The second numerical index of Banach spaces

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Where is contained?

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On a second numerical index for Banach spaces
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Introduction

Section 1

1 Introduction
Some notation

\( X \) real or complex Banach space

\( B_X \) closed unit ball

\( S_X \) unit sphere

\( X^* \) topological dual

\( \mathcal{L}(X) \) Banach algebra of all bounded linear operators from \( X \) to \( X \)

\( \mathcal{L}(X,Y) \) Banach space of all bounded linear operators from \( X \) to \( Y \)

\( \Pi(X) = \{(x,x^*) \in S_X \times S_{X^*} : x^*(x) = 1\} \)
Definitions

Numerical range and numerical radius (Bauer, Lumer, early 60’s)

\( X \) Banach space, \( T \in \mathcal{L}(X) \)

\[
V(T) = \{ x^*(Tx) : (x, x^*) \in \Pi(X) \}
\]

\[
v(T) = \sup\{|\lambda| : \lambda \in V(T)\}
\]

\[
v(T) = \sup\{|x^*(Tx)| : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}
\]

Obviously one has \( v(T) \leq \|T\| \)

Numerical index (Lumer, 1968)

\( X \) Banach space

\[
n(X) = \inf\{v(T) : T \in S_{\mathcal{L}(X)}\} = \max\{k \geq 0 : k\|T\| \leq v(T)\}
\]

- \( 0 \leq n(X) \leq 1 \);
- \( v \) and \( \| \cdot \| \) are equivalent norms iff \( n(X) > 0 \);
Some known results

- \( n(C(K)) = n(L_1(\mu)) = 1 \) (Duncan-McGregor-Pryce-White, 1970)
- \( n(X) = 1 \) iff \( \max_{|w|=1} \|\text{Id} + wT\| = 1 + \|T\| \forall T \in \mathcal{L}(X) \) (Duncan et al., 1970)
- \( \{n(X) : X \text{ is a complex Banach space}\} = [1/e, 1] \)
  \( \{n(X) : X \text{ is a real Banach space}\} = [0, 1] \) (Duncan et al., 1970)
- \( n(L_p(\mu)) = \inf\{n(\ell^m_p) : m \in \mathbb{N}\} \) for \( \mu \) so that \( \dim(L_p(\mu)) = \infty \) (Aksoy-Eddari-Khamsi, 2007)
- \( n(L_p(\mu)) > 0 \) for \( p \neq 2 \) (Martín-Merí-Popov, 2011)

Let \( \{X_\lambda : \lambda \in \Lambda\} \) be an arbitrary family of Banach spaces. Then

\[
\begin{align*}
n\left(\bigoplus_{\lambda \in \Lambda} X_\lambda\right)_{c_0} &= n\left(\bigoplus_{\lambda \in \Lambda} X_\lambda\right)_{\ell_1} = n\left(\bigoplus_{\lambda \in \Lambda} X_\lambda\right)_{\ell_\infty} = \inf\{n(X_\lambda) : \lambda \in \Lambda\} \\
n\left(\bigoplus_{\lambda \in \Lambda} X_\lambda\right)_{\ell_p} &\leq \inf\{n(X_\lambda) : \lambda \in \Lambda\}
\end{align*}
\]

(Martín-Payá, 2000)
Some known results

- Let $X$ be a Banach space, $L$ locally compact Hausdorff, $K$ compact Hausdorff, $\Omega$ completely regular Hausdorff, and $\mu$ positive measure. Then
  \[
  n\left(C_0(L, X)\right) = n\left(C_b(\Omega, X)\right) = n\left(L_1(\mu, X)\right) = n(X) \quad \text{(Martín-Payá, 2000)}
  \]
  \[
  n\left(L_\infty(\mu, X)\right) = n(X) \quad \text{(Martín-Villena, 2003)}
  \]
  \[
  n\left(C_w(K, X)\right) = n(X) \quad \text{(López-Martín-Merí, 2007)}
  \]

- $n(\cdot)$ is continuous with respect to the Banach-Mazur distance (Finet-Martín-Payá, 2003)

- $n(X^*) \leq n(X)$ holds for every Banach space $X$ and the inequality can be strict (Boyko-Kadets-Martín-Werner, 2007)

- $X$ real with $\dim(X) = \infty$ and $n(X) = 1 \implies \ell_1 \subset X^*$ (Avilés-Kadets-Martín-Merí-Shepelska, 2010)
Spear operators (Ardalani, 2014)

\( \mathcal{L}(X, Y) \) is a spear operator iff
\[
\max_{|w|=1} \|G + wT\| = 1 + \|T\| \quad \forall T \in \mathcal{L}(X, Y).
\]

**Some examples:**
- \( \text{Id}_X \) when \( n(X) = 1 \),
- the Fourier transform,
- the inclusion \( A(\mathbb{D}) \hookrightarrow C(\mathbb{T}) \).

**Some consequences:**
- \( G : X \rightarrow Y \) real spear with infinite rank \( \Rightarrow X^* \supset \ell_1 \),
- \( G : X \rightarrow Y \) real spear, infinite rank, \( X \) RNP \( \Rightarrow Y \supset c_0 \) or \( Y \supset \ell_1 \),
- \( G : X \rightarrow Y \) spear, \( \dim(X) > 1 \)
  \( \Rightarrow X^* \) not strictly convex nor smooth, \( B_X \) does not contain WLUR points.

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**References**

V. Kadets, M. Martín, J. Merí, A. Pérez
Spear operators between Banach spaces
Lecture Notes of Mathematics 2205 (2018)
Possible extension II: numerical index with respect to an operator

**Numerical range and radius with respect to $G$ (Ardalani, 2014)**

$G \in \mathcal{L}(X, Y)$ with $\|G\| = 1$, $T \in \mathcal{L}(X, Y)$

$$V_G(T) = \bigcap_{\delta > 0} \{ y^*(Tx) : x \in S_X, y^* \in S_{Y^*}, \Re y^*(Gx) > 1 - \delta \}$$

$$v_G(T) = \sup \{ |\lambda| : \lambda \in V_G(T) \}$$

★ $v_{\text{Id}}(T) = v(T) \ \forall T \in \mathcal{L}(X)$

**Numerical index with respect to $G$**

$$n_G(X, Y) = \inf \{ v_G(T) : T \in S_{\mathcal{L}(X, Y)} \} = \max \{ k \geq 0 : k\|T\| \leq v_G(T) \}$$

★ $n_{\text{Id}}(X, X) = n(X)$. 

Work in progress...
Section 2

2. The second numerical index

- Relationship with absolute sums
- Spaces with absolute norm and $n'(X) = 1$
- Vector valued spaces
- Duality
The base field does matter for the numerical index

(Bohnenblust-Karlin, Glickfeld-1970)

\[ n(X) \geq \frac{1}{e} \text{ for every complex Banach space } X \]

Examples in the real case

- \( n(H) = 0 \) for \( H \) real Hilbert space with \( \dim(H) \geq 2 \)
- \( n(X_\mathbb{R}) = 0 \) for \( X \) complex Banach space
- But there is \( X \) such that \( n(X) = 0 \) and \( v \) is a norm \hspace{1cm} (Martín-Payá, 2000)

In the first two cases there is \( T \in \mathcal{L}(X) \setminus \{0\} \) with \( v(T) = 0 \):

- \( (x_1, x_2, x_3, \ldots) \mapsto (-x_2, x_1, 0, \ldots) \),
- \( x \mapsto ix \)

Observation

\[ v(T) = 0 \iff \exp(\rho T) \text{ is an onto isometry for every } \rho \in \mathbb{R} \]
The second numerical index

Lie Algebra

Let $X$ be a real Banach space.

$$\mathcal{Z}(X) := \{ S \in \mathcal{L}(X) : v(S) = 0 \}$$

(it is a closed subspace of $\mathcal{L}(X)$)

Then, for all $T + \mathcal{Z}(X) \in \mathcal{L}(X)/\mathcal{Z}(X)$ we may consider two norms:

$$\| T + \mathcal{Z}(X) \| := \inf \{ \| T - S \| : S \in \mathcal{Z}(X) \}$$

$$v(T + \mathcal{Z}(X)) := \inf \{ v(T - S) : S \in \mathcal{Z}(X) \} = v(T)$$

It is immediate that $v(T) \leq \| T + \mathcal{Z}(X) \|$ for every $T \in \mathcal{L}(X)$

Second numerical index

$$n'(X) := \inf \{ v(T) : T \in \mathcal{L}(X), \| T + \mathcal{Z}(X) \| = 1 \}$$

$$= \max \{ k \geq 0 : k\| T + \mathcal{Z}(X) \| \leq v(T) \ \forall T \in \mathcal{L}(X) \}$$

Obviously $0 \leq n'(X) \leq 1$
The second numerical index

**Observations**

- If $\mathcal{Z}(X) = \{0\}$ (in particular if $n(X) > 0$), then $n'(X) = n(X)$
- $n(X) \leq n'(X)$
- On $\mathcal{L}(X)/\mathcal{Z}(X)$, both $\| \cdot + \mathcal{Z}(X) \|$ and $v(\cdot)$ are norms

**Some examples**

- $n'(X) > 0$ when $X$ is finite-dimensional
- But there is a Banach space $X$ with $n(X) = 0$ and $n'(X) = 0$

**Further observation**

There is no third numerical index
Main example

**Theorem**

Let $H$ be a Hilbert space. Then, $n'(H) = 1$.

**Proof**

Fixed $T \in \mathcal{L}(H)$ we have to show that

$$v(T) = \|T + \mathcal{Z}(H)\| = \left\| \frac{T + T^*}{2} \right\|$$

**Facts**

- $S \in \mathcal{Z}(H) \iff S = -S^*$
- $T = T^* \implies v(T) = \|T\|$
Absolute norm on $\mathbb{R}^m$ and absolute sum of Banach spaces

**Absolute norm**

A norm $\| \cdot \|$ on $\mathbb{R}^m$ is absolute if

- $\|(a_1, \ldots, a_m)\| = \|(|a_1|, \ldots, |a_m|)\|$ for every $(a_1, \ldots, a_m) \in \mathbb{R}^m$.
- $\|e_k\| = 1$ for every $k = 1, \ldots, m$ where $e_k = (0, \ldots, 0, 1_k, 0, \ldots, 0)$.

**Absolute sum**

Let $E$ be $\mathbb{R}^m$ endowed with an absolute norm. We write $[X_1 \oplus \cdots \oplus X_m]_E$ for the $E$-sum of the Banach spaces $X_1, \ldots, X_m$. That is, the space $X_1 \times \cdots \times X_m$ endowed with the complete norm $\|(x_1, \ldots, x_m)\| = \|(\|x_1\|, \ldots, \|x_m\|)\|_E$.

When $E$ is $\mathbb{R}^2$ endowed with an absolute norm $\| \cdot \|_a$ we just write $X_1 \oplus_a X_2 = [X_1 \oplus X_2]_E$. 
Relationship of $n'$ with absolute sums

**Proposition**

Let $X = X_1 \oplus_a X_2$, where $\oplus_a \neq \oplus_2$ is an absolute sum. Then,

$$n'(X) \leq \min\{n'(X_1), n'(X_2)\}.$$

**Corollary**

Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of Banach spaces, $1 \leq p \leq \infty$ with $p \neq 2$. Then

$$n'(\left[ \bigoplus_{\lambda \in \Lambda} X_\lambda \right]_{\ell_p}) \leq \inf\{n'(X_\lambda) : \lambda \in \Lambda\}.$$

**Examples (equality does not hold)**

$$n'(\ell_2^2 \oplus_\infty \mathbb{R}) \leq \frac{\sqrt{3}}{2} < 1 \quad \text{and} \quad n'(\ell_2^2 \oplus_1 \mathbb{R}) \leq \frac{\sqrt{3}}{2} < 1$$
Relationship of $n'$ with absolute sums

**Proposition**

Let $X_1$, $X_2$ be Banach spaces and write $X = X_1 \oplus_\infty X_2$ or $X = X_1 \oplus_1 X_2$.

- If $n(X_1) > 0$ and $n(X_2) > 0$, then $n'(X) = n(X) = \min\{n(X_1), n(X_2)\}$.
- If $n(X_1) > 0$ and $n(X_2) = 0$, then $n'(X) \geq \min\{n(X_1), \frac{n'(X_2)}{n'(X_2)+1}\}$.
- If $n(X_1) = 0$ and $n(X_2) = 0$, then

$$n'(X) \geq \min\left\{\frac{n'(X_1)}{n'(X_1)+1}, \frac{n'(X_2)}{n'(X_2)+1}\right\}.$$

**Example**

$$\frac{1}{2} \leq n'(\ell^2_2 \oplus_\infty \mathbb{R}) \leq \frac{\sqrt{3}}{2} \quad \text{and} \quad \frac{1}{2} \leq n'(\ell^2_2 \oplus_1 \mathbb{R}) \leq \frac{\sqrt{3}}{2}$$
A family of examples

Example
For every $\theta \in (0, 1/2]$, there is a four-dimensional Banach space $X_\theta$ such that $n(X_\theta) = 0$ and $n'(X_\theta) = \theta$.

Let $Y_\theta$ be a two-dimensional space with $n(Y_\theta) = \theta$ and take $X_\theta = Y_\theta \oplus \ell_2^\infty$. Then:

- $n(X_\theta) \leq n(\ell_2^2) = 0$
- $n'(X_\theta) \leq n'(Y_\theta) = n(Y_\theta) = \theta$
- $n'(X_\theta) \geq \min \left\{ n(Y_\theta), \frac{n'(\ell_2^2)}{n'(\ell_2^2) + 1} \right\} = \min \{ \theta, \frac{1}{2} \} = \theta$

More examples (low dimensions)
- $\dim(X) = 2$, $n(X) = 0 \implies n'(X) = 1$,
- $\{n'(X) : n(X) = 0, \dim(X) = 3\} \supset [1/e, 1/2]$ and it is NOT an interval,
- $\{n'(X) : n(X) = 0, \dim(X) = 4\} \supset (0, 1/2]$. 
$n'$ is not continuous with respect Banach-Mazur distance

Example

For $1 < p < \infty$, let $X_p = \ell_p^2 \oplus_p \ell_2^2$ (observe that $n(X_p) = 0$ for every $p$).

- Then $n'(X_p) \leq n'(\ell_p^2) = n(\ell_p^2)$ for $p \neq 2$
- Therefore $\lim_{p \to 2} n'(X_p) \leq \lim_{p \to 2} n(\ell_p) = 0$
- On the other hand $n'(X_2) = n'(\ell_2^4) = 1$

Another example

For $1 < p < \infty$, let $X_p = \ell_p^2 \oplus_1 \ell_2^2$ (observe that $n(X_p) = 0$ for every $p$).

- Then $n'(X_p) \leq n'(\ell_p^2) = n(\ell_p^2)$ for $p \neq 2$
- Therefore $\lim_{p \to 2} n'(X_p) \leq \lim_{p \to 2} n(\ell_p) = 0$
- On the other hand $\frac{1}{2} \leq n'(X_2) < 1$

Observation

Continuity of $n'(\cdot)$ holds if $\mathcal{Z}(X)$ does not change
Relationship of the indices with absolute sums (revisited)

Let $E$ be $\mathbb{R}^m$ endowed with an absolute norm, let $X_1, \ldots, X_m$ be Banach spaces and $X = [X_1 \oplus \cdots \oplus X_m]_E$.

**Diagonal operator**

$S \in \mathcal{L}(X)$ is **diagonal** if $P_k S I_j = 0$ for $j, k \in \{1, \ldots, m\}$ with $j \neq k$.

If moreover $S \in \mathcal{Z}(X)$, it follows that

$$P_k S I_k \in \mathcal{Z}(X_k) \quad \forall k \in \{1, \ldots, m\}.$$ 

**Positive elements and positive operators**

- $a \in E$ is positive if $a_k \geq 0$ for every $k = 1, \ldots, m$.
- $U \in \mathcal{L}(E)$ is positive if $U(a)$ is positive for every positive $a \in E$. 

Let $E$ be $\mathbb{R}^m$ endowed with an absolute norm, let $X_1, \ldots, X_m$ be Banach spaces and $X = \left[ X_1 \oplus \cdots \oplus X_m \right]_E$.

**Proposition (Martín-Merí-Popov-Randrianantoanina, 2011)**

$$n(X) \leq \min\{n(X_1), \ldots, n(X_m)\}$$

**Proposition**

Suppose that every $S \in \mathcal{Z}(X)$ is diagonal. Then,

$$n'(X) \leq \min\{n'(X_1), \ldots, n'(X_m)\}$$
The second numerical index

Relationship of the indices with positive operators on $E$

Let $E$ be $\mathbb{R}^m$ endowed with an absolute norm, let $X_1, \ldots, X_m$ be Banach spaces and $X = [X_1 \oplus \cdots \oplus X_m]_E$.

**Proposition**

For every positive operator $U \in \mathcal{L}(E)$ one has

$$n(X) \leq \frac{v(U)}{\|U\|}$$

**Proposition**

Suppose that every $S \in \mathcal{Z}(X)$ is diagonal. Let $U \in \mathcal{L}(E)$ be a positive operator so that there is $a \in E$ satisfying $\|Ua\| = \|U\|$ and $\text{supp}(Ua) \cap \text{supp}(a) = \emptyset$. Then,

$$n'(X) \leq \frac{v(U)}{\|U\|}$$
Spaces with absolute norm and $n'(X) = 1$

**Theorem**

Let $X$ be $\mathbb{R}^m$ endowed with an absolute norm. Suppose that $n(X) = 0$ and $n'(X) = 1$. Then, $X$ is a Hilbert space.

**Observation**

The result is more general and it can be extended to Banach spaces with (long) one-unconditional basis.

**Sketch of the proof**

- As $n(X) = 0$ and $X$ has an absolute norm, we may use a result of Rosenthal (80’s) to obtain:
  - There are $\ell \in \mathbb{N}$, Hilbert spaces $H_1, \ldots, H_\ell$, and $E = (\mathbb{R}^\ell, | \cdot |_a)$ such that $X = \left[ H_1 \oplus \cdots \oplus H_\ell \right]_E$ and
    - $\dim(H_1) \geq 2$
    - Every $S \in \mathcal{Z}(X)$ is diagonal (with respect to this decomposition)
- Suppose for contradiction that $\ell \neq 1$. 
Spaces with absolute structure and $n'(X) = 1$

Sketch of the proof

- For $k \in \{2, \ldots, \ell\}$ let $E_{1,k}$ be the linear span of $\{e_1, e_k\}$ in $E$. Using that $n'(X) = 1$, it is possible to prove that $E_{1,k} = \ell_2^2$.

- Fix $j \in \{2, \ldots, \ell\}$ and show that the positive operator $U \in \mathcal{L}(E)$ given by $U = e_1^* \otimes e_j$ satisfies

$$\|U\| = 1 \quad \text{and} \quad v(U) < 1.$$ 

- Therefore $n'(X) \leq \frac{v(U)}{\|U\|} < 1$ which gives the desired contradiction.
Vector valued spaces

**Proposition**

Let $X$ be a Banach space, $L$ locally compact Hausdorff, $K$ compact Hausdorff, $\Omega$ completely regular Hausdorff, and $\mu$ positive measure. Then

- $n'(C_0(L, X)) \leq n'(X)$
- $n'(C_w(K, X)) \leq n'(X)$
- $n'(C_b(\Omega, X)) \leq n'(X)$
- $n'(L_1(\mu, X)) \leq n'(X)$
- $n'(L_\infty(\mu, X)) \leq n'(X)$

**Example**

Let $K$ be a compact Hausdorff topological space with at least two points. Then

$$n'(C(K, \ell_2^2)) \leq \frac{\sqrt{3}}{2} < 1.$$
Duality

Observation

Let $X$ be a Banach space. If every element in $Z(X^*)$ is the transpose of an element in $Z(X)$ then $n'(X^*) \leq n'(X)$

Proposition

Suppose that one of the following holds

- The norm of $X^*$ is Fréchet-smooth on a dense set (e.g. $X = \ell_\infty$);
- $B_X$ is the closed convex hull of the $w-\| \cdot \|$ continuity points of Id (in particular, $X$ RNP, $X$ CPCP, $X$ LUR, $X$ has a Kadec norm, $X = X_1 \widehat{\otimes}_\pi X_2$ where $X_1, X_2$ RNP, or $X = L(R)$ where $R$ is reflexive);
- $X^* \nsubseteq \ell_1$;
- $X$ is isomorphic to a subspace of a separable $L$-embedded space;
- $X$ is the (unique) predual of a von Neumann algebra.

Then $n'(X^*) \leq n'(X)$
Duality II

On the other hand,

Example
Given $0 \leq \alpha \leq \beta \leq 1/2$, there is a Banach space $X_{\alpha,\beta}$ with $n(X_{\alpha,\beta}) = 0$ such that

$$n'(X_{\alpha,\beta}) = \beta \quad \text{and} \quad n'(X_{\alpha,\beta}^*) = \alpha.$$
Section 3

3 An application to BPB-property for numerical radius
**An application**

**Definition (Guirao-Kozhushkina, 2013; Kim-Lee-Martín, 2014)**

Let $X$ be a Banach space.

- $X$ has the **Bishop-Phelps-Bollobás property for numerical radius** if for every $0 < \varepsilon < 1$, there is $\eta(\varepsilon) > 0$ such that whenever $T \in L(X)$ and $(x, x^*) \in \Pi(X)$ satisfy $v(T) = 1$ and $|x^*Tx| > 1 - \eta(\varepsilon)$, there exist $S \in L(X)$ and $(y, y^*) \in \Pi(X)$ such that

  $$v(S) = |y^*Sy| = 1, \quad \|T - S\| < \varepsilon, \quad \|x - y\| < \varepsilon, \quad \text{and} \quad \|x^* - y^*\| < \varepsilon.$$  

- $X$ has the **weak-Bishop-Phelps-Bollobás property for numerical radius** if for every $0 < \varepsilon < 1$, there is $\eta(\varepsilon) > 0$ such that whenever $T \in L(X)$ and $(x, x^*) \in \Pi(X)$ satisfy $v(T) = 1$ and $|x^*Tx| > 1 - \eta(\varepsilon)$, there exist $S \in L(X)$ and $(y, y^*) \in \Pi(X)$ such that

  $$v(S) = |y^*Sy|, \quad \|T - S\| < \varepsilon, \quad \|x - y\| < \varepsilon, \quad \text{and} \quad \|x^* - y^*\| < \varepsilon.$$
An application

**Proposition (Kim-Lee-Martín,2014)**

$X$ Banach space with $n(X) > 0$. Then, the weak-Bishop-Phelps-Bollobás property for numerical radius implies the Bishop-Phelps-Bollobás property for numerical radius.

**Actually...**

$X$ Banach space with $n'(X) > 0$. Then, the weak-Bishop-Phelps-Bollobás property for numerical radius implies the Bishop-Phelps-Bollobás property for numerical radius.

**Proposition (Kim-Lee-Martín,2014)**

$X$ uniformly convex and uniformly smooth $\implies X$ has the weak-Bishop-Phelps-Bollobás property for numerical radius.

**Corollary**

Hilbert spaces have the Bishop-Phelps-Bollobás property for numerical radius.
Open problems

Section 4

4 Open problems
Some open problems

- Which is the set of values of \( n'(X) \) for Banach spaces \( X \) with \( n(X) = 0 \)? Does it cover the interval \([0, 1]\)?
  - We know that it covers the interval \([0, 1/2]\) and contains 1.
  - It can be done (except for the value cero) with four-dimensional spaces.
  - If \( \dim(X) = 2 \) and \( n(X) = 0 \) then \( X = \ell_2^2 \).
  - If \( \dim(X) = 3 \) and \( n(X) = 0 \) then \( X = \ell_2^2 \oplus_a \mathbb{R} \).
  - In this case we know that it is NOT an interval.

- Is \( n'(X \oplus_2 Y) \leq \min\{n'(Y), n'(W)\} \)?

- Let \( \mu \) be a positive measure, \( X \) a Banach space and \( 1 < p < \infty \). Is it true that \( n'(L_p(\mu, X)) \leq n'(X) \)?

- Is \( n'(X^*) \leq n'(X) \) for every Banach space \( X \)?

- Are Hilbert spaces the unique Banach spaces \( X \) with \( n(X) = 0 \) and \( n'(X) = 1 \)?

- \( X \) complex, what is the meaning of \( n'(X_R) \)?

- \( X = \mathbb{C} \oplus_a \mathbb{C} \). What is the value of \( n'(X_R) \)?
  - \( \oplus_a = \oplus_2 \implies n'(X_R) = 1 \),
  - \( \oplus_a = \oplus_1 \implies \frac{1}{2} \leq n'(X_R) \leq \frac{\sqrt{3}}{2} \).