Spear operators and the numerical index with respect to an operator

Miguel Martín

UNIVERSIDAD DE GRANADA

(work in progress with V. Kadets, J. Merí, A. Pérez, and A. Quero)

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Some notation

\( X, Y \) real or complex Banach spaces

\( K \) base field, \( \mathbb{R} \) or \( \mathbb{C} \)

\( B_X \) closed unit ball

\( S_X \) unit sphere

\( X^* \) topological dual

\( L(X, Y) \) Banach space of all bounded linear operators from \( X \) to \( Y \)

\( L(X) \) Banach algebra of all bounded linear operators from \( X \) to \( X \)
A walk through the “classical” numerical index
Definitions

**Numerical range for Hilbert spaces (Toeplitz, 1918)**

*H* Hilbert space, $(\cdot | \cdot)$ inner product, $T \in L(H)$

$$W(T) = \{(Tx | x): x \in H, (x | x) = 1\}$$

- It is a convex subset of $\mathbb{K}$

**Numerical range and numerical radius (Bauer, Lumer, early 1960’s)**

*X* Banach space, $T \in L(X)$

$$V(T) = \{x^*(Tx): x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}$$

$$v(T) = \sup\{|\lambda|: \lambda \in V(T)\}$$

$$= \sup\{|x^*(Tx)|: x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}$$

- $V(T)$ is connected not necessarily convex,
- $\overline{V(T)}$ contains the spectrum of $T$,
- obviously, $v(T) \leq \|T\|$ for every $T \in L(X)$. 
Numerical index (Lumer, 1968)

Let $X$ be a Banach space. The numerical index $n(X)$ is defined as:

$$n(X) = \inf \{ v(T) : T \in S_{L(X)} \} = \max \{ k \geq 0 : k \|T\| \leq v(T) \}$$

- $0 \leq n(X) \leq 1$
- $v$ and $\| \cdot \|$ are equivalent norms iff $n(X) > 0$

Possible values of the numerical index

- For complex Banach spaces: $[e^{-1}, 1]$
- For real Banach spaces: $[0, 1]$
Some known results

- $H$ Hilbert space, $n(H) = 0$ in real case and $n(H) = 1/2$ in complex case.

- $n(C(K)) = n(L_1(\mu)) = 1$ (Duncan-McGregor-Pryce-White, 1970)

- $n(X) = 1$ iff $\max_{|w|=1} \|\text{Id} + wT\| = 1 + \|T\| \forall T \in L(X)$ (Duncan et al., 1970)

- Let $\{X_\lambda : \lambda \in \Lambda\}$ be an arbitrary family of Banach spaces. Then

$$n\left(\bigoplus_{\lambda \in \Lambda} X_\lambda\right)_{c_0} = n\left(\bigoplus_{\lambda \in \Lambda} X_\lambda\right)_{l_1} = n\left(\bigoplus_{\lambda \in \Lambda} X_\lambda\right)_{l_\infty} = \inf_{\lambda \in \Lambda} n(X_\lambda)$$

$$n\left(\bigoplus_{\lambda \in \Lambda} X_\lambda\right)_{l_p} \leq \inf_{\lambda \in \Lambda} n(X_\lambda)$$

(Martín-Payá, 2000)
Some known results

- $X$ Banach space, $K$ compact Hausdorff, $\mu$ positive measure

  \[ n\left(C(K, X)\right) = n\left(L_1(\mu, X)\right) = n(X) \]  
  \[ n\left(L_\infty(\mu, X)\right) = n(X) \]  
  (Martín-Payá, 2000)

  \[ n\left(L_p(\mu)\right) = n(\ell_p) \text{ if } \dim L_p(\mu) = \infty \]  
  (EdDari-Khamsi, 2006)

  \[ n\left(L_p(\mu)\right) > 0 \text{ for } p \neq 2 \]  
  (Martín-Merí-Popov, 2011)

- $n(X^*) \leq n(X)$

  and the inequality can be strict  
  (Boyko-Kadets-Martín-Werner, 2007)

- $X$ separable (WCG)

  \[ \implies \{n(X, \lVert \cdot \rVert) : \lVert \cdot \rVert \text{ equivalent norm}\} \supseteq \begin{cases} [0, 1] & \text{real case} \\ [1/e, 1] & \text{complex case} \end{cases} \]  
  (Finet-Martín-Payá, 2003)

- $X$ real, $\dim(X) = \infty$, $n(X) = 1 \implies X^* \supseteq \ell_1$  
  (Avilés, Kadets, Martín, Merí, Shepelska, 2010)
Extending the concept of numerical range
Spatial numerical range

**Bauer–Lumer (spatial) Numerical range**

$X$ Banach space, $T \in L(X)$,

$$V(T) = \{ x^*(Tx) : x \in S_X, x^* \in S_{X^*}, x^* (\text{Id} \cdot x) = 1 \}$$

$\star \ G \in L(X,Y)$ with $\|G\| = 1$, $T \in L(X,Y)$, how to define $V_G(T)$?

The first idea (not working):

$$V_G(T) = \{ y^*(Tx) : x \in S_X, y^* \in S_{Y^*}, y^* (G \cdot x) = 1 \}$$

**(Approximate spatial) Numerical range with respect to $G$** (Ardalani, 2014)

$X$, $Y$ Banach spaces, $G \in L(X,Y)$ with $\|G\| = 1$, $T \in L(X,Y)$

$$V_G(T) = \bigcap_{\delta > 0} \{ y^*(Tx) : x \in S_X, y^* \in S_{Y^*}, \text{Re} \ y^* (Gx) > 1 - \delta \}$$

For $G = \text{Id}$, by the Bishop–Phelps–Bollobás theorem

$$V_{\text{Id}}(T) = \overline{V(T)} \quad \text{for every } T \in L(X)$$
Intrinsic Numerical range

(Bonsall-Duncan, 1971)

Let $X$ be a Banach space. Then for every $T \in L(X)$

$$\overline{co} \, V(T) = \{ \Phi(T) : \Phi \in L(X)^*, \|\Phi\| = \Phi(\text{Id}) = 1 \}.$$ 

Consequently, $v(T) = \max\{|\Phi(T)| : \Phi \in L(X)^*, \|\Phi\| = \Phi(\text{Id}) = 1\}$.

Intrinsic (or algebraic) numerical range

$X$ Banach space, $T \in L(X)$,

$$\widetilde{V}(T) = \{ \Phi(T) : \Phi \in L(X)^*, \|\Phi\| = \Phi(\text{Id}) = 1 \}$$

Intrinsic numerical range with respect to $G$

$X$, $Y$ Banach spaces, $G \in L(X, Y)$ with $\|G\| = 1$, $T \in L(X, Y)$

$$\widetilde{V}_G(T) = \{ \Phi(T) : \Phi \in L(X, Y)^*, \|\Phi\| = \Phi(G) = 1 \}$$
The relationship

Two possible numerical ranges

Let $X, Y$ be Banach spaces, $G \in L(X, Y)$ with $\|G\| = 1$, $T \in L(X, Y)$.

\[
V_G(T) = \bigcap_{\delta > 0} \{y^*(Tx): x \in S_X, y^* \in S_{Y^*}, \Re y^*(Gx) > 1 - \delta\}
\]

\[
\tilde{V}_G(T) = \{\Phi(T): \Phi \in L(X, Y)^*, \|\Phi\| = \Phi(G) = 1\}
\]

Relationship (Martín, 2016)

Let $X, Y$ be Banach spaces, $G \in L(X, Y)$ with $\|G\| = 1$, then

\[
\tilde{V}_G(T) = \text{co} V_G(T) \quad \text{for every } T \in L(X, Y)
\]

Both concepts produce the same numerical radius:

Numerical radius with respect to $G$

Let $X, Y$ be Banach spaces, $G \in L(X, Y)$ with $\|G\| = 1$, $T \in L(X, Y)$.

\[
v_G(T) = \sup \{|\lambda|: \lambda \in V_G(T)\} = \sup \{|\lambda|: \lambda \in \tilde{V}_G(T)\}
\]
Numerical index with respect to an operator: definition
Numerical index with respect to an operator

**Numerical index with respect to $G$**

$X, Y$ Banach spaces, $G \in L(X, Y)$ with $\|G\| = 1$,

$$n_G(X, Y) = \inf \{ v_G(T) : T \in S_{L(X, Y)} \} = \max \{ k \geq 0 : k \|T\| \leq v_G(T) \}$$

**We recuperate the classical numerical index**

$$n_{Id}(X, X) = n(X)$$

**Characterization**

For $k \in [0, 1]$, TFAE:

- $n_G(X, Y) \geq k$.
- $\inf_{\delta > 0} \sup \{ |y^*(Tx)| : x \in S_X, y^* \in S_{Y^*}, \Re y^*(Gx) > 1 - \delta \} \geq k \|T\| \ \forall T \in L(X, Y),$
- $\max_{|\theta|=1} \|G + \theta T\| \geq 1 + k \|T\| \ \forall T \in L(X, Y).$

**Consequence**

$$n_G(X, Y) > 0 \iff G \text{ is a (geometrically) unitary element of } L(X, Y)$$
Numerical index with respect to an operator: examples and properties
Some interesting examples I

Set of values
There exists $X$ (real and complex versions) such that
\[ \left\{ n_G(X, X) : G \in L(X, X), \|G\| = 1 \right\} = [0, 1]. \]

Hilbert spaces
$H_1, H_2$ Hilbert spaces of dimension at least two,
- Real case: $n_G(H_1, H_2) = 0$ for all $G \in L(H_1, H_2)$ with $\|G\| = 1$,
- Complex case: $n_G(H_1, H_2) \in \{0, 1/2\}$ for all $G \in L(H_1, H_2)$ with $\|G\| = 1$.

Actually...
$G \in L(X, Y)$ with $\|G\| = 1$, if $X$ or $Y$ is a real Hilbert space
\[ \implies n_G(X, Y) = 0. \]
★ There are more spaces with this property...
Some interesting examples II

\( \mathbb{L}_p \)-spaces

\( G \in \mathbb{L}(X, Y) \) with \( \|G\| = 1 \), if \( X \) or \( Y \) is a \( \mathbb{L}_p(\mu) \)-space (\( 1 < p < \infty \)),

\[ n_G(X, Y) \leq \begin{cases} 
\sup_{t \in [0,1]} \frac{|t^{p-1} - t|}{1 + t^p} & \text{real case} \\
q^{-1/p} p^{-1/q} & \text{complex case}
\end{cases} \]

Spaces of integrable functions

\( \mu_1, \mu_2 \) \( \sigma \)-finite measures,

\( n_G(L_1(\mu_1), L_1(\mu_2)) \in \{0, 1\} \) for all \( G \in \mathbb{L}(L_1(\mu_1), L_1(\mu_2)) \) with \( \|G\| = 1 \).

Spaces of essentially bounded functions

\( \mu_1, \mu_2 \) \( \sigma \)-finite measures,

\( n_G(L_\infty(\mu_1), L_\infty(\mu_2)) \in \{0, 1\} \) for all \( G \in \mathbb{L}(L_\infty(\mu_1), L_\infty(\mu_2)) \) with \( \|G\| = 1 \).
Proposition

Let \( \{ X_\lambda : \lambda \in \Lambda \} \), \( \{ Y_\lambda : \lambda \in \Lambda \} \) be two families of Banach spaces and let \( G_\lambda \in L(X_\lambda, Y_\lambda) \) with \( \| G_\lambda \| = 1 \) for every \( \lambda \in \Lambda \). Let \( E \) be one of the Banach spaces \( c_0 \), \( \ell_\infty \) or \( \ell_1 \), let \( X = \bigoplus_{\lambda \in \Lambda} X_\lambda \) \( E \) and \( Y = \bigoplus_{\lambda \in \Lambda} Y_\lambda \) \( E \) and define the operator \( G : X \to Y \) by

\[
G \left[ (x_\lambda)_{\lambda \in \Lambda} \right] = (G_\lambda x_\lambda)_{\lambda \in \Lambda}
\]

for every \( (x_\lambda)_{\lambda \in \Lambda} \in \bigoplus_{\lambda \in \Lambda} X_\lambda \) \( E \). Then

\[
n_G(X, Y) = \inf_{\lambda} n_{G_\lambda}(X_\lambda, Y_\lambda).
\]

Moreover, for \( 1 < p < \infty \)

\[
n_G \left( \bigoplus_{\lambda \in \Lambda} X_\lambda \ell_p , \bigoplus_{\lambda \in \Lambda} Y_\lambda \ell_p \right) \leq \inf_{\lambda} n_{G_\lambda}(X_\lambda, Y_\lambda).
\]
Composition operators

**Theorem**

Let $X$, $Y$ be Banach spaces, and $G \in L(X,Y)$ with $\|G\| = 1$.

- $K$ compact, consider $\tilde{G}: C(K,X) \rightarrow C(K,Y)$ given by $\tilde{G}(f) = G \circ f$; then
  
  $$n_{\tilde{G}}(C(K,X),C(K,Y)) = n_G(X,Y).$$

- $\mu$ measure, consider $\tilde{G}: L_1(\mu,X) \rightarrow L_1(\mu,Y)$ given by $\tilde{G}(f) = G \circ f$; then
  
  $$n_{\tilde{G}}(L_1(\mu,X),L_1(\mu,Y)) = n_G(X,Y).$$

- $\mu$ $\sigma$-finite, consider $\tilde{G}: L_\infty(\mu,X) \rightarrow L_\infty(\mu,Y)$ given by $\tilde{G}(f) = G \circ f$; then
  
  $$n_{\tilde{G}}(L_\infty(\mu,X),L_\infty(\mu,Y)) = n_G(X,Y).$$

Besides, for vector-valued $L_p$-spaces one inequality holds:

$$n_{\tilde{G}}(L_p(\mu,X),L_p(\mu,Y)) \leq n_G(X,Y)$$

for $1 < p < \infty$, $\tilde{G}$ defined analogously.
Spear operators
Examples of spear operators

Spear operator (Ardalani, 2014; Kadets, Martín, Merí, Pérez, 2018)

\[ G \text{ spear operator} \iff n_G(X,Y) = 1 \iff \max_{|\theta|=1} \|G + \theta T\| = 1 + \|T\| \quad \forall T \in L(X,Y). \]

Some interesting examples of spear operators

- Fourier transform (for example, \( \mathcal{F} : L_1(\mathbb{R}) \rightarrow C_0(\mathbb{R}) \));
- The inclusion \( A(\mathbb{D}) \hookrightarrow C(\mathbb{T}) \);
- The identity operator on \( C(K), L_1(\mu) \ldots \)
- \( G : X \rightarrow c_0 \) spear iff \( \left| x^{**}(G^*(e_n)) \right| = 1 \) for \( n \in \mathbb{N} \) and \( x^{**} \in \text{ext} (B_{X^{**}}) \);
- \( G : \ell_1 \rightarrow Y \) spear iff \( \left| y^*(G(e_n)) \right| = 1 \) for \( n \in \mathbb{N} \) and \( y^* \in \text{ext} (B_{Y^*}) \);
- If \( \dim(X) < \infty \), \( G \) spear iff \( \left| y^*(Gx) \right| = 1 \) for \( y^* \in \text{ext} (B_{Y^*}) \) and \( x \in \text{ext} (B_X) \);
- If \( \dim(Y) < \infty \), \( G \) spear iff \( \left| x^{**}(G^*(y^*)) \right| = 1 \) for \( x^{**} \in \text{ext} (B_{X^{**}}) \) and \( y^* \in \text{ext} (B_{X^*}) \);
Spear operator (Ardalani, 2014; Kadets, Martín, Merí, Pérez, 2018)

\[ G \text{ spear operator } \iff n_G(X, Y) = 1 \iff \max_{|\theta|=1} \|G + \theta T\| = 1 + \|T\| \forall T \in L(X, Y). \]

Remark

To work with spear operators, two other concepts are introduced:
- lush operator,
- the alternative Daugavet property (aDP),

★ Both are geometric properties (related to \( G \))
★ They are related as follows:

\[
\begin{align*}
\text{lush operator} & \iff \text{spear operator} \iff \text{operator with aDP} \\
\text{SCD operator} & \\
(X \text{ RNP, } X \nsubseteq \ell_1, Y \text{ Asplund...})
\end{align*}
\]
Spear operators: consequences

Some isomorphic and isometric consequences

Let $X$, $Y$ be Banach spaces, $G \in L(X,Y)$ a spear operator,

- If $\dim(G(X)) = \infty$, then $X^* \supset \ell_1$,
- If $X^*$ is strictly convex, then $X = K$,
- If $X^*$ is smooth, then $X = K$,
- If $B_X$ contains a WLUR point, then $X = K$,
- If $Y^*$ is strictly convex, then $Y = K$,
- If $B_Y$ contains a WLUR point, then $Y = K$.

Norm attainment

- If $G$ is lush, $G$ attains its norm; actually:

\[ B_X = \overline{\text{co}} \{ x \in S_X : \|Gx\| = 1 \}, \]

- There are examples of aDP operators which do not attain the norm,
- What about spear operators?
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