Vector space structure in the set of norm attaining functionals

(V. Kadets, G. López, M. Martín, and D. Werner)

Workshop on Infinite Dimensional Analysis
to celebrate the 60th birthday of Domingo García
Bibliography

V. Kadets, G. López, and M. Martín
Some geometric properties of Read’s space

V. Kadets, G. López, M. Martín, and D. Werner
Equivalent norms with an extremely nonlineable set of norm attaining functionals

C. Read
Banach spaces with no proximinal subspaces of codimension 2
*Israel J. Math.* (to appear)

M. Rmoutil
Norm-attaining functionals need not contain 2-dimensional subspaces
Roadmap of the talk

1. Preliminaries
   - Lineability of $\mathcal{NA}(X)$
   - Proximinality

2. Read’s and Rmoutil’s results

3. Our construction
   - A direct approach to (G)
   - Modest subspaces
   - Main theorem
   - Consequences

4. Open problems and limitations of the construction
Norm attaining functionals

Norm attaining functionals

\( x^* \in X^* \) attains its norm when

\[ \exists x \in X, \|x\| = 1 : x^*(x) = \|x^*\| \]

\( \star \) \( NA(X) = \{x^* \in X^* : x^* \text{ attains its norm}\} \)

First results

- \( \dim(X) < \infty \implies NA(X) = X^* \) (Heine-Borel),
- \( X \) reflexive \( \implies NA(X) = X^* \) (Hahn-Banach),
- \( X \) non-reflexive \( \implies NA(X) \neq X^* \) (James),
- \( NA(X) \) is always norm dense in \( X^* \) (Bishop-Phelps).

Examples

- \( NA(c_0) = c_{00} \leq \ell_1 \),
- \( NA(\ell_1) = \{x \in \ell_\infty : \|x\|_\infty = \max_n \{|x(n)|\}\} \).
**Lineability**

**Examples**
- \( \text{NA}(c_0) = c_{00} \leq \ell_1 \),
- \( \text{NA}(\ell_1) = \{ x \in \ell_\infty : \|x\|_\infty = \max_n \{|x(n)|\} \} \).

- Note that \( \text{NA}(c_0) \) is a linear space, but \( \text{NA}(\ell_1) \) is not.
- However, \( \text{NA}(\ell_1) \) contains the infinite-dimensional space \( c_0 \).

**Lineability**

Recall that a subset \( S \) of a vector space \( V \) is called **lineable** if \( S \cup \{0\} \) contains an infinite-dimensional subspace.
Lineability of $\mathcal{NA}(X)$

Main question

Lineability of $\mathcal{NA}(X)$?

More concretely,

Problems (G. Godefroy, 2001)

$(G_\infty)$ Does $\mathcal{NA}(X)$ always contain an infinite-dimensional linear subspace?

$(G)$ Does $\mathcal{NA}(X)$ always contain a linear subspace of dimension 2?

The case of dimension 1 is taken care of by the Hahn-Banach theorem!

Note that $(G_\infty)$ holds in all classical spaces.
Proximinality

Proximinal subspace

$Y \subseteq X$ is proximal iff

$$\forall x \in X \ \exists y_0 \in Y : \|x - y_0\| = \inf\{\|x - y\| : y \in Y\} = \text{dist}(x, Y)$$

- $Y$ proximal iff $Q(B_X) = B_{X/Y}$  
  ($Q : X \rightarrow X/Y$ quotient map)

- $x^* \in \text{NA}(X) \iff \ker x^*$ proximal.

Problem (I. Singer, 1974)

(S) Is there always a proximal subspace of codimension 2?
The two main problems

(S) Does there always exist a proximinal subspace of codimension 2?

(G) Does $\text{NA}(X)$ always contain a linear subspace of dimension 2?

Important result (Garkavi, 1967)

$$Y \leq X \text{ proximinal of finite codimension} \implies Y^\perp \subset \text{NA}(X).$$

Therefore...

If (S) is true, then (G) is true.

The converse result is not true

There exist $X$ and finite codimensional $Y$ such that $Y^\perp \subset \text{NA}(X)$ but $Y$ is not proximinal (Phelps, 1963)
Read’s and Rmoutil’s theorems

**Theorem (Read, 2013)**

There is a counterexample $X_R$ to (S).

As (S) $\Rightarrow$ (G), $X_R$ is a natural candidate for a counterexample to (G).

Actually,

**Theorem (Rmoutil, 2015)**

- $X/Y$ strictly convex and $Y^\perp \subset NA(X) \implies Y$ proximinal.
- $\dim X_R/Y = 2 \implies X_R/Y$ strictly convex.
- Consequently, $X_R$ is also a counterexample to (G).

A simplification of Rmoutil’s proof by Kadets/López/Martín:

**Proposition**

$X_R^{**}$ is strictly convex; hence *all* quotients of $X_R$ are strictly convex.
Read’s construction

$X_R$ is a renorming of $c_0$:

Let $\Omega = \{(s_n): (s_n) \text{ has finite support, all } s_n \in \mathbb{Q}\} \subset \ell_1$.
Enumerate $\Omega = \{u_1, u_2, \ldots\}$ so that every element is repeated infinitely often.
Take a sequence of integers $(a_n)$ such that

$$a_k > \max \text{ supp } u_k, \quad a_k \geq \|u_k\|_{\ell_1}.$$  

Renorm $c_0$ by

$$p(x) = \|x\|_{\infty} + \sum_k 2^{-a_k^2} |\langle u_k - e_{a_k}, x \rangle|.$$  

Then Read shows that $(c_0, p)$ fails (S), and Rmoutil shows, relying on Read’s work, that $(c_0, p)$ fails (G).

The proof of Read’s theorem is not trivial at all!!!!!
A new, direct approach to (G)

We four are more used to norm-attainment than to proximinality, so we changed the point of view:

We want to show directly that certain Banach spaces have a renorming failing (G) and hence have a renorming failing (S).

Let $R: X \rightarrow \ell_1$ be continuous; we renorm $X$ by

$$p(x) = \|x\| + \|Rx\|_{\ell_1}.$$ 

More precisely, let $[Rx](n) = 2^{-n}v^*_n(x)$, $(v^*_n) \subset B_{X^*}$, so

$$p(x) = \|x\| + \sum_{n=1}^{\infty} \frac{v^*_n(x)}{2^n}.$$ 

(Note that Read's renorming is of this type.)

Aim

Under suitable assumptions, the $v^*_n$ can be chosen so that $(X, p)$ fails (G) (and hence fails (S)).
A tentative calculation

\[ p(x) = \|x\| + \sum 2^{-n}|v_n^*(x)|. \] Then \( B(x^*, p^*) = B_X + \sum 2^{-n}[-v_n^*, v_n^*] \) (Minkowski sum).

Let \( x^* \in NA_1(X, p) \) be norm attaining at \( x \); then

\[ x^* = x_0^* + \sum 2^{-n}t_n v_n^* \]

for some \( x_0^* \in NA_1(X) \) and \( t_n = \text{sign } v_n^*(x) \) whenever \( v_n^*(x) \) is nonzero.

Write the same decomposition for \( y^* \in NA_1(X, p) \), norm attaining at \( y \):

\[ y^* = y_0^* + \sum 2^{-n}t'_n v_n^*. \]

Let’s try to prove that \( x^* + y^* \notin NA(X, p) \): Otherwise we would have a similar decomposition for \( z^* = (x^* + y^*)/\|x^* + y^*\| \):

\[ z^* = z_0^* + \sum 2^{-n}s_n v_n^*. \]

Sort the items, setting \( \lambda = \|x^* + y^*\| \):

\[ 0 = x^* + y^* - \lambda z^* = [x_0^* + y_0^* - \lambda z_0^*] + \left[ \sum (t_n + t'_n - \lambda s_n) v_n^* \right] \]
Wish list

\[ 0 = [x_0^* + y_0^* - \lambda z_0^*] + \left[ \sum (t_n + t'_n - \lambda s_n)v_n^* \right] \]

We now wish to select the \( v_n^* \) to be sort of “orthogonal” to \( \text{span}(\text{NA}(X)) \) (which contains the first bracket) so that both brackets vanish.

In addition we wish the \( v_n^* \) to have some Schauder basis character so that we can deduce from \( \sum (t_n + t'_n - \lambda s_n)v_n^* = 0 \) that all \( t_n + t'_n - \lambda s_n = 0 \).

Finally we wish the support points \( x \) and \( y \) to be distinct, and we wish the span of the \( v_n^* \) to be dense enough to separate \( x \) and \( y \) for many \( n \), i.e., \( v_n^*(x) < 0 < v_n^*(y) \) and thus \( t_n + t'_n = 0 \) fairly often, while at the same time \( s_n \neq 0 \) for at least one of those \( n \).

This contradiction would show that \( x^* + y^* \notin \text{NA}(X, p) \).
Modest subspaces

Definition: operator range, (weak*) modest subspace

- $V, W$ Banach spaces, $T : V \rightarrow W$ injective. Then $T(V)$ is called an operator range.
- $Z \leq W$ is modest if there is a separable dense operator range $Y$ with $Y \cap Z = \{0\}$.
- If $W$ is a dual space, then $Z \leq W$ is weak* modest if there is a separable weak* dense operator range $Y$ with $Y \cap Z = \{0\}$.

Note that the choice of $V$ in the definition of a modest subspace is at our discretion since

$$E, F\text{ separable} \implies \exists \text{ continuous injection } S : E \rightarrow F \text{ with dense range.}$$

Example

$$\{(s_n) : (s_n) \text{ has finite support}\} \text{ is modest in } \ell_1.$$
**Main Theorem**

**Theorem**

If span(NA(X)) is weak* modest, then X has a renorming that fails (G) and, consequently, fails (S). (We call such an equivalent norm a Read norm.)

Recall ansatz: $p(x) = \|x\| + \sum 2^{-n}|v_n^*(x)|$; how to choose the $v_n^*$?

**Lemma**

Let $Y \leq X^*$ be a separable operator range. Then there is an injective operator $S: \ell_1 \to X^*$ such that, for $v_n^* = S(e_n)$, the set $\{v_n^*/\|v_n^*\|\}$ is dense in $S_Y$.

With this choice of $v_n^*$ it is possible to fulfill our wishes: the $v_n^*$ are “orthogonal” to $\text{NA}(X)$ (wish #1), they are an injective image of a Schauder basis (wish #2) and sufficiently dense (wish #4). As for wish #3, if $x = y$, then $x \neq -y$ and one should look at $x^* - y^*$!

Thus we can show that for linearly independent $x^*, y^* \in \text{NA}(X, p)$ of norm 1, at most one of $x^* \pm y^*$ can be in $\text{NA}(X, p)$.
First consequence

**Example (we recuperate Read’s and Rmoutil’s results)**

$c_0$ admits a Read norm, that is, a norm failing (G) and hence failing (S).

Indeed, $\text{NA}(c_0) = c_{00}$ is modest in $\ell_1$.

**Note**

The original construction by Read is NOT a particular case of ours:

Indeed, both norms are of the form $p(x) = \|x\| + \sum 2^{-n} |v_n^*(x)|$, but

- in the original Read’s construction, the $v_n^*$’s belong to $\text{NA}(c_0)$,
- in our construction, the $v_n^*$’s are “orthogonal” to $\text{NA}(c_0)$. 
More consequences I

Proposition

A separable Banach space containing a copy of $c_0$ admits a Read norm.

Indeed, renorm $X$ so that $X = c_0 \oplus_\infty E$; then $X^* = \ell_1 \oplus_1 E^*$ and $\text{NA}(X) \subset \text{NA}(c_0) \oplus_1 E^*$. The latter can be shown to be contained in a weak* modest subspace.

Example

$C[0, 1]$ admits an equivalent Read norm.

Norms with additional properties

$X$ separable containing $c_0$. Then for each $0 < \varepsilon < 2$ there is a Read norm $p_\varepsilon$ on $X$ with the following properties:

- $p_\varepsilon$ is strictly convex and smooth,
- $p_\varepsilon^*$ is strictly convex,
- $p_\varepsilon^*$ is $(2 - \varepsilon)$-rough; i.e., every slice of $B(X, p_\varepsilon)$ has diameter $\geq 2 - \varepsilon$,
- If moreover $X^*$ is separable, then $p_\varepsilon^{**}$ is strictly convex.
More consequences II

Theorem

A Banach space containing a copy of $c_0$ which has a countable system of norming functionals admits a Read norm.

\[ \{ x_n^* \} \text{ is a norming system if } x \mapsto \sup_n |x_n^*(x)| \text{ is an equivalent norm. Such a space is isomorphic to a closed subspace of } \ell_\infty \text{ and vice versa.} \]

Example

\( \ell_\infty \) admits an equivalent Read norm.

Norms with additional properties

\( X \) containing \( c_0 \) which has a countable system of norming functionals. Then for each \( 0 < \varepsilon < 2 \) there is a Read norm \( p_\varepsilon \) on \( X \) so that

- \( p_\varepsilon \) is strictly convex,
- \( p_\varepsilon^* \) is \((2 - \varepsilon)\)-rough; i.e., every slice of \( B(x,p_\varepsilon) \) has diameter \( \geq 2 - \varepsilon \),
- actually, every convex combination of slices (hence every relatively weakly open subset) of \( B(x,p_\varepsilon) \) has diameter \( \geq 2 - \varepsilon \).
Open problems

Open problem

Does every separable non-reflexive Banach space admit an equivalent Read norm?
- $\ell_\infty(\Gamma)$ with $\Gamma$ uncountable does not admit a Read norm

Some remarks

- Our construction needs $\text{span}(\text{NA}(X))$ to be “small” (weak-star modest).
- This is not always possible: if $X$ RNP, then $\text{span}(\text{NA}(X)) = X^*$ (Bourgain).
- Actually, if $\text{NA}(X)$ is residual, then $\text{span}(\text{NA}(X)) = X^*$.

An stronger result

$X$ separable, $\text{span}(\text{NA}(X))$ second category $\implies \text{span}(\text{NA}(X)) = X^*$.

Two concrete problems

- Does $\ell_1$ admit a Read norm? (observe that $\text{span}(\text{NA}(X)) = X^*$ for every $X \simeq \ell_1$)
- Does $L_1[0, 1]$ admit a norm such that $\text{span}(\text{NA}(X))$ is weak-star modest? (observe that $\text{NA}(L_1[0, 1])$ is first category but $\text{span}(\text{NA}(L_1[0, 1])) = L_1[0, 1]^*$)