ON REMOTALITY FOR CONVEX SETS IN BANACH SPACES

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Abstract. We show that every infinite dimensional Banach space has a closed and bounded convex set that is not remotal. This settles a problem raised by Sababheh and Khalil in [8].

1. Introduction

Let $X$ be a real Banach space and let $E \subset X$ be a bounded set. We write $\text{ext}(E)$ for the set of extreme points of $E$ and $\text{co}(E)$ for the closed (in the norm topology) convex hull of $E$. If $\tau$ is a locally convex topology in $X$, we will write $\text{co}_\tau(E)$ to denote the $\tau$-closed convex hull of $E$. We denote by $B_X$ the closed unit ball of $X$.

The set $E$ is said to be remotal from a point $x \in X$, if there exists a point $e_0 \in E$ such that $D(x, E) = \sup\{\|x - e\| : e \in E\} = \|x - e_0\|$. The point $e_0$ is called a farthest point of $E$ from $x$. $E$ is said to be remotal (densely remotal) if it is remotal from all (on a dense set) $x \in X$. Let $F(x, E) = \{e \in E : D(x, E) = \|x - e\|\}$. In general this set can be empty. A well known result of Lau ([5]) says that any weakly compact set is densely remotal. It seems to be open, the question of whether every infinite dimensional Banach space has a closed and bounded convex set that is not remotal. This question was actually raised in [8] and some partial positive answers were given in [8] and [7] in the case of reflexive Banach spaces and Banach spaces that fail the Schur property. The aim of this note is to give a positive answer to this question. We follow the notation and terminology of [8] and [7].

Let us outline the content of this paper. Let $X$ be an infinite dimensional Banach space and let $X^*$ be its topological dual. Using a classical integral representation theorem, we first show that $X^*$ has a weak*-compact convex set $K$ that is not remotal. This should be compared with [2, Proposition 1] where the authors exhibited a weak*-compact convex set $C \subset \ell^1$ that has no farthest points. To prove the general result, we use a stronger form of integral representation theorem for closed convex bounded sets with the Radon-Nikodym property (RNP for short) due to Edgar ([4], see [6, Theorem 16.12]). Let $E \subset X$ be a weakly closed and bounded set. An interesting problem is that open is to determine conditions on $\text{co}(E)$ so that $\text{co}(E)$ is remotal from $x$ implies that $E$ is remotal from $x$. We will give an example showing that $E$ being norm closed in a reflexive space is not enough for the validity of Theorem A in [8].

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2. Main result

We first prove a weak*-version of [8, Theorem A]. In order to produce a weak*-compact convex non-remotal set, it is enough to show that if \( E \) is a weak*-compact set having no vector of maximum length, then the same is true of \( \overline{E}^{\text{weak}^*} \) (weak*-closed convex hull). For a compact convex set \( K \subset X^* \) and for a probability measure \( \mu \) on \( K \), let \( \gamma(\mu) \in K \) denote its resultant (or weak integral) with the property

\[
[\gamma(\mu)](x) = \int_K k(x) \, d\mu(x) \quad (x \in X).
\]

We refer to [3, 6] for the results on integral representations we use here.

**Theorem 1.** Let \( X \) be an infinite dimensional Banach space. Let \( E \subset X^* \) be a weak*-closed and bounded set having no vector of maximum length. Then the weak*-closed convex hull \( K \) of \( E \) has no vector of maximum length. Equivalently, if \( E \) is not remotal from a point \( x \in X \), then neither is \( K \).

**Proof.** Let \( M = D(0, E) = \sup\{\|e\| : e \in E\} = \sup\{\|k\| : k \in K\} \). Suppose that there exists \( x_0^* \in K \) such that \( \|x_0^*\| = M \). Let \( \mu \) be a probability measure on \( K \) with \( \mu(E) = 1 \) and such that \( \gamma(\mu) = x_0^* \) (see [6, Proposition 1.1]). We fix \( \varepsilon > 0 \) and take \( x \in X \) such that \( \|x\| = 1 \) and \( x_0^*(x) > M - \varepsilon \). Now,

\[
M - \varepsilon < x_0^*(x) = \int_K x^*(x) \, d\mu(x^*) = \int_E x^*(x) \, d\mu(x^*) \leq \int_E \|x^*\| \, d\mu \leq M.
\]

Letting \( \varepsilon \downarrow 0 \), we get that \( \int_E \|x^*\| \, d\mu(x^*) = M \) and so, \( M = \|e\| \) \( \mu \)-a.e. Hence \( M = \|e\| \) for some \( e \in E \). A contradiction. The last part of the statement is equivalent to the first one just by translation. \( \square \)

**Corollary 2.** Let \( X \) be an infinite dimensional Banach space. Then there exists a weak*-compact convex set \( K \subset X^* \) that is not remotal.

**Proof.** Since \( X \) is infinite dimensional, by the well-known Josefson-Nissenzweig theorem (see [3, p. 219]), there exists a sequence \( \{x_n^*\}_{n \geq 1} \) of unit vectors such that \( x_n^* \to 0 \) in the weak*-topology. Consider the set

\[
E = \left\{ \frac{n}{n+1} x_n^* : n \in \mathbb{N} \right\} \cup \{0\},
\]

which is clearly a weak*-compact set having no vector of maximum length. Thus, by the above theorem, the weak*-closed convex hull \( K \) of \( E \) does not have vectors of maximum length, so \( K \) is not remotal from 0. \( \square \)

**Remark 3.** The arguments in Theorem 1 and Corollary 2 also work in the case of a weakly compact set \( E \) and its closed convex hull \( K = \overline{E} \) (actually, the argument simplifies in this case and \( \varepsilon \) is not necessary). Thus, in a Banach space \( X \) that fail the Schur property, by taking a sequence \( \{x_n\}_{n \geq 1} \) of unit vectors which converges to 0 in the weak topology, we get that the set

\[
K = \overline{\left\{ \left\{ \frac{n}{n+1} x_n : n \in \mathbb{N} \right\} \cup \{0\} \right\}}
\]
is nonremotal from 0 (alternatively, the set does not have any vector of maximal length). This gives an alternative proof of the main result from [7].

**Remark 4.** From the above arguments it is easy to see that for a weak*-compact set $E \subset X^*$ and for any $x^* \in X^*$, if the set $F(x^*, K)$ of farthest points in the weak*-closed convex hull $K$ of $E$ to $x^*$ is non-empty, then it has a point of $E$. However the method of proof in [7] has the advantage that it shows that there is an extreme point of $K$ in $F(x^*, K)$. Then by Milman’s theorem [3, p. 151], such an extreme point is also in $E$.

The following easy example shows that the hypothesis of weak*-closedness can not be omitted on the set $E$ in Theorem 1 (weak-compactness in the case of Remark 3).

**Example 5.** Let $\{e_n\}_{n \geq 1}$ denote the canonical vector basis in $\ell^2$. Let $X = K \oplus_{\infty} \ell^2$, where $K = \mathbb{R}$ or $K = \mathbb{C}$ is the base field and $\oplus_{\infty}$ means the $\ell^\infty$-direct sum. Consider the set

$$ E = \left\{ \begin{pmatrix} n \\ n+1 \end{pmatrix} e_n : n \in \mathbb{N} \right\}. $$

Then $E$ is a norm closed set which is not remotal from 0. Since $\overline{\text{co}}(E) = \overline{\text{co}}^\text{weak}(E)$ by Mazur’s theorem and $\{e_n\}_{n \geq 1} \rightharpoonup 0$ in the weak topology, $(1, 0) \in \overline{\text{co}}(E)$ and so, $\overline{\text{co}}(E)$ is remotal from 0.

**Remark 6.** Let $X$ be a Banach space and let $E \subset X$ be a weakly closed and bounded set. We do not know if remotality of $K = \overline{\text{co}}(E)$ from a point always implies that of $E$. Since any strongly exposed point of $K$ clearly lies in $E$, the answer is affirmative if the farthest point in $K$ is actually strongly exposed. We may also ask whether the above question has a positive answer for RNP sets (see [1, § 3] for these concepts).

We are now able to present the main result of our paper.

**Theorem 7.** Let $X$ be an infinite dimensional Banach space. Then, there exists a closed and bounded convex set $K$ that is not remotal.

**Proof.** As before, we will construct a closed and bounded set $E$ which is not remotal from 0 and show that $K = \overline{\text{co}}(E)$ is also not remotal from 0.

In view of Remark 3 (or of [7]), we may assume without loss of generality that $X$ has the Schur property. Since $X$ is infinite dimensional, by Rosenthal’s $\ell_1$ Theorem (see [3, § XI]), $X$ contains an isomorphic copy of $\ell^1$. Let $\| \cdot \|$ denote the norm on $X^*$. Now we will be done if we can construct in every Banach space $Y = (\ell_1, \| \cdot \|)$ isomorphic to $\ell_1$, a closed convex bounded set $K \subseteq Y$ which is not remotal from 0. Let us write $\tau$ for the weak*-topology of $\ell_1$ as dual of $c_0$ inherited in $Y$. This is a locally convex topology on $Y$ weaker than the norm topology and any $\tau$-closed norm-bounded set is compact in this topology. Observe now that $\| \cdot \|$ is not necessarily weak*-lower semi-continuous (i.e. $Y$ may not be a dual space) so, on the one hand, Corollary 2 does not apply and, on the other hand, $B_Y$ may not be $\tau$-closed.

Let $\{e_n\}_{n \geq 1}$ be the canonical basis of $\ell^1$. Consider the set

$$ E = \left\{ \frac{n}{n+1} e_n \|e_n\| : n \in \mathbb{N} \right\} \cup \{0\} \subseteq B_Y $$
which is \( \tau \)-compact since \( \{e_n\}_{n \geq 1} \) \( \tau \)-converges to 0 and \( \| \cdot \| \) is equivalent to the usual norm of \( \ell_1 \). We consider the set \( K = \overline{co}\, (E) \subseteq Y \), which is \( \tau \)-compact since it is \( \tau \)-closed and norm-bounded (indeed, \( E \) is contained in the \( \tau \)-closed set \( MB_{\ell_1} \) for some \( M > 0 \), so \( K \subset MB_{\ell_1} \)).

**Claim.** \( K \subseteq B_Y \). Indeed, since \( \ell_1 \) (and so \( Y \)) has the RNP, \( K \) is a set with the RNP. Therefore, we have \( K = \overline{co}\, (\text{ext}(K)) \) (closure in norm, see [1, §3]). As \( K \) and \( E \) are \( \tau \)-compact, Milman’s theorem gives us that \( \text{ext}(K) \subseteq E \) (see [3, p. 151]). Therefore, we have

\[
K = \overline{co}\, (\text{ext}(K)) \subseteq \overline{co}(E) \subseteq \overline{co}^{\tau}(E) = K,
\]

so \( K = \overline{co}(E) \subseteq B_Y \) as claimed.

Suppose \( K \) is nonremotal from \( 0 \) in \( Y \). As \( D(0, E) = 1 \) and \( K \subseteq B_Y \), we also have \( D(0, K) = 1 \). Therefore, there is a vector \( y_0 \in K \) with \( \| y_0 \| = 1 \), and we may pick a functional \( y_0^* \in Y^* \) with

\[
\| y_0^* \| = 1 \quad \text{and} \quad y_0^*(y_0) = 1.
\]

As \( K \) is a separable closed convex bounded set with the RNP, Edgar’s integral representation theorem ([14], see [6, Theorem 16.12]), gives us that there exists a probability measure \( \mu \) on \( K \) with \( \mu(\text{ext}(K)) = 1 \) (so \( \mu(E) = 1 \)) such that

\[
1 = y_0^*(y_0) = \int_K y_0^*(y) \, d\mu(y) = \int_E y_0^*(y) \, d\mu(y) \leq \int_E \|y\| \, d\mu(y) \leq 1.
\]

Therefore, \( \|y\| = 1 \) \( \mu \)-a.e. in \( E \), which is clearly false. Thus we get a contradiction and \( K \) is nonremotal from 0.

Since remotality from 0 is equivalent to having a vector of maximal norm, we get the following corollary.

**Corollary 8.** Let \( X \) be an infinite-dimensional Banach space. Then there is a closed convex set \( K \) contained in the open unit ball of \( X \) such that \( \sup\{\|x\| : x \in K\} = 1 \).

**Remark 9.** Similar to Remark 4 (see also Remark 6), let us note that for a separable weakly closed and bounded set \( E \) such that its closed convex hull \( K \) has the RNP, our arguments show that if \( F(x, K) \neq \emptyset \) then it has an extreme point of \( K \).

**Remark 10.** Going into the details of the proofs of Remark 3 and Theorem 7, one realizes that for every infinite-dimensional Banach space \( X \), there is a locally convex Hausdorff topology \( \tau \), which is weaker than the norm topology and such that there is a \( \tau \)-compact convex set \( K \) which is not remotal (from 0). Indeed, if \( X \) does not have the Schur property, then the set \( K \) is actually weak compact. Otherwise, \( X \) contains a subspace \( Y \) isomorphic to \( \ell_1 \), and the set \( K \subseteq Y \) is compact for the topology \( \tau' \) of \( Y \) which it inherits from the weak* topology of \( \ell_1 \) as dual of \( c_0 \). Since we may extend the topology \( \tau' \) of \( Y \) to a locally convex Hausdorff topology \( \tau \) of \( X \) (still weaker than the norm topology of \( X \)), we get that \( K \) is \( \tau \)-compact, as desired.

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References


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