. In particular, the existence of the spectral gap for the Dirichlet problem is stated, but the proof is postponed to Section 7. In Section 4 we define the operator that appears naturally in our problems, and we compute its linearization. Section 5 is devoted to the study of the linearized operator and its spectrum. Finally, in Section 6 we use a local bifurcation result to conclude the proof.

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2. STATEMENT OF OUR MAIN RESULT AND COMMENTS

Let us first set some notations. Given a symmetry group G acting on \mathbb{R}^N , we say that $\Omega \subset \mathbb{R}^N$ is G -symmetric if it is invariant under the action of the group G . In such case, we can define the Sobolev spaces of G -symmetric functions as follows:

$$H_G^1(\Omega) = \{u \in H^1(\Omega) : u = u \circ g \ \forall g \in G\},$$

$$H_{0,G}^1(\Omega) = \{u \in H_0^1(\Omega) : u = u \circ g \ \forall g \in G\},$$

and by $H_G^{-1}(\Omega)$ the dual space of $H_{0,G}^1(\Omega)$. We will use the same kind of notations for functions defined in $\partial\Omega$. In particular:

$$H_G^{1/2}(\partial\Omega) = \{u \in H^{1/2}(\partial\Omega) : u = u \circ g \ \forall g \in G\}.$$

We denote by $B_R \subset \mathbb{R}^N$ the ball of radius R centered at 0, and we may also write \mathbb{S}^{N-1} instead of ∂B_1 . If Ω is radially symmetric, we shall denote the spaces of radially symmetric functions as:

$$H_r^1(\Omega) = \{u \in H^1(\Omega) : u(x) = u(|x|) \text{ a.e. } x \in \Omega\},$$

$$H_{0,r}^1(\Omega) = \{u \in H_0^1(\Omega) : u(x) = u(|x|) \text{ a.e. } x \in \Omega\}.$$

For a function $u \in H^1(\Omega)$, we denote $\|u\| = \left(\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 \right)^{1/2}$ its Sobolev norm. Other norms will be clear from the subscript. In the case of the Holder regularity we can define the following spaces:

$$C_G^{k,\alpha}(\Omega) = \{u \in C^{k,\alpha}(\Omega) : u = u \circ g \ \forall g \in G\},$$

$$C_G^{k,\alpha}(\partial\Omega) = \{u \in C^{k,\alpha}(\partial\Omega) : u = u \circ g \ \forall g \in G\}.$$

Moreover, we will denote by $C_{G,m}^{k,\alpha}(\mathbb{S}^{N-1})$ the set of functions in $C_G^{k,\alpha}(\mathbb{S}^{N-1})$ whose mean is 0. Given a positive function $w \in C_G^{2,\alpha}(\mathbb{S}^{N-1})$ let us denote

$$B_w := \left\{ x \in \mathbb{R}^N : 0 \leq |x| < w \left(\frac{x}{|x|} \right) \right\}.$$

and B_w^c its complement in \mathbb{R}^N .

We denote $\Delta_{\mathbb{S}^{N-1}}$ the Laplace-Beltrami operator in \mathbb{S}^{N-1} , and $\{\mu_i\}_{i \in \mathbb{N}}$ its eigenvalues, i.e. $\mu_i = i(i + N - 2)$. From now on, we will fix a symmetry group G with the following property:

- (G) G leaves invariant the origin and, denoting by $\{\mu_{i_k}\}_{k \in \mathbb{N}}$ the eigenvalues of $\Delta_{\mathbb{S}^{N-1}}$ restricted to G -symmetric functions and m_k their multiplicity, we require $i_1 \geq 2$ and m_1 odd.

We are now able to state the main result of this paper, from which Theorem 1.1 follows immediately.

Theorem 2.1. *Let $N \in \mathbb{N}$, $N \geq 2$, $1 < p < \frac{N+2}{N-2}$ ($p > 1$ if $N = 2$). Let G be a group of symmetries of \mathbb{R}^N satisfying (G). Then there exist $R_* = R_*(i_1, p) > 0$, a sequence of non-zero functions $v_n \in C_{G,m}^{2,\alpha}(\mathbb{S}^{N-1})$ converging to 0, and a sequence of positive real numbers R_n converging to R_* such that the overdetermined problem:*

$$\begin{cases} -\Delta u + u - u^p = 0, & \text{in } B_{R_n(1+v_n)}^c \\ u = 0 & \text{on } \partial B_{R_n(1+v_n)} \\ \frac{\partial u}{\partial \nu} = cte & \text{on } \partial B_{R_n(1+v_n)} \end{cases}$$

admits a positive bounded solution $u \in C_G^{2,\alpha}(B_{R_n(1+v_n)}^c) \cap H_{0,G}^1(B_{R_n(1+v_n)}^c)$.

Remark 2.2. There are many examples of groups G satisfying (G). For instance, if $1 \leq m \leq N-1$, the group $G = O(m) \times O(N-m)$ satisfies that $i_1 = 2$ and $m_1 = 1$. Indeed, in this case, the corresponding eigenvalue is given as the restriction to \mathbb{S}^{N-1} of the 2-homogeneous harmonic polynomial:

$$p(x) = (N-m)(x_1^2 + \cdots + x_m^2) - m(x_{m+1}^2 + \cdots + x_N^2).$$

In dimension 2, we can take as G any dihedral group \mathbb{D}_k , $k \geq 3$. In this case, $i_1 = k$ and $m_1 = 1$. In dimension 3 we can take G as the group of isometries of the tetrahedron ($i_1 = 3$ and $m_1 = 1$), the octahedron ($i_1 = 4$ and $m_1 = 1$) or the icosahedron ($i_1 = 6$ and $m_1 = 1$), see [16].

Remark 2.3. One could ask whether two different groups G_1, G_2 give rise to different domains Ω . The answer is (partially) affirmative. Indeed, define $G = \langle G_1, G_2 \rangle$, and denote:

- (1) $\{\mu_{i_k}\}$ the eigenvalues of $\Delta_{\mathbb{S}^{N-1}}$ restricted to G_1 -symmetric functions,
- (2) $\{\mu_{j_k}\}$ the eigenvalues of $\Delta_{\mathbb{S}^{N-1}}$ restricted to G_2 -symmetric functions,
- (3) $\{\mu_{l_k}\}$ the eigenvalues of $\Delta_{\mathbb{S}^{N-1}}$ restricted to G -symmetric functions.

Clearly, $l_1 \geq \max\{i_1, j_1\}$. If $l_1 > \min\{i_1, j_1\}$, then the two groups G_1 and G_2 give rise to different domains Ω . In particular, this is true if $i_1 \neq j_1$. In fact the value of the bifurcation radius R_* is different; this is due to the fact that the value R_* is strictly increasing with respect to i_1 , as can be seen in the proof of Lemma 6.4.

2.1. On the validity of the BCN conjecture for other type of nonlinearities. As commented in the introduction, Theorem 2.1 gives a counterexample to the Berestycki-Caffarelli-Nirenberg conjecture in exterior domains. We end this section by discussing its validity for other types of nonlinear terms $f(u)$.

Proposition 2.4. *Let Ω be a bounded smooth domain and u be a bounded solution of the problem:*

$$(2.1) \quad \begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega^c, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial v}{\partial \nu} = c & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega^c. \end{cases}$$

Suppose that $f : [0, +\infty) \rightarrow \mathbb{R}$ is a C^1 function such that there exists $L > 0$ with:

- (1) $f(t) > 0$ for all $t \in (0, L)$,
- (2) $f(t) < 0$ for all $t > L$,
- (3) $f(L) = 0$ and $f'(L) < 0$,
- (4) If $f(0) = 0$ then $f'(0) > 0$.

Then Ω is a ball and u is radially symmetric and increasing.

Proof. The proof relies on the result [20] on exterior domains. But in order to apply it, we need to show that $u(x) < \lim_{|x| \rightarrow +\infty} u(x) = L$. This is done in two steps.

Step 1. $u(x) < L$ for all $x \in \Omega^c$.

Assume that $\alpha = \sup u > L$, and take $x_n \in \Omega^c$ such that $u(x_n) \rightarrow \alpha$. Since u is subharmonic in the domain $\{u(x) > L\}$, then it cannot reach its supremum, so x_n is unbounded. Define $u_n(x) = u(x + x_n)$. Clearly u_n is bounded in L^∞ , and hence also its C^k norm is bounded in L^∞ on compact sets, for every $k \in \mathbb{N}$. As a consequence, $u_n \rightarrow v$ in C^k sense on compact sets of \mathbb{R}^N , where $0 \leq v(x) \leq \alpha$ is an entire solution of the equation $\Delta v + f(v) = 0$, and $v(0) = \alpha > L$. But v is strictly subharmonic in the domain $\{v(x) > L\}$, then it cannot reach its supremum. Hence $\alpha \leq L$. Moreover, observe that $u \equiv L$ is a constant solution of the problem $\Delta u + f(u) = 0$. By the maximum principle, those two solutions cannot be at contact, concluding the proof.

Step 2. $\lim_{|x| \rightarrow +\infty} u(x) = L$.

Let us define $\beta = \liminf_{|x| \rightarrow +\infty} u(x)$. This implies that there exists $x_n \in \Omega^c$, $|x_n| \rightarrow +\infty$, such that $u(x_n) \rightarrow \beta$. As before, define $u_n(x) = u_n(x + x_n)$, $u_n \rightarrow v$ in C^k sense on compact sets of \mathbb{R}^N , where $\beta \leq v(x) \leq L$ is an entire solution of the equation $\Delta v + f(v) = 0$. In particular v is superharmonic, and $v(0) = \beta = \inf v \in [0, L]$. Hence $v \equiv \beta$, which in turn implies that $\beta = 0$ or $\beta = L$. We now exclude the possibility $\beta = 0$. In such case $f(0) = 0$ and u_n converges uniformly to 0 on compact sets. Then, given any $\varepsilon > 0$, $R > 0$, we have that $u_n(x) \in (0, \varepsilon)$ in $B(0, R)$ for large n . By hypothesis (4), $f(u) \geq \frac{f'(0)}{2}u$ for any $u \in (0, \varepsilon)$, provided that $\varepsilon > 0$ is chosen sufficiently small. We can

also choose $R > 0$ such that the first eigenvalue of the Laplace operator under Dirichlet boundary conditions satisfies $\lambda_1 = \lambda_1(B_R) < \frac{f'(0)}{2}$. Then u_n is a positive solution of the inequality $-\Delta u_n > \lambda_1 u_n$ in B_R . But this is not possible, so we conclude that $\beta = L$.

We are now in conditions to apply the result of [20] to the function $v(x) = L - u(x)$ to conclude the proof. \square

The above result applies for instance to the case of the Allen-Cahn nonlinearity $f(u) = u - u^3$. The following proposition, probably known in the literature, represents an other example when the BCN conjecture works for exterior domains.

Proposition 2.5. *Let Ω be a bounded smooth domain and u be a bounded solution of the problem:*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega^c, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial v}{\partial \nu} = c & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega^c. \end{cases}$$

Then Ω is a ball and u is radially symmetric and increasing.

Proof. Here also the proof relies on the application of the result [20] to $v(x) = L - u(x)$, where $L = \lim_{|x| \rightarrow +\infty} u(x)$. This is possible because

- (1) The limit $\lim_{|x| \rightarrow +\infty} u(x)$ exists for harmonic functions defined outside of a compact set, and then L is well defined (see for example Theorems 4.9 and 4.10 of [5]).
- (2) v is positive, because u is harmonic and then $u(x) < L$ for all $x \in \Omega^c$.

This concludes the proof. \square

Remark 2.6. *There exist nonlinearities such that, for any bounded smooth domain $\Omega \subset \mathbb{R}^N$, the problem*

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega^c, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega^c, \end{cases}$$

does not admit any bounded solution. In this case we could say that the BCN conjecture holds in a degenerate way. This happens, for instance, under the following assumptions:

- (1) $f(t) < 0$ for any $t > 0$. This can be seen by defining $\beta = \sup u > 0$, $x_n \in \Omega^c$ such that $u(x_n) \rightarrow \beta$ and, arguing as in the proof of Proposition 2.4, we would pass to a nonconstant subharmonic function attaining its supremum.
- (2) $f(t) \geq \varepsilon t$ for any $t > 0$ and some $\varepsilon > 0$, see for example [22, Proposition 3.1].
- (3) $f(t) \geq c > 0$ for any $t > 0$. Indeed, for any ball $B(x_0, R)$, we have that $v(x) = \frac{c}{2N}(R^2 - |x - x_0|^2)$ solves the equation $\Delta v + c = 0$, with 0 boundary data. By the

maximum principle, if such ball is contained in Ω^c , then a solution of (??) would satisfy $u \geq v$. But as R becomes large v becomes larger, giving a contradiction.

The previous results, together with [21], implies the following:

Corollary 2.7. *In dimension 2, under the assumptions of Propositions 2.4 and 2.5 or Remark 2.6, then the BCN conjecture holds.*

We remark also that the case of the original Serrin's overdetermined problem, i.e. the case where f is a constant, is completely solved for exterior domains by the previous results.

We finish this section with a comment on the case of affine functions $f(u)$. In view of Propositions 2.4 and 2.5 and Remark 2.6, the only case left is $f(u) = au + b$ with $a > 0$ and $b < 0$. This case seems to remain as an interesting and hard open problem.

To fix ideas, let us take $N = 3$. Then, the function $u(x) = \frac{\sin|x|}{|x|} + \frac{2}{3\pi}$ is a solution of the problem:

$$\begin{cases} \Delta u + u - \frac{2}{3\pi} = 0 & \text{in } B^c, \\ u = 0 & \text{on } \partial B, \\ \frac{\partial v}{\partial \nu} = c \neq 0 & \text{on } \partial B, \\ u > 0 & \text{in } B^c. \end{cases}$$

Here $B = B_{\frac{3\pi}{2}}$. This is of course a radial solution to the overdetermined problem, but it is not monotone along the radius, as the solution of (1.3). Actually it oscillates infinitely many times around its limit $\frac{2}{3\pi}$. As a consequence a symmetry proof via monotonicity, in the spirit of [1, 20, 28], cannot be performed. It would be interesting to use this solution to build other counterexamples to the BCN conjecture, as we do in this paper, but this seems difficult. The solution is explicit but it does not belong to the Sobolev space $H^1(B^c)$, and its Morse index is infinity, even when we restrict ourselves to radially symmetric functions.

3. PRELIMINARY RESULTS

As commented in the introduction, we will prove Theorem 2.1 by means of a bifurcation argument to show the existence of such domain Ω close to the exterior of a ball. For that, we shall need some facts of the Dirichlet problem; given any $p > 1$, consider:

$$(3.1) \quad \begin{cases} -\Delta u + u - u^p = 0, & u > 0 & \text{in } B_R^c, \\ u = 0 & & \text{on } \partial B_R. \end{cases}$$

It will be convenient to make a change of scale and pass to the equivalent problem:

$$(3.2) \quad \begin{cases} -\lambda \Delta u + u - u^p = 0, & u > 0 & \text{in } B_1^c, \\ u = 0 & & \text{on } \partial B_1, \end{cases}$$

where $\lambda = \frac{1}{R^2}$. In the proposition below we list some known properties of this problem.

Proposition 3.1. *There follows:*

- a) *For any $\lambda > 0$, there exists a radially symmetric C^∞ solution of (3.2). This solution increases in the radius up to a certain maximum, and then it decreases and converges to 0 at infinity.*
- b) *The positive and radial solution to (3.2) is unique: we denote it by u_λ .*
- c) *Let us define the linearized operator $L_\lambda : H_{0,G}^1(B_1^c) \rightarrow H_G^{-1}(B_1^c)$,*

$$(3.3) \quad L_\lambda(\phi) = -\lambda\Delta\phi + \phi - pu_\lambda^{p-1}\phi,$$

and consider the eigenvalue problem:

$$L_\lambda(\phi) = \tau\phi.$$

In the space of radially symmetric functions $H_{0,r}^1(B_R^c)$ this problem has a unique negative eigenvalue and no zero eigenvalues. In other words, u_λ is nondegenerate in $H_{0,r}^1(B_1^c)$ and has Morse index 1. We denote by $z_\lambda \in H_{0,r}^1(B_1^c)$ (normalized by $\|z_\lambda\| = 1$) the positive eigenfunction with negative eigenvalue, i.e.

$$(3.4) \quad \begin{cases} -\Delta z_\lambda + z_\lambda - pu_\lambda^{p-1}z_\lambda = \tau_0 z_\lambda & \text{in } B_1^c, \\ z_\lambda = 0 & \text{on } \partial B_1, \end{cases}$$

where $\tau_0 = \tau_0(\lambda) < 0$. Moreover z_λ is a C^∞ function.

Proof. Statement a) is quite well known and has been proved in [10], for instance. The results b) and c) are more recent and have been obtained in [12, 31]. \square

Let us define the bilinear operator associated to (3.3): $Q_\lambda : H_{0,G}^1(B_1^c) \times H_{0,G}^1(B_1^c) \rightarrow \mathbb{R}$,

$$(3.5) \quad Q_\lambda(\psi_1, \psi_2) = \int_{B_1^c} \lambda \nabla \psi_1 \cdot \nabla \psi_2 + \psi_1 \psi_2 - pu_\lambda^{p-1} \psi_1 \psi_2.$$

By Proposition 3.1, c), Q_λ is positive definite for $\psi \in H_{0,r}^1(B_1^c)$ with $\int_{B_1^c} \psi z_\lambda = 0$. In next lemma we show that this property may fail if we do not impose radial symmetry. This might be known in the literature, but we have not been able to find a specific reference.

Proposition 3.2. *Let G be a symmetry group satisfying hypothesis (G). Then there exists $\varepsilon > 0$ such that for any $\lambda \in (0, \varepsilon)$, there exists $\psi \in H_{0,G}^1(B_1^c)$ such that*

- (1) $\int_{B_1^c} \psi z_\lambda = 0$.
- (2) $Q_\lambda(\psi, \psi) < 0$.

Remark 3.3. The proof of Proposition 3.2 can be adapted to show that the Morse index of u_λ in $H_0^1(B_1^c)$ diverges as $\lambda \rightarrow 0$. This is in contrast with what happens in the radial case. Hence, one expects the existence of infinitely many branches of nonradial solutions to the problem (3.2) bifurcating from u_λ . As far as we know, this result has not been

explicitly written in the literature. In any case, the existence of this kind of solutions is outside the scope of this paper.

Proof of Proposition 3.2. Let μ_{i_1} be the first eigenvalue of the operator $\Delta_{\mathbb{S}^{N-1}}$ restricted to G -symmetric functions, and ϕ one of the corresponding normalized eigenfunctions. Let us define the function ψ in polar coordinates: $\psi(r, \theta) = u_\lambda(r)\phi(\theta)$, with $\theta \in \mathbb{S}^{N-1}$. Since z_λ is radially symmetric, $\int_{B_1^c} \psi z_\lambda = 0$. Observe now that

$$|\nabla \psi|^2 = u'_\lambda(r)^2 \phi^2(\theta) + \frac{1}{r^2} u_\lambda(r)^2 |\nabla_\theta \phi(\theta)|^2.$$

Therefore,

$$\begin{aligned} Q_\lambda(\psi, \psi) &= \int_1^{+\infty} r^{N-1} (\lambda u'_\lambda(r)^2 + u_\lambda(r)^2 - p|u_\lambda(r)|^{p+1}) \int_{\mathbb{S}^{N-1}} \phi^2(\theta) d\theta dr \\ &\quad + \lambda \int_1^{+\infty} r^{N-1} \frac{1}{r^2} u_\lambda(r)^2 \int_{\mathbb{S}^{N-1}} |\nabla_\theta \phi(\theta)|^2 d\theta dr \\ &= \int_1^{+\infty} \left(\lambda u'_\lambda(r)^2 + u_\lambda(r)^2 - p|u_\lambda(r)|^{p+1} + \lambda \frac{\mu_{i_1}}{r^2} u_\lambda(r)^2 \right) r^{N-1} dr. \end{aligned}$$

If we multiply the equation in (3.2) (with $u = u_\lambda$) by u_λ and integrate, we obtain:

$$\int_1^{+\infty} (\lambda u'_\lambda(r)^2 + u_\lambda(r)^2) r^{N-1} dr = \int_1^{+\infty} |u_\lambda(r)|^{p+1} r^{N-1} dr.$$

Plugging this identity in the above expression, we obtain:

$$\begin{aligned} Q_\lambda(\psi, \psi) &= \int_1^{+\infty} \left((1-p) (\lambda u'_\lambda(r)^2 + u_\lambda(r)^2) + \lambda \frac{\mu_{i_1}}{r^2} u_\lambda(r)^2 \right) r^{N-1} dr \\ &\leq \int_1^{+\infty} (1-p + \lambda \mu_{i_1}) u_\lambda(r)^2 r^{N-1} dr, \end{aligned}$$

and this last quantity is negative if $\lambda < \varepsilon := \frac{p-1}{\mu_{i_1}}$. \square

In view of Proposition 3.2, the operator L_λ may be degenerate if we consider non radially symmetric functions. However, as a consequence of next proposition, we conclude that L_λ is nondegenerate for large values of λ .

Proposition 3.4. *There exists $M > 0$ such that for any $\lambda > M$, $Q_\lambda(\psi, \psi) > 0$ for any $\psi \in H_{0,G}^1(B_1^c)$ such that $\int_{B_1^c} \psi z_\lambda = 0$.*

The proof of this proposition is postponed to Section 7.

We define:

$$(3.6) \quad \Lambda_0 = \sup \left\{ \lambda > 0 : Q_\lambda(\psi, \psi) \leq 0 \text{ for some } \psi \in H_{0,G}^1(B_1^c) \setminus \{0\}, \int_{B_1^c} \psi z_\lambda = 0 \right\},$$

Observe that Proposition 3.4 implies that the set above is bounded from above, whereas Proposition 3.2 implies $\Lambda_0 > 0$. For $\lambda > \Lambda_0$, the Dirichlet problem (3.2) is nondegenerate, in the sense that the operator L_λ has trivial kernel. The following result is rather standard but, since the domain under consideration is unbounded, we prefer to state it and include its proof.

Lemma 3.5. *Assume that the operator L_λ has trivial kernel. Then:*

- a) *The operator L_λ is an isomorphism.*
- b) *Given $v \in H_G^{1/2}(\mathbb{S}^{N-1})$, there exists a unique solution $\psi_v \in H_G^1(B_1^c)$ of the problem:*

$$(3.7) \quad \begin{cases} -\lambda \Delta \psi_v + \psi_v - pu_\lambda^{p-1} \psi_v = 0 & \text{in } B_1^c, \\ \psi_v = v & \text{on } \partial B_1. \end{cases}$$

Proof. We observe that the operator

$$\psi \rightarrow \lambda \Delta \psi + \psi$$

is an isomorphism from $H_{0,G}^1(B_1^c) \rightarrow H_G^{-1}(B_1^c)$. Moreover, the operator

$$\psi \rightarrow pu_\lambda^{p-1} \psi$$

is a compact operator from $H_{0,G}^1(B_1^c) \rightarrow H_G^{-1}(B_1^c)$, because $u_\lambda(x)$ tends to 0 when $|x| \rightarrow +\infty$ (see Proposition 3.1). Our operator L_λ is the sum of the two previous operators, and since it has trivial kernel by assumption, we conclude that it is an isomorphism.

In order to prove b), take $\phi \in H_G^1(B_1^c)$ such that $\phi|_{\partial B_1} = v$. Observe that $\xi = -\lambda \Delta \phi + \phi - pu_\lambda^{p-1} \phi$ is a element of $H_G^{-1}(B_1^c)$ in the sense that:

$$(\xi, \psi) = \int_{B_1^c} \lambda \nabla \phi \cdot \nabla \psi + \phi \psi - pu_\lambda^{p-1} \phi \psi.$$

for all $\psi \in H_{0,G}^1(B_1^c)$. By a), we can find $\theta \in H_{0,G}^1(B_1^c)$ with $L_\lambda(\theta) = \xi$. Then $\phi - \theta$ is a solution of (3.7). \square

4. THE DIRICHLET-TO-NEUMANN OPERATOR AND ITS LINEARIZATION

The main result of this section is the following:

Proposition 4.1. *Assume that $\lambda > \Lambda_0$, where Λ_0 is given by (3.6). Then, for all function $v \in C_{G,m}^{2,\alpha}(\mathbb{S}^{N-1})$ whose norm is sufficiently small, there exists a unique positive solution $u = u(\lambda, v) \in C^{2,\alpha}(B_{1+v}^c) \cap H_{0,G}^1(B_{1+v}^c)$ to the problem*

$$(4.1) \quad \begin{cases} -\lambda\Delta u + u - u^p = 0 & \text{in } B_{1+v}^c \\ u = 0 & \text{on } \partial B_{1+v}. \end{cases}$$

In addition u depends smoothly on the function v , and $u = u_\lambda$ when $v \equiv 0$.

Proof. Let $v \in C_{G,m}^{2,\alpha}(\mathbb{S}^{N-1})$. Instead of working on a domain depending on the function v , it will be more convenient to work on the fixed domain B_1^c , endowed with a new metric depending on the function v . This can be achieved by considering the diffeomorphism $Y : B_1^c \rightarrow B_{1+v}^c$ given by

$$(4.2) \quad Y(y) := \left(1 + \chi(y) v \left(\frac{y}{|y|} \right) \right) y$$

where χ is a cut-off function such that:

$$\chi(y) = \begin{cases} 0 & |y| \geq 3/2, \\ 1 & |y| \leq 5/4. \end{cases}$$

Hence the coordinates we consider from now on are $y \in B_1^c$ and in these coordinates the new metric g can be written as

$$g = \delta_{ij} + \sum_{i,j} C^{ij} dy_i dy_j,$$

where the coefficients $C^{ij} \in C_G^{1,\alpha}(B_1^c)$ are functions of y depending on v and the first partial derivatives of v . Moreover, $C^{ij} \equiv 0$ when $v = 0$ and the maps $v \mapsto C^{ij}(v)$ are smooth. Up to some multiplicative constant, the problem we want to solve can now be rewritten in the form

$$(4.3) \quad \begin{cases} -\lambda\Delta_g \hat{u} + \hat{u} - \hat{u}^p = 0 & \text{in } B_1^c \\ \hat{u} = 0 & \text{on } \partial B_1. \end{cases}$$

When $v \equiv 0$, the metric g is nothing but the Euclidean metric and a solution of (4.3) is therefore given by $\hat{u} = u_\lambda$. In the general case, the relation between the function u and the function \hat{u} is simply given by

$$Y^* u = \hat{u}.$$

For $\psi \in H_{0,G}^1(B_1^c)$ we define

$$N(v, \psi) := -\lambda\Delta_g(u_\lambda + \psi) + (u_\lambda + \psi) - [(u_\lambda + \psi)^+]^p.$$

where $(u_\lambda + \psi)^+$ denotes the positive part of the function $u_\lambda + \psi$. We have

$$N(0, 0) = 0.$$

The mapping N is a smooth map from a neighborhood of $(0, 0)$ in $C_G^{2,\alpha}(\mathbb{S}^{N-1}) \times H_{0,G}^1(B_1^c)$ into $H_G^{-1}(B_1^c)$. The partial differential of N with respect to ψ , computed at $(0, 0)$, is given by

$$D_\psi N|_{(0,0)}(\psi) = -\lambda \Delta \psi + \psi - p u_\lambda^{p-1} \psi.$$

Since $\lambda > \Lambda_0$, $D_\psi N|_{(0,0)} : H_{0,G}^1(B_1^c) \rightarrow H_G^{-1}(B_1^c)$ is an isomorphism by Lemma 3.5. Therefore, the Implicit Function Theorem implies that, for v in a neighborhood of 0 in $C_G^{2,\alpha}(\mathbb{S}^{N-1})$, there exists of $\psi(\lambda, v) \in H_{0,G}^1(B_1^c)$ such that $N(v, \psi(v, \lambda)) = 0$. The regularity of $u = u_\lambda + \psi(v, \lambda)$ follows from classical Schauder regularity theory, whereas the fact that u is positive comes from the maximum principle. \square

For any $\lambda > \Lambda_0$, after canonical identification of ∂B_{1+v} with \mathbb{S}^{N-1} , we can take an open set $\mathcal{U} \in (\Lambda_0, +\infty) \times C_{G,m}^{2,\alpha}(\mathbb{S}^{N-1})$ containing $(\Lambda_0, +\infty) \times \{0\}$, as the domain of the operator $F : \mathcal{U} \rightarrow C_{G,m}^{1,\alpha}(\mathbb{S}^{N-1})$ defined by

$$(4.4) \quad F(\lambda, v) = \frac{\partial u(\lambda, v)}{\partial \nu} - \frac{1}{\text{Vol}(\partial B_{1+v})} \int_{\partial B_{1+v}} \frac{\partial u(\lambda, v)}{\partial \nu}$$

where ν denotes the unit normal vector field to ∂B_{1+v} pointing to the interior of B_{1+v} and $u(\lambda, v)$ is the solution of (4.1) provided by Proposition 4.1.

Observe that $F(\lambda, v) = 0$ if and only if $\frac{\partial u}{\partial \nu}$ is constant at the boundary ∂B_{1+v} . Clearly, $F(\lambda, 0) = 0$ for all $\lambda \in (\Lambda_0, +\infty)$. Our purpose is to find a bifurcation branch from those solutions, so that we get $F(\lambda, v_\lambda) = 0$, with $v_\lambda \in C_{G,m}^{2,\alpha}(\mathbb{S}^{N-1})$ small, but different from 0.

We will compute now the Frechet derivative of the operator F . Before this, we state a useful lemma.

Lemma 4.2. *Let $v \in C_{G,m}^{2,\alpha}(\mathbb{S}^{N-1})$ and $\psi = \psi_v \in C_G^{2,\alpha}(B_1^c) \cap H_G^1(B_1^c)$ be a solution of (3.7). Then:*

$$\int_{B_1^c} \psi z_\lambda = 0, \quad \int_{\partial B_1} \frac{\partial \psi}{\partial \nu} = 0.$$

Proof. We multiply the equation in (3.4) by ψ , the equation in (3.7) by z_λ , and integrate by parts to obtain:

$$\tau_0 \int_{B_1^c} z_\lambda \psi = \int_{\partial B_1} \left(\frac{\partial \psi}{\partial \nu} z_\lambda - \frac{\partial z}{\partial \nu} \psi \right).$$

We know that $z_\lambda = 0$ and $\frac{\partial z_\lambda}{\partial \nu}$ is constant on ∂B_1 (recall that z_λ is radially symmetric) $\psi = v$ on ∂B_1 . The first identity follows immediately.

Now, define $\kappa_\lambda \in H_r^1(B_1^c)$ as the unique solution of the problem:

$$(4.5) \quad \begin{cases} -\Delta \kappa_\lambda + \kappa_\lambda - p u_\lambda^{p-1} \kappa_\lambda = 0 & \text{in } B_1^c, \\ \kappa_\lambda = 1 & \text{on } \partial B_1. \end{cases}$$

The existence of such solution is guaranteed for any $\lambda \in \mathbb{R}$ by Proposition (3.1) c) and Lemma 3.5 b). We multiply (4.5) by ψ , (3.7) by κ_λ and integrate by parts to conclude:

$$0 = \int_{\partial B_1} \left(\frac{\partial \psi}{\partial \nu} \kappa_\lambda - \frac{\partial \kappa_\lambda}{\partial \nu} \psi \right).$$

We know that $k_\lambda = 1$ and $\frac{\partial k_\lambda}{\partial \nu}$ is constant on ∂B_1 (k_λ is also radially symmetric), and that $\psi = v$ on ∂B_1 . The second identity follows immediately. \square

We define the operator $H_\lambda : C_{G,m}^{2,\alpha}(\mathbb{S}^{N-1}) \rightarrow C_{G,m}^{1,\alpha}(\mathbb{S}^{N-1})$,

$$(4.6) \quad H_\lambda(v) = \frac{\partial \psi_v}{\partial \nu} - (N-1)v.$$

Here ψ_v is given by Lemma 3.5, b). Observe that by Schauder Elliptic Estimates, if $v \in C_{G,m}^{2,\alpha}(\partial B_1)$, $\psi_v \in C_G^{2,\alpha}(B_1^c)$, and then the operator is well defined.

Proposition 4.3. *The map F defined in 4.4 is a C^1 operator in a neighborhood of $(\lambda, 0)$ for all $\lambda > \Lambda_0$, and $D_v F|_{(\lambda,0)} = H_\lambda$.*

Proof. The operator F is a C^1 operator by Proposition 4.1 (the function u depends smoothly on v). The linear operator obtained by linearizing F with respect to v at $(\lambda, 0)$ is then given by the directional derivative

$$F'(w) = \lim_{s \rightarrow 0} \frac{F(\lambda, s w) - F(\lambda, 0)}{s} = \lim_{s \rightarrow 0} \frac{F(\lambda, s w)}{s}.$$

Writing $v = s w$, we consider the diffeomorphism $Y : B_1^c \rightarrow B_{1+v}^c$ given in (4.2). We set \hat{g} the induced metric, so that $\hat{u} = Y^* u$ is the solution (smoothly depending on the real parameter s) of

$$\begin{cases} -\lambda \Delta_{\hat{g}} \hat{u} + \hat{u} - \hat{u}^p = 0 & \text{in } B_1^c \\ \hat{u} = 0 & \text{on } \partial B_1. \end{cases}$$

We remark that $\hat{u}_\lambda := Y^* u_\lambda$ is a solution of

$$-\lambda \Delta_{\hat{g}} \hat{u}_\lambda + \hat{u}_\lambda - \hat{u}_\lambda^p = 0$$

in a neighborhood of B_1^c (note that u_λ is radial and then can be extended in a neighborhood of ∂B_1), and

$$\hat{u}_\lambda(y) = u_\lambda((1 + s w) y),$$

on ∂B_1 . Writing $\hat{u} = \hat{u}_\lambda + \hat{\psi}$ we find that

$$(4.7) \quad \begin{cases} -\lambda \Delta_{\hat{g}} \hat{\psi} + (\hat{u}_\lambda + \hat{\psi}) - (\hat{u}_\lambda + \hat{\psi})^p - \hat{u}_\lambda + \hat{u}_\lambda^p = 0 & \text{in } B_1^c \\ \hat{\psi} = -\hat{u}_\lambda & \text{on } \partial B_1 \end{cases}$$

Obviously $\hat{\psi}$ is a smooth functions of s . When $s = 0$, we have $u = u_\lambda$. Therefore, $\hat{\psi} = 0$ and $\hat{u}_\lambda = u_\lambda$ when $s = 0$. We set

$$\dot{\psi} = \partial_s \hat{\psi}|_{s=0}.$$

Differentiating (4.7) with respect to s and evaluating the result at $s = 0$, we obtain

$$\begin{cases} -\lambda \Delta \dot{\psi} + (1 + p u_\lambda^{p-1}) \dot{\psi} = 0 & \text{in } B_1^c \\ \dot{\psi} = -\partial_r u_\lambda w & \text{on } \partial B_1 \end{cases}$$

where we have set $r := |y|$. To summarize, we have proved that

$$\hat{u} = \hat{u}_\lambda + s \partial_r u_\lambda \psi + \mathcal{O}(s^2)$$

where ψ is the solution of (3.7). In particular, in $B_{5/4}^c$, we have

$$\begin{aligned} \hat{u}(y) &= u_\lambda((1 + s w(y/|y|)) y) + s \partial_r u_\lambda \psi(y) + \mathcal{O}(s^2) \\ &= u_\lambda(y) + s \partial_r u_\lambda (r w(y/|y|) + \psi) + \mathcal{O}(s^2) \end{aligned}$$

In order to compute the normal derivative of the function \hat{u} when the normal is computed with respect to the metric \hat{g} , we use polar coordinates $y = r \theta$ where $\theta \in \mathbb{S}^{N-1}$. Then the metric \hat{g} can be expanded in $B_{5/4} \setminus B_1$ as

$$\hat{g} = (1 + s w)^2 dr^2 + 2 s (1 + s w) r dw dr + r^2 (1 + s w)^2 \hat{e} + s^2 r^2 dw^2$$

where \hat{e} is the metric on \mathbb{S}^{N-1} induced by the Euclidean metric. It follows from this expression that the unit normal vector field to ∂B_1 for the metric \hat{g} is given by

$$\hat{\nu} = ((1 + s w)^{-1} + \mathcal{O}(s^2)) \partial_r + \mathcal{O}(s) \partial_{\theta_j}$$

where ∂_{θ_j} are vector fields induced by a parameterization of \mathbb{S}^{N-1} . Using this, we conclude that

$$\hat{g}(\nabla \hat{u}, \hat{\nu}) = \partial_r u_\lambda + s (w \partial_r^2 u_\lambda + \partial_r \psi) + \mathcal{O}(s^2)$$

on ∂B_1 . The result then follows at once from the fact that $\partial_r u_\lambda$ and $\partial_r^2 u_\lambda$ are constant on ∂B_1 , while the terms w and $\partial_r \psi$ have mean 0 on ∂B_1 , and

$$-\lambda(\partial_r^2 u_\lambda + (N-1)\partial_r u_\lambda) = 0$$

on ∂B_1 . This completes the proof of the proposition. \square

5. STUDY OF THE LINEARIZED OPERATOR

In view of Proposition 4.3, a bifurcation might appear only for values of λ so that H_λ becomes degenerate (H_λ was defined in (4.6)). We shall see that this is indeed the case for some $\lambda > \Lambda_0$. Let us define the first eigenvalue of the operator H_λ as

$$\sigma_1(H_\lambda) = \inf \left\{ \int_{\mathbb{S}^{N-1}} v H_\lambda(v) : v \in C_{G,m}^{2,\alpha}(\mathbb{S}^{N-1}), \int_{\mathbb{S}^{N-1}} v^2 = 1 \right\} \in [-\infty, +\infty).$$

The main result of this section is the following:

Proposition 5.1. *There exists $\Lambda_2 > \Lambda^* > \Lambda_0$ (Λ_0 is given in (3.6)) such that:*

- (1) *if $\lambda \geq \Lambda_2$ then $\sigma_1(H_\lambda) > 0$;*
- (2) *$\sigma_1(H_{\Lambda^*}) = 0$;*
- (3) *there exists a sequence of real numbers λ_n , increasing and converging to Λ^* , such that $\sigma_1(H_{\lambda_n}) < 0$.*

In order to prove Proposition 5.1, let us define the following bilinear forms: $T_\lambda : H_G^{1/2}(\mathbb{S}^{N-1}) \times H_G^{1/2}(\mathbb{S}^{N-1}) \rightarrow \mathbb{R}$ defined by

$$T_\lambda(v_1, v_2) = \int_{\mathbb{S}^{N-1}} v_1 \frac{\partial \psi_{v_2}}{\partial \nu},$$

and $\tilde{T}_\lambda : H_G^{1/2}(\mathbb{S}^{N-1}) \times H_G^{1/2}(\mathbb{S}^{N-1}) \rightarrow \mathbb{R}$ defined by

$$\tilde{T}_\lambda(v_1, v_2) = \int_{\mathbb{S}^{N-1}} v_1 \frac{\partial \psi_{v_2}}{\partial \nu} - \int_{\mathbb{S}^{N-1}} (N-1)v_1 v_2.$$

Observe that $T_\lambda, \tilde{T}_\lambda$ are symmetric. Moreover, it is clear that:

$$\sigma_1(H_\lambda) = \inf \left\{ \int_{\mathbb{S}^{N-1}} \tilde{T}_\lambda(v, v) : v \in C_{G,m}^{2,\alpha}(\mathbb{S}^{N-1}), \int_{\mathbb{S}^{N-1}} v = 0, \int_{\mathbb{S}^{N-1}} |v|^2 = 1 \right\}.$$

We also define the bilinear form $\tilde{Q}_\lambda : H_G^1(B_1^c) \times H_G^1(B_1^c) \rightarrow \mathbb{R}$, by

$$(5.1) \quad \tilde{Q}_\lambda(\psi_1, \psi_2) = Q_\lambda(\psi_1, \psi_2) - \lambda(N-1) \int_{\partial B_1} \psi_1 \psi_2,$$

where Q_λ has been defined in (3.5). It is easy to verify that

$$T_\lambda(v_1, v_2) = \frac{1}{\lambda} Q_\lambda(\psi_{v_1}, \psi_{v_2}), \quad \tilde{T}_\lambda(v_1, v_2) = \frac{1}{\lambda} \tilde{Q}_\lambda(\psi_{v_1}, \psi_{v_2}).$$

The following lemma relates $\sigma_1(H_\lambda)$ with the bilinear form \tilde{Q} .

Lemma 5.2. *For any $\lambda > \Lambda_0$ we have*

$$\sigma_1(H_\lambda) = \inf \left\{ \frac{1}{\lambda} \tilde{Q}(\psi, \psi) : \psi \in E, \int_{\partial B_1} \psi^2 = 1 \right\}.$$

where

$$(5.2) \quad E = \left\{ \psi \in H_G^1(B_1^c), \int_{\partial B_1} \psi = 0, \int_{B_1^c} \psi z_\lambda = 0 \right\}.$$

Moreover this infimum is attained.

Proof. Fix $\lambda > \Lambda_0$. First we prove that

$$(5.3) \quad \gamma_1 := \inf \left\{ Q_\lambda(\psi, \psi) : \psi \in E, \int_{\partial B_1} \psi^2 = 1 \right\}.$$

is achieved. Take $\psi_n \in E$ such that $Q_\lambda(\psi_n, \psi_n) \rightarrow \gamma_1 \in [-\infty, +\infty)$. We show that ψ_n is bounded by contradiction; if $\|\psi_n\| \rightarrow +\infty$, define $\phi_n = \|\psi_n\|^{-1}\psi_n$, and we can assume that up to a subsequence $\phi_n \rightharpoonup \phi_0$. Observe that $\int_{\partial B_1} \phi_n^2 \rightarrow 0$, which implies that $\phi_0 \in H_{0,G}^1(B_1^c)$. We also point out that

$$\int_{B_1^c} u_\lambda^{p-1} \phi_n^2 \rightarrow \int_{B_1^c} u_\lambda^{p-1} \phi_0^2.$$

Now, let us consider two cases:

Case 1: $\phi_0 = 0$. In such case,

$$Q_\lambda(\psi_n, \psi_n) = \|\psi_n\|^2 \int_{B_1^c} \left(\lambda |\nabla \phi_n|^2 + \phi_n^2 - p u_\lambda^{p-1} \phi_n^2 \right) \rightarrow +\infty,$$

which is impossible.

Case 2: $\phi_0 \neq 0$. In this case,

$$\begin{aligned} \liminf_{n \rightarrow +\infty} Q_\lambda(\psi_n, \psi_n) &= \liminf_{n \rightarrow +\infty} \|\psi_n\|^2 \int_{B_1^c} \left(\lambda |\nabla \phi_n|^2 + \phi_n^2 - p u_\lambda^{p-1} \phi_n^2 \right) \\ &\geq \liminf_{n \rightarrow +\infty} \|\psi_n\|^2 Q_\lambda(\phi_0, \phi_0), \end{aligned}$$

but $Q_\lambda(\phi_0, \phi_0) > 0$ since $\lambda > \Lambda_0$. This is again a contradiction.

Therefore, ψ_n is bounded, so up to a subsequence we can pass to the weak limit $\psi_n \rightharpoonup \psi$.

As before,

$$1 = \int_{\partial B_1} \psi_n^2 \rightarrow \int_{\partial B_1} \psi^2, \quad \int_{B_1^c} u_\lambda^{p-1} \psi_n^2 \rightarrow \int_{B_1^c} u_\lambda^{p-1} \psi^2.$$

Then ψ is a minimizer for γ_1 , and in particular $\gamma_1 > -\infty$.

Now we observe that under the constraints $\psi \in E$, and $\int_{\partial B_1} \psi^2 = 1$ we have

$$(5.4) \quad \tilde{Q}_\lambda(\psi, \psi) = Q_\lambda(\psi, \psi) - \lambda(N-1)$$

and in particular, also

$$\inf \left\{ \frac{1}{\lambda} \tilde{Q}_\lambda(\psi, \psi) : \psi \in E, \int_{\partial B_1} \psi^2 = 1 \right\}.$$

is achieved.

Let $\psi \in E$ be the minimizer such that $Q_\lambda(\psi, \psi) = \gamma_1$. By the Lagrange multiplier rule, there exist $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$ so that for any $\rho \in H_G^1(B_1^c)$,

$$\int_{B_1^c} \left(\nabla \psi \cdot \nabla \rho + \psi \rho - p u_\lambda^{p-1} \psi \rho - \alpha_0 \rho z_\lambda \right) = \int_{\partial B_1} \rho (\alpha_1 \psi + \alpha_2).$$

Taking $\rho = \psi$, we conclude that $\alpha_1 = \gamma_1$. Moreover, taking $\rho = z_\lambda$ and $\rho = \kappa_\lambda$ (recall the definitions of z_λ and κ_λ in (3.4) and (4.5)), we conclude that $\alpha_0 = 0$ and $\alpha_2 = 0$, respectively. In other words, ψ is a (weak) solution of the equation:

$$(5.5) \quad \begin{cases} -\lambda \Delta \psi + \psi - p u_\lambda^{p-1} \psi = 0 & \text{in } B_1^c, \\ \frac{\partial \psi}{\partial \nu} = \gamma_1 \psi & \text{on } \partial B_1. \end{cases}$$

By the regularity theory, $\psi \in C_G^{2,\alpha}(B_1^c)$.

Now recall that $T_\lambda(v, v) = \frac{1}{\lambda} Q_\lambda(\psi_v, \psi_v)$. By Lemma 4.2 $\psi_v \in C_G^{2,\alpha}(B_1^c) \cap E$, and then

$$\gamma_1 \leq \inf \left\{ \lambda T_\lambda(v, v) : v \in C_G^{2,\alpha}(\mathbb{S}^{N-1}), \int_{\mathbb{S}^{N-1}} v = 0, \int_{\mathbb{S}^{N-1}} |v|^2 = 1 \right\}$$

Moreover, γ_1 is achieved at a certain $\psi \in C_G^{2,\alpha}(B_1^c)$, which solves (5.5). In particular, denoting $v = \psi|_{\partial B_1}$, we conclude that $\lambda T_\lambda(v, v) = \gamma_1$. Then we have

$$(5.6) \quad \gamma_1 = \inf \left\{ \lambda T_\lambda(v, v) : v \in C_G^{2,\alpha}(\mathbb{S}^{N-1}), \int_{\mathbb{S}^{N-1}} v = 0, \int_{\mathbb{S}^{N-1}} |v|^2 = 1 \right\}.$$

Now we observe that under the constraints $\int_{\mathbb{S}^{N-1}} v = 0$, and $\int_{\mathbb{S}^{N-1}} |v|^2 = 1$ we have

$$\tilde{T}_\lambda(v, v) = T_\lambda(v, v) - (N-1)$$

and then the result follows at once from (5.3), (5.4) and (5.6). \square

The previous lemma leads us to the study of the bilinear form \tilde{Q}_λ . The first key result for our purposes is the following:

Proposition 5.3. *There exists $M > \Lambda_0$ such that for any $\lambda > M$, $\tilde{Q}_\lambda(\psi, \psi) > 0$ for any $\psi \in E \setminus \{0\}$, where E is the subspace defined in (5.2).*

The proof of this proposition is somehow delicate and it is postponed to Section 7. We point out that this is the only point where the assumption $p < \frac{N+2}{N-2}$ (if $N > 2$) is needed.

Let us define now:

$$(5.7) \quad \Lambda^* = \sup\{\lambda > 0 : \tilde{Q}_\lambda(\psi, \psi) < 0 \text{ for some } \psi \in E\}.$$

By Proposition 5.3, $\Lambda^* < +\infty$. Moreover, since $\tilde{Q}_\lambda(\psi, \psi) = Q_\lambda(\psi, \psi)$ for all $\psi \in H_{0,G}^1(B_1^c)$, we have also that $\Lambda^* \geq \Lambda_0$. The last main ingredient to prove Proposition 5.1 is the following:

Lemma 5.4. $\Lambda^* > \Lambda_0$

Proof. It suffices to show that for $\lambda = \Lambda_0$, $\tilde{Q}_\lambda(\psi, \psi) < 0$ for some $\psi \in E$. Reasoning by contradiction, assume that \tilde{Q}_λ is semipositive definite in E . By definition of Λ_0 , there exists $\psi_0 \in H_{0,G}^1(B_1^c)$, with $Q_\lambda(\psi_0, \psi_0) = 0$, and ψ_0 is a solution of (3.3). We have $\psi_0 \in E$ and by our assumptions it is also a minimizer for \tilde{Q}_λ when defined in E . By the Lagrange multiplier rule, there exist α_0 and $\alpha_1 \in \mathbb{R}$ so that for any $\rho \in H_G^1(B_1^c)$,

$$\int_{B_1^c} \left(\nabla \psi \cdot \nabla \rho + \psi \rho - p u_\lambda^{p-1} \psi \rho - \alpha_0 \rho z_\lambda \right) = \alpha_1 \int_{\partial B_1} \rho$$

Taking $\rho = z_\lambda$ and $\rho = \kappa_\lambda$ (recall the definitions of z_λ and κ_λ in (3.4) and (4.5)), we conclude that $\alpha_0 = 0$ and $\alpha_1 = 0$, respectively. In other words, ψ is a (weak) solution of the equation:

$$\begin{cases} -\lambda \Delta \psi_0 + \psi_0 - p u_\lambda^{p-1} \psi_0 = 0 & \text{in } B_1^c, \\ \frac{\partial \psi_0}{\partial \eta} - (N-1) \psi_0 = 0 & \text{on } \partial B_1. \end{cases}$$

Since $\psi_0 = 0$ on ∂B_1 , we have $\frac{\partial \psi_0}{\partial \nu} = 0$ on ∂B_1 . By unique continuation we should have $\psi_0 = 0$, but this is a contradiction. \square

We are now able to give the proof of the main proposition of this section.

Proof. (Proposition 5.1.) Assertion (1) follows immediately from Proposition 5.3. Statements (2) and (3) follow by the definition of Λ^* in (5.7) and Lemma 5.4. \square

6. THE BIFURCATION ARGUMENT

In order to use a local bifurcation result we need to rewrite our problem in a more convenient way. For that, the following lemma will be essential.

Lemma 6.1. *There exists $\varepsilon > 0$ such that for any $\lambda \in (\Lambda^* - \varepsilon, +\infty)$, the operator*

$$\begin{aligned} H_\lambda + Id & : C_{G,m}^{2,\alpha}(\mathbb{S}^{N-1}) \rightarrow C_{G,m}^{1,\alpha}(\mathbb{S}^{N-1}) \\ v & \mapsto H_\lambda(v) + v \end{aligned}$$

is invertible.

Proof. It suffices to prove that the operator

$$v \rightarrow H_\lambda(v) + \sigma v$$

defined in $C_{G,m}^{2,\alpha}(\mathbb{S}^{N-1})$ is invertible for all $\sigma > -\sigma_1(H_\lambda)$. Equivalently, we can prove that the operator

$$v \rightarrow \frac{\partial \psi_v}{\partial \nu} \Big|_{\partial B_1} + \gamma v$$

defined in $C_{G,m}^{2,\alpha}(\mathbb{S}^{N-1})$ is invertible for all $\gamma > -\gamma_1$, where γ_1 is defined in (5.3). Then, define the bilinear form $Q_{\lambda,\gamma} : E \times E \rightarrow \mathbb{R}$ as

$$Q_{\lambda,\gamma}(\psi_1, \psi_2) = Q_\lambda(\psi_1, \psi_2) + \lambda \gamma \int_{\partial B_1} \psi_1 \psi_2,$$

and the bilinear form $T_{\lambda,\gamma} : H_G^{1/2}(\mathbb{S}^{N-1}) \times H_G^{1/2}(\mathbb{S}^{N-1}) \rightarrow \mathbb{R}$ as

$$T_{\lambda,\gamma}(v_1, v_2) = T_\lambda(v_1, v_2) + \gamma \int_{\mathbb{S}^{N-1}} v_1 v_2.$$

Since $\gamma > \gamma_1$, those bilinear forms are positive definite. We claim that they are indeed coercive. Let us start with $Q_{\lambda,\gamma}$, and show that:

$$\alpha := \inf\{Q_{\lambda,\gamma}(\psi, \psi) : \psi \in E, \|\psi\| = 1\} > 0.$$

Take $\psi_n \in E$, $\|\psi_n\| = 1$, $Q_{\lambda,\gamma}(\psi_n, \psi_n) \rightarrow \alpha$, and assume that $\psi_n \rightharpoonup \psi_0$. If the convergence is strong, then the infimum α is attained, which implies that $\alpha > 0$. Otherwise,

$$\begin{aligned} \alpha &= \limsup_{n \rightarrow +\infty} \int_{B_1^c} \lambda |\nabla \psi_n|^2 + \psi_n^2 - p u_\lambda^{p-1} \psi_n^2 + \gamma \int_{\partial B_1} \psi_n^2 \\ &> \int_{B_1^c} \lambda |\nabla \psi_0|^2 + \psi_0^2 - p u_\lambda^{p-1} \psi_0^2 + \gamma \int_{\partial B_1} \psi_0^2 \geq 0. \end{aligned}$$

Hence $Q_{\lambda,\gamma}$ is coercive. Now, observe that:

$$T_{\lambda,\gamma}(v, v) = \int_{\partial B_1} \left[v \frac{\partial \psi_v}{\partial \nu} + \gamma v^2 \right] = \frac{1}{\lambda} Q_{\lambda,\gamma}(\psi_v, \psi_v) \geq c \|\psi_v\|_{H^1(B_1^c)}^2 \geq c' \|v\|_{H^{1/2}(\mathbb{S}^{N-1})}^2,$$

where we have used the trace estimate in the last inequality. Therefore $T_{\lambda,\gamma}$ is coercive. By the Lax-Milgram Theorem, the operator

$$v \rightarrow \frac{\partial \psi_v}{\partial \nu} \Big|_{\partial B_1} + \gamma v$$

is invertible for all $\gamma > -\gamma_1$ in the spaces $H_G^{1/2}(\mathbb{S}^{N-1}) \rightarrow H_G^{-1/2}(\mathbb{S}^{N-1})$. By the regularity theory and the fact that the mean property is preserved, it is invertible also in the spaces $C_{G,m}^{2,\alpha}(\mathbb{S}^{N-1}) \rightarrow C_{G,m}^{1,\alpha}(\mathbb{S}^{N-1})$. \square

According to Proposition 5.1, we can take $\Lambda_1 \in (\Lambda_0, \Lambda^*)$ sufficiently close to Λ^* so that $\sigma_1(H_{\Lambda_1}) < 0$. We define $G : [\Lambda_1, \Lambda_2] \times \mathcal{V} \rightarrow \mathcal{W}$ by

$$(6.1) \quad G(\lambda, v) = F(\lambda, v) + v.$$

Here $\mathcal{V} \subset C_{G,m}^{2,\alpha}(\mathbb{S}^{N-1})$ and $\mathcal{W} \subset C_{G,m}^{1,\alpha}(\mathbb{S}^{N-1})$ are open neighborhoods of the 0 function, and Λ_2 is given by Proposition 5.1. By Lemma 6.1, taking Λ_1 close enough to Λ^* we can assume that $D_v G|_{(\lambda,0)}$ is an isomorphism for all $\lambda \in [\Lambda_1, \Lambda_2]$. By using the Inverse Function Theorem, we can further restrict \mathcal{V} and \mathcal{W} so that $G(\lambda, \cdot)$ is invertible for all $\lambda \in [\Lambda_1, \Lambda_2]$.

Define now $R : [\Lambda_1, \Lambda_2] \times \mathcal{W} \rightarrow \mathcal{W}$, $R(\lambda, w) = w - \tilde{w}$, where \tilde{w} is such that $G(\lambda, \tilde{w}) = w$. We point out that R has the form of identity plus a compact operator. Clearly, $F(\lambda, v) = 0 \Leftrightarrow R(\lambda, v) = 0$. Hence Theorem 2.1 follows if we show local bifurcation of solutions of the equation $R(\lambda, v) = 0$.

We have

$$D_w R|_{(\lambda,0)}(w) = w - D_w G|_{(\lambda,0)}^{-1}(w).$$

Hence

$$(6.2) \quad D_w R|_{(\lambda,0)}(w) = \mu w \Leftrightarrow H_\lambda(w) = \frac{\mu}{1-\mu}w.$$

By the proof of Lemma 6.1, $\mu < 1$ if $\lambda \geq \Lambda_1$. Therefore $D_w R|_{(\lambda,0)}(w)$ has the same number of negative eigenvalues as H_λ .

In this framework we can use a local bifurcation result by Krasnoselskii.

Theorem 6.2. (see for instance [17], [II.3.2]). *Let $F : [a, b] \times \mathcal{Z} \rightarrow X$ be C^1 map defined in $\mathcal{Z} \subset X$ a neighborhood of the origin in the Banach Space X . Assume that F is given by $F(\lambda, x) = x - K(\lambda, x)$ where $K(\lambda, \cdot)$ is a compact map. Assume moreover, that $D_x F|_{(a,0)}$ and $D_x F|_{(b,0)}$ are isomorphisms of X . Denote by $i_{D_x F}(a)$ and $i_{D_x F}(b)$ their indices, that is, the number of negative eigenvalues (counted with algebraic multiplicity). Assume finally that $i_{D_x F}(a) - i_{D_x F}(b)$ is an odd integer. Then every neighborhood of $[a, b] \times \{0\}$ contains solutions of $F(\lambda, x) = 0$, with $\lambda \in (a, b)$, $x \in X$, $x \neq 0$.*

Remark 6.3. The above version of the Krasnoselskii bifurcation result differs slightly from the classical one; usually one imposes the existence of an unique value $\lambda \in (a, b)$ such that the derivative $D_x F|_{(\lambda,0)}$ is degenerate. Under this assumption, one concludes bifurcation at the point $(\lambda, 0)$. The version we give above follows immediately from the proof of the classical Krasnoselskii bifurcation result, which is based on a change of the Leray-Schauder degree of the 0 solution. A drawback of this version is that we cannot identify exactly the bifurcation point.

We now apply Theorem 6.2 to $R(\lambda, w)$. For $\lambda = \Lambda_2$, $i_{D_v R}(\Lambda_2) = 0$ by Proposition 5.1. Therefore we just need to show the validity of the following Lemma.

Lemma 6.4. *$i_{D_v R}(\Lambda_1)$ is odd if Λ_1 is chosen sufficiently close to Λ^* .*

Proof. In view of (6.2), it suffices to prove that H_{Λ^*} has a kernel with odd multiplicity. For any $\psi \in E$, there exist functions $\psi_0, \psi_{k,j}$ defined in $[1, +\infty)$ such that we can write

$$\psi(r, \theta) = \psi_0(r) + \sum_{k=1}^{+\infty} \sum_{j=1}^{\tilde{m}_k} \psi_{k,j}(r) \zeta_{k,j}(\theta),$$

where $r = |x|$, $\theta = \frac{x}{|x|}$ and $\zeta_{k,j}$ are the G -symmetric spherical harmonics (normalized to 1 in the L^2 -norm) with eigenvalue μ_{i_k} of multiplicity m_k . Then the quadratic form $\psi \rightarrow \tilde{Q}_\lambda(\psi, \psi)$ defined in E can be written as

$$(6.3) \quad \tilde{Q}_\lambda(\psi, \psi) = \tilde{Q}_{\lambda,0}(\psi_0) + \sum_{k=1}^{+\infty} \sum_{j=0}^{\tilde{m}_k} \tilde{Q}_{\lambda,k}(\psi_{k,j})$$

where for a function $\phi : (1, +\infty) \rightarrow \mathbb{R}$ we denote

$$\begin{aligned} \tilde{Q}_{\lambda,k}(\phi) &= \int_1^{+\infty} \left(\lambda \phi' \phi' + \phi^2 - p u_\lambda^{p-1} \phi^2 \right) r^{N-1} dr - (N-1) \phi(1)^2 \\ &\quad + \mu_{i_k} \int_1^{+\infty} \phi^2 r^{N-3} dr \end{aligned}$$

choosing by convention that $\mu_{i_0} = 0$. Since $\psi \in E$ we have that $\psi_0(1) = 0$ and that ψ_0 is orthogonal to the function z_λ restricted to the radial variable. By Proposition 3.1 we have $\tilde{Q}_{\lambda,0}(\psi_0) > 0$. For $\lambda = \Lambda^*$, the bilinear form \tilde{Q}_λ is positive semi-definite in $E \times E$, and then from (6.3) we have that all the quadratic forms $\tilde{Q}_{\lambda,k}$ are positive semi-definite. Moreover, it is clear that

$$\tilde{Q}_{\lambda,k_1}(\phi) < \tilde{Q}_{\lambda,k_2}(\phi)$$

if $1 \leq k_1 < k_2$. We know also that there exists a $\psi \in E$ such that $\tilde{Q}_\lambda(\psi, \psi) = 0$. Therefore $\tilde{Q}_{\lambda,1}$ is positive semi-definite, and $\tilde{Q}_{\lambda,k}$ are positive definite for $k > 1$. This implies that the kernel of H_{Λ^*} has dimension equal to m_1 , which is odd by assumption (G). \square

7. PROOF OF PROPOSITIONS 3.4 AND 5.3

Observe that the bilinear form \tilde{Q}_λ defined in (5.1), when restricted to functions in $H_{0,G}^1(B_1^c)$, is nothing but Q_λ (recall (3.5)). Hence Proposition 3.4 follows immediately from Proposition 5.3.

In order to prove Proposition 5.3, we shall consider the problem in the form (3.1); that is, we aim to prove that $\hat{Q}_R : H_G^1(B_R^c) \times H_G^1(B_R^c) \rightarrow \mathbb{R}$,

$$\hat{Q}_R(\psi_1, \psi_2) = \int_{B_R^c} \nabla \psi_1 \cdot \nabla \psi_2 + \psi_1 \psi_2 - p u_R^{p-1} \psi_1 \psi_2 - \frac{N-1}{R} \int_{\partial B_R} \psi_1 \psi_2$$

is positive definite if $R > 0$ is sufficiently small, when ψ_1, ψ_2 belong to the space:

$$E_R = \left\{ \psi \in H_G^1(B_R^c), \int_{\partial B_R} \psi = 0, \int_{B_R^c} \psi z_R = 0 \right\}.$$

Here u_R and z_R stand for

$$u_R(x) = u_\lambda \left(\frac{x}{R} \right), \quad z_R(x) = z_\lambda \left(\frac{x}{R} \right), \quad \lambda = R^{-2}.$$

The strategy of the proof is to make $R = R_n \rightarrow 0$ and assume that \hat{Q}_R is not positive definite in E_R to reach a contradiction. For that, the behavior of u_R, z_R as $R \rightarrow 0$ is needed. This result might be known, but we have not been able to find a specific reference.

Lemma 7.1. *Let u_n be the positive radial solution of (3.1) for $R = R_n \downarrow 0$, and $z_n = \|z_{R_n}\|^{-1} z_{R_n}$. Let us consider those functions extended to \mathbb{R}^N by 0. Then, $u_n \rightarrow U$ and $z_n \rightarrow Z$ in $H^1(\mathbb{R}^N)$, where U is the radial ground state solution of problem:*

$$(7.1) \quad -\Delta U + U = U^p, \quad U > 0, \quad \text{in } \mathbb{R}^N,$$

and Z is the normalized positive eigenfunction corresponding to the negative eigenvalue of the linearized problem, that is,

$$(7.2) \quad -\Delta Z + Z - pU^{p-1}Z = \tau Z, \quad \text{in } \mathbb{R}^N,$$

with $\tau < 0$.

Proof. Let us define the energy functional associated to (3.1):

$$I(u) = \frac{1}{2} \left(\int_{B_{R_n}^c} |\nabla u|^2 + u^2 \right) - \frac{1}{p+1} \int_{B_{R_n}^c} |u|^{p+1}.$$

It is well known that,

$$I(u_n) = \inf \{ \max \{ I(tu) : t > 0 \}, u \in H_{0,r}^1(B_{R_n}^c) \} > 0,$$

see for instance [30]. Since $H_{0,r}^1(B_{R_n}^c) \subset H_{0,r}^1(B_{R_{n+1}}^c)$ (up to extension by 0), then $I(u_n)$ is decreasing in n . In particular, $I(u_n)$ is bounded. Moreover, multiplying (3.1) by u_n

and integrating, we obtain that $DI_{u_n}(u_n) = 0$. What follows is standard (see for instance [4]); first, observe that:

$$(p+1)I(u_n) = (p+1)I(u_n) - DI_{u_n}(u_n) = \frac{p-1}{2}\|u_n\|^2,$$

from which u_n is bounded. Passing to a subsequence, we can assume that $u_n \rightharpoonup u_0$ in H^1 sense. Multiplying (3.1) by $\phi \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$, we conclude that u_0 is a weak solution of the problem:

$$(7.3) \quad -\Delta u_0 + u_0 = u_0^p \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Since u_0 is in the Sobolev class, the singularity is removable. Multiplying (3.1) by u_n and (7.3) by u_0 , we have:

$$\|u_n\|^2 = \int_{\mathbb{R}^N} |u_n|^{p+1}, \quad \|u_0\|^2 = \int_{\mathbb{R}^N} |u_0|^{p+1}.$$

By [29], $u_n \rightarrow u_0$ strongly in L^{p+1} , so that $\|u_n\| \rightarrow \|u_0\|$. From this we conclude that $u_n \rightarrow u_0$ strongly in $H^1(\mathbb{R}^N)$ and u_0 is a nontrivial positive solution of (7.1). By uniqueness ([15]), $u_0 = U$. Regarding z_n , it is a radially symmetric function solving:

$$(7.4) \quad \begin{cases} -\Delta z_n + z_n - pu_n^{p-1}z_n = \tau_n z_n & \text{in } B_{R_n}^c, \\ z_n = 0 & \text{on } \partial B_{R_n}, \end{cases}$$

with $\tau_n < 0$. Since $\|z_n\| = 1$, z_n converges weakly to some z_0 . Multiplying the above equation by z_n we get:

$$(7.5) \quad \int_{\mathbb{R}^N} |\nabla z_n|^2 + (1 - \tau_n)z_n^2 - pu_n^{p-1}z_n^2 = 0.$$

By compact embedding of radial functions ([29]), for instance, we conclude that:

$$(7.6) \quad \int_{\mathbb{R}^N} u_n^{p-1}z_n^2 \rightarrow \int_{\mathbb{R}^N} U^{p-1}z_0^2.$$

This implies in particular that z_0 is not zero. Moreover, $\liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^N} z_n^2 \geq \int_{\mathbb{R}^N} z_0^2$. In particular τ_n is bounded, and we can assume $\tau_n \rightarrow \tau_0 \leq 0$. Then, z_0 is a weak solution of

$$-\Delta z_0 + z_0 - pU^{p-1}z_0 = \tau_0 z_0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Since z_0 belongs to the Sobolev class, the singularity is removable, and it is an entire solution; hence $z_0 = Z$. In particular,

$$\int_{\mathbb{R}^N} |\nabla z_0|^2 + (1 - \tau_0)z_0^2 - pU^{p-1}z_0^2 = 0.$$

This, together with (7.5) and (7.6), allows us to conclude that $z_n \rightarrow z_0$ strongly, concluding the proof. \square

The following lemmas will be of use:

Lemma 7.2. *For any $f \in C_0^\infty(\mathbb{R})$, the following inequality holds:*

$$r^{N-2}f(r)^2 \leq \frac{1}{\lambda} \int_r^{+\infty} f'(s)^2 s^{N-1} ds + (2 - N + \lambda) \int_r^{+\infty} f(s)^2 s^{N-3} ds,$$

where $N \geq 2$, $\lambda > 0$ and $r > 0$.

Proof. Observe that:

$$r^{N-2}f(r)^2 = - \int_r^{+\infty} (2s^{N-2}f(s)f'(s) + (N-2)s^{N-3}f(s)^2) ds.$$

We now estimate the first term in the right hand side by using Cauchy-Schwartz inequality:

$$\begin{aligned} \int_r^{+\infty} 2s^{N-2}|f(s)||f'(s)| ds &= \int_r^{+\infty} 2s^{\frac{N-3}{2}}|f(s)|s^{\frac{N-1}{2}}|f'(s)| ds \\ &\leq \lambda \int_r^{+\infty} s^{N-3}f(s)^2 ds + \frac{1}{\lambda} \int_r^{+\infty} s^{N-1}f'(s)^2 ds. \end{aligned}$$

This lemma follows at once. \square

Lemma 7.3. *Let G be a group of symmetries satisfying (G). Then,*

$$\frac{1}{R} \int_{\partial B_R} \psi(x)^2 ds_x \leq \frac{1}{N} \int_{B_R^c} |\nabla \psi(x)|^2 dx$$

for any $\psi \in H_G^1(B_R^c)$ with $\int_{\partial B_R} \psi(x) ds_x = 0$.

Proof. By density arguments, we can assume that $\psi \in C_0^\infty(B_R^c)$. We decompose it in Fourier series:

$$\psi(r, \theta) = \sum_{i=0}^{\infty} \psi_k(r) \phi_k(\theta),$$

where ϕ_k are the eigenfunctions of $\Delta_{\mathbb{S}^{N-1}}$ under G -symmetry. Observe that $\phi_0(\theta) = 1$ and $\psi_0(\frac{1}{R}) = 0$. Therefore it suffices to prove the inequality for the summands $\psi_k(r)\phi_k(\theta)$, $i \geq 1$. Observe that:

$$\int_{B_R^c} |\nabla(\psi_k(r)\phi_k(\theta))|^2 dr d\theta = \int_R^{+\infty} (\psi_k'(r)^2 r^{N-1} + \mu_{i_k} \psi_k(r)^2 r^{N-3}) dr \int_{\partial B_1} \phi_k(\theta)^2 d\theta.$$

Moreover,

$$\frac{1}{R} \int_{\partial B_R} |\psi_k(r)\phi_k(\theta)|^2 ds_x = R^{N-2} \psi_k(R)^2 \int_{\partial B_1} \phi_k(\theta)^2 d\theta.$$

By assumption (G), $\mu_{i_1} \geq \mu_2 = 2N$. Now it suffices to take $\lambda = N$ in Lemma 7.2 to conclude. \square

We are now able to prove Proposition 5.3.

Proof. (Proposition 5.3) Take $R = R_n \rightarrow 0$, denote $B_n = B_{R_n}$, $u_n = u_{R_n}$ and $z_n = z_{R_n}$, and define:

$$\chi_n = \inf \left\{ \hat{Q}_R(\psi, \psi) : \psi \in H_G^1(B_n^c), \int_{\partial B_n} \psi = 0, \int_{B_n^c} \psi z_n = 0, \int_{B_n^c} |\psi|^2 = 1 \right\}.$$

Assume, by contradiction, that $\chi_n \leq 0$. The proof will be divided in several steps:

Step 1: We show here that χ_n is attained.

Take ψ_k a minimizing sequence for χ_n . If ψ_k is unbounded in the H^1 norm, define $\phi_k = \|\psi_k\|^{-1} \psi_k$. Then,

$$0 \leq \int_{B_n^c} |\nabla \phi_k|^2 + (1 - \chi_n) |\phi_k|^2 - p u_n^{p-1} |\phi_k|^2 - \frac{N-1}{R_n} \int_{\partial B_n} |\phi_k|^2 \rightarrow 0.$$

But $\phi_k \rightarrow 0$ in the L^2 norm, so that $\int_{B_n^c} u_n^{p-1} |\phi_k|^2 \rightarrow 0$ as $k \rightarrow +\infty$. Moreover, $\phi_k \rightarrow 0$ in H^1 , so $\int_{\partial B_n} |\phi_k|^2 \rightarrow 0$, yielding a contradiction. Hence ψ_k is bounded in H^1 , so that we can assume that $\psi_k \rightarrow \psi$. Then,

$$\int_{B_n^c} u_n^{p-1} |\psi_k|^2 \rightarrow \int_{B_n^c} u_n^{p-1} |\psi|^2, \quad \int_{\partial B_n} |\psi_k|^2 \rightarrow \int_{\partial B_n} |\psi|^2.$$

Above we have used the fact that u_n decays to 0 at infinity and the fact that the embedding $H^1(B_n^c) \hookrightarrow L^2(\partial B_n)$ is compact. We conclude that the convergence is strong and that ψ is a minimizer for χ_n .

Step 2: We pass now to the limit.

Let us denote by ψ_n the minimizer of χ_n renormalized with respect to the H^1 norm. Observe that ψ_n is a solution of the equation:

$$(7.7) \quad -\Delta \psi_n + \psi_n - p u_n^{p-1} \psi = \chi_n \psi \quad \text{in } B_n^c.$$

Moreover,

$$(7.8) \quad \int_{B_n^c} |\nabla \psi_n|^2 + (1 - \chi_n) |\psi_n|^2 - p u_n^{p-1} |\psi_n|^2 - \frac{N-1}{R_n} \int_{\partial B_n} |\psi_n|^2 = 0.$$

By a Cantor diagonal process, $\psi_n \rightarrow \psi_0 \in H_G^1(B_r^c)$ for any $r > 0$, where $\psi_0 \in H^1(\mathbb{R}^N)$ (recall that $H_0^1(\mathbb{R}^N \setminus \{0\}) = H^1(\mathbb{R}^N)$).

Step 3: We show here that

$$\int_{B_n^c} u_n^{p-1} |\psi_n|^2 \rightarrow \int_{\mathbb{R}^N} U^{p-1} |\psi_0|^2.$$

Indeed, given any $\varepsilon > 0$, $\psi_n \rightharpoonup \psi_0$ in $H^1(B_\varepsilon^c)$, which implies that $\psi_n^2 \rightharpoonup \psi_0^2$ in $L^{\frac{p+1}{2}}$. Moreover $u_n \rightarrow U$ in $H^1(\mathbb{R}^N)$, hence:

$$\int_{B_\varepsilon^c} u_n^{p-1} |\psi_n|^2 \rightarrow \int_{B_\varepsilon^c} U^{p-1} |\psi_0|^2.$$

Apply now the Hölder inequality:

$$\int_{B_\varepsilon \setminus B_n} u_n^{p-1} |\psi_n|^2 \leq \left(\int_{B_\varepsilon} u_n^{p+1} \right)^{\frac{p-1}{p+1}} \left(\int_{B_n^c} |\psi_n|^{p+1} \right)^{\frac{2}{p+1}}.$$

Recall that $u_n \rightarrow U$ in L^{p+1} so that

$$\int_{B_\varepsilon} |u_n|^{p+1} \leq \int_{\mathbb{R}^N} |u_n - U|^{p+1} + \int_{B_\varepsilon} |U|^{p+1} \leq C\varepsilon^N,$$

by choosing sufficiently large n . Since ε is arbitrary, we conclude the proof of step 2.

Step 4: We get now the desired contradiction.

By Lemma 7.3,

$$\frac{N-1}{R_n} \int_{\partial B_n} |\psi_n|^2 \leq \frac{N-1}{N} \int_{B_n^c} |\nabla \psi_n|^2.$$

This, together with Step 2 and (7.8), implies that $\psi_0 \neq 0$. In particular,

$$\liminf_{n \rightarrow +\infty} \int_{B_n^c} |\psi_n|^2 \geq \int_{B_n^c} |\psi_0|^2 > 0.$$

Plugging this information in (7.8), and taking into account Lemma 7.3, we conclude that χ_n is bounded. Let us assume that $\chi_n \rightarrow \chi_0 \leq 0$. By (7.7), ψ_0 is a nontrivial weak solution of the problem:

$$-\Delta \psi_0 + \psi_0 - pU^{p-1}\psi_0 = \chi_0 \psi_0, \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Since $\psi_0 \in H^1(\mathbb{R}^N)$, the singularity is removable and ψ is a weak solution in the whole \mathbb{R}^N . Since ψ_0 is G -symmetric, the only possibility is $\psi_0 = kZ$, $k \neq 0$ (see [15]). Observe now that $\int_{B_n^c} \psi_n z_n = 0$. By the same arguments as in Step 2, we conclude that

$$\int_{B_n^c} \psi_n z_n \rightarrow \int_{\mathbb{R}^N} \psi_0 Z,$$

which yields the desired contradiction. \square

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