Norm attaining compact operators

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Roadmap of the talk

1. Introducing the topic
2. An quick overview on norm attaining operators
3. Norm attaining compact operators
4. Further developments
5. Bibliography
Introducing the topic

Section 1

1. Introducing the topic
   - Notation
   - Short introduction
Introducing the topic

Section 1

1. Introducing the topic
   - Notation
   - Short introduction
**Notation**

Let $X, Y$ be real or complex Banach spaces.

- **$\mathbb{K}$** base field $\mathbb{R}$ or $\mathbb{C}$,

- $B_X = \{ x \in X : \|x\| \leq 1 \}$ closed unit ball of $X$,

- $S_X = \{ x \in X : \|x\| = 1 \}$ unit sphere of $X$,

- $\mathcal{L}(X, Y)$ bounded linear operators from $X$ to $Y$,
  - $\|T\| = \sup \{ \|T(x)\| : x \in S_X \}$ for $T \in \mathcal{L}(X, Y)$,

- $\mathcal{K}(X, Y)$ compact linear operators from $X$ to $Y$,

- $\mathcal{F}(X, Y)$ bounded linear operators from $X$ to $Y$ with finite rank (i.e. dimension of the range is finite),

- $X^* = \mathcal{L}(X, \mathbb{K})$ topological dual of $X$. 

Introducing the topic

Section 1

Introducing the topic

- Notation
- Short introduction
Norm attaining functionals

\( x^* \in X^* \) attains its norm when

\[ \exists x \in S_X : |x^*(x)| = \|x^*\| \]

\( \star \) \( \text{NA}(X, \mathbb{K}) = \{x^* \in X^* : x^* \text{ attains its norm}\} \)

First examples

- \( \text{dim}(X) < \infty \implies \text{NA}(X, \mathbb{K}) = \mathcal{L}(X, \mathbb{K}) \) (Heine-Borel).
- \( X \) reflexive \( \implies \text{NA}(X, \mathbb{K}) = \mathcal{L}(X, \mathbb{K}) \) (Hahn-Banach).
- \( X \) non-reflexive \( \implies \text{NA}(X, \mathbb{K}) \neq \mathcal{L}(X, \mathbb{K}) \) (James),
- but \( \text{NA}(X, \mathbb{K}) \) always separates the points of \( X \) (Hahn-Banach).

- \( \text{NA}(c_0, \mathbb{K}) = c_{00} \leq \ell_1 \),
- \( \text{NA}(\ell_1, \mathbb{K}) = \{x \in \ell_\infty : \|x\|_\infty = \max_n \{|x(n)|\} \} \leq \ell_\infty \), not subspace, contains \( c_0 \),
- \( \text{NA}(X, \mathbb{K}) \) may not contain two-dimensional subspaces (Rmoutil, 2017).
Norm attaining operators

$T \in \mathcal{L}(X, Y)$ attains its norm when

$$\exists \ x \in S_X : \|T(x)\| = \|T\|$$

$\star \ NA(X, Y) = \{T \in \mathcal{L}(X, Y): T \text{ attains its norm}\}$

First examples

- $\text{dim}(X) < \infty \implies NA(X, Y) = \mathcal{L}(X, Y)$ for every $Y$ (Heine-Borel).
- $NA(X, Y) \neq \emptyset$ (Hahn-Banach),
- $X$ reflexive $\implies \mathcal{K}(X, Y) \subseteq NA(X, Y)$ for every $Y$ (we will comment),
- $X$ non-reflexive $\implies \mathcal{K}(X, Y) \not\subseteq NA(X, Y)$ for any $Y$ (James),
- $\text{dim}(X) = \infty \implies NA(X, c_0) \neq \mathcal{L}(X, c_0)$ (see M.-Merí-Payá, 2006).
The problem of density of norm attaining functionals

Problem

Is $\text{NA}(X, K)$ always dense in $X^*$?

Theorem (E. Bishop & R. Phelps, 1961)

The set of norm attaining functionals is dense in $X^*$ (for the norm topology).

Problem

Is $\text{NA}(X, Y)$ always dense in $\mathcal{L}(X, Y)$?

The answer is No, and this is the origin of the study of norm attaining operators.

Modified problem

When is $\text{NA}(X, Y)$ dense in $\mathcal{L}(X, Y)$?

The study of this problem was initiated by J. Lindenstrauss in 1963, who provided the first negative and positive examples.
Section 2

2 An quick overview on norm attaining operators

- First results: Lindenstrauss
- The relation with the RNP: Bourgain
- Counterexamples for property B: Gowers and Acosta
- Some results on pairs of classical spaces
- Main open problems
Section 2

An quick overview on norm attaining operators

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Lindenstrauss' seminal paper of 1963

Negative answer
There are bounded linear operators which cannot be approximated by norm-attaining operators:
- the domain can be $c_0$ (usual norm),
- the range can be any strictly convex renorming of $c_0$,
- the domain and the range may coincide.

The result for $c_0$ (we will give a detailed proof later)

$$Y \text{ strictly convex}, \ T \in \text{NA}(c_0, Y) \implies T e_n = 0 \text{ for } n \text{ big enough}$$

Observation
- The question then is for which $X$ and $Y$ the density holds.
- As this problem is too general, Lindenstrauss introduced two properties.
Lindenstrauss properties A and B

Definition

X, Y Banach spaces,
- X has (Lindenstrauss) property A iff \( \overline{\text{NA}(X, Z)} = \mathcal{L}(X, Z) \quad \forall Z \)
- Y has (Lindenstrauss) property B iff \( \overline{\text{NA}(Z, Y)} = \mathcal{L}(Z, Y) \quad \forall Z \)

Examples

- If X is finite-dimensional, then X has property A,
- Actually, reflexive spaces have property A,
- \( \ell_1 \) has property A,
- \( c_0 \) fails property A,
- K has property B (Bishop-Phelps theorem),
- every Y such that \( c_0 \subset Y \subset \ell_\infty \) has property B,
- finite-dimensional polyhedral spaces have property B,
- every strictly convex renorming of \( c_0 \) fails property B.
An quick overview on norm attaining operators

Section 2

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The RNP and property A

Theorem (Bourgain, 1977)
Radon Nikodým Property $\implies$ property A.

Non-linear Bourgain-Stegall variational principle (Stegall, 1978)

Let $X, Y$ be Banach spaces, $C \subset X$ a bounded subset-dentable, $\varphi : C \to Y$ uniformly bounded such that $x \mapsto \|\varphi(x)\|$ is upper semicontinuous.

Then for every $\delta > 0$, there exists $x_0^* \in X^*$ with $\|x_0^*\| < \delta$ and $y_0 \in S_Y$ such that the function $x \mapsto \|\varphi(x) + x_0^*(x)y_0\|$ attains its supremum on $C$.

Theorem (Bourgain, 1977)

$X$ separable with property A $\implies$ $B_X$ is dentable.
The RNP and properties A and B

A refinement of Bourgain’s result (Huff, 1980)

Let $X$ be a Banach space failing the RNP. Then there exist $X_1$ and $X_2$ equivalent renorming of $X$ such that

$$\text{NA}(X_1, X_2) \text{ is NOT dense in } \mathcal{L}(X_1, X_2).$$

Main consequence

Every renorming of $X$ has property A $\iff$ $X$ has the RNP.

Another consequence

Every renorming of $Y$ has property B $\implies$ $Y$ has the RNP.
An quick overview on norm attaining operators

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Gowers’ result and Acosta’s result

**Observation**

It was an open question in the 1970’s and 1980’s whether

\[ \text{RNP} \implies \text{property B} \]

But...

**Theorem (Gowers, 1990)**

\( \ell_p \) does not have property B for any \( 1 < p < \infty \).

**Theorem (Acosta, 1999)**

Every infinite-dimensional strictly convex space fails property B.

**Consequence**

\( Y \) separable, every renorming of \( Y \) has property B \( \implies \) \( Y \) is finite-dimensional
An quick overview on norm attaining operators

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Pairs of classical spaces

Example (Johnson-Wolfe, 1979)

In the real case, \( \text{NA}(C(K_1), C(K_2)) \) is dense in \( \mathcal{L}(C(K_1), C(K_2)) \).

Example (Iwanik, 1979)

\( \text{NA}(L_1(\mu), L_1(\nu)) \) is dense in \( \mathcal{L}(L_1(\mu), L_1(\nu)) \).

Examples (Schachermayer, 1983)

\( \text{NA}(C(K), L_p(\mu)) \) is dense in \( \mathcal{L}(C(K), L_p(\mu)) \) for \( 1 \leq p < \infty \).

Example (Finet-Payá, 1998)

\( \text{NA}(L_1[0, 1], L_\infty[0, 1]) \) is dense in \( \mathcal{L}(L_1[0, 1], L_\infty[0, 1]) \).

Example (Schachermayer, 1983)

\( \text{NA}(L_1[0, 1], C[0, 1]) \) is NOT dense in \( \mathcal{L}(L_1[0, 1], C[0, 1]) \).
Section 2

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Main open problems

The main open problem
★ Do finite-dimensional spaces have Lindenstrauss property B?

(Stunning) open problem
Do finite-dimensional Hilbert spaces have Lindenstrauss property B?

Open problem
Characterize the topological compact spaces $K$ such that $C(K)$ has property B.

Open problem
$X$ Banach space without the RNP, does there exists a renorming of $X$ such that $\text{NA}(X, X)$ is not dense in $L(X, X)$?

Remark
If $X \asymp Z \oplus Z$, then the answer to the question above is positive (use Bourgain-Huff).
Norm attaining compact operators

Section 3

3 Norm attaining compact operators
- Posing the problem for compact operators
- The easiest negative example
- More negative examples
- Positive results on property AK
- Positive results on property BK
- Open Problems
Section 3

Norm attaining compact operators

- Posing the problem for compact operators
  - The easiest negative example
  - More negative examples
  - Positive results on property AK
  - Positive results on property BK
  - Open Problems
Posing the problem for compact operators

Question

Can every compact operator be approximated by norm-attaining operators?

Observations

- In all the negative examples of the previous section, the authors constructed NON COMPACT operators which cannot be approximated by norm attaining operators.
- Actually, the idea of the proofs is to use that the operator which is not going to be approximated is not compact or, even, it is an isomorphism.
- In most examples, it was even known that compact operators attaining the norm are dense.

Where was it explicitly possed?

More observations on compact operators

Question
Can every compact operator be approximated by norm-attaining operators?

Observations

- If $X$ is reflexive, then ALL compact operators from $X$ into $Y$ are norm attaining. (Indeed, compact operators carry weak convergent sequences to norm convergent sequences.)

- It is known from the 1970’s that whenever $X = C_0(L)$ or $X = L_1(\mu)$ (and $Y$ arbitrary) or $Y = L_1(\mu)$ or $Y^* \equiv L_1(\mu)$ (and $X$ arbitrary), $\Rightarrow \ NA(X, Y) \cap K(X, Y)$ is dense in $K(X, Y)$. 
Section 3

Norm attaining compact operators

- Posing the problem for compact operators
- The easiest negative example
- More negative examples
- Positive results on property AK
- Positive results on property BK
- Open Problems
Extending a result by Lindenstrauss

$X$, $Y$ Banach spaces, $T \in \mathcal{L}(X,Y)$ and $x_0 \in S_X$ with $\|T\| = \|Tx_0\| = 1$.

- If $x_0$ is not extreme point of $B_X$, there is $z \in X$ such that $\|x_0 \pm z\| \leq 1$, so $\|Tx_0 \pm Tz\| \leq 1$.
- If $Tx_0$ is an extreme point of $B_Y$, then $Tz = 0$.
Extending a result by Lindenstrauss

$X$, $Y$ Banach spaces, $T \in \mathcal{L}(X, Y)$ and $x_0 \in S_X$ with $\|T\| = \|Tx_0\| = 1$.

- If $x_0$ is not extreme point of $B_X$, there is $z \in X$ such that $\|x_0 \pm z\| \leq 1$, so $\|Tx_0 \pm Tz\| \leq 1$.
- If $Tx_0$ is an extreme point of $B_Y$, then $Tz = 0$.

Geometrical lemma (abstract version of a Lindenstrauss’ result)

$X$, $Y$ Banach spaces. Suppose that

- for every $x_0 \in S_X$, $\text{lin}\{z \in X : \|x_0 \pm z\| \leq 1\}$ has finite codimension,
- $Y$ is strictly convex.

Then, $\text{NA}(X, Y) \subseteq \mathcal{F}(X, Y)$.

First consequence (recalling, Lindenstrauss, 1963)

- $\text{NA}(c_0, Y) \subseteq \mathcal{F}(c_0, Y)$ if $Y$ is strictly convex.
- Therefore, $c_0$ fails property A.
Extending a result by Lindenstrauss (II)

Proposition (extension of Lindenstrauss result)

\[ X \leq c_0. \text{ For every } x_0 \in S_X, \text{ lin}\{z \in X : \|x_0 \pm z\| \leq 1\} \text{ has finite codimension.} \]

Proof.

- as \( x_0 \in c_0, \) there exists \( m \) such that \( |x_0(n)| < 1/2 \) for every \( n \geq m; \)
- let \( Z = \{z \in X : z(i) = 0 \text{ for } 1 \leq i \leq m\} \) (finite codimension in \( X\));
- for \( z \in Z \) with \( \|z\| \leq 1/2, \) one has \( \|x_0 \pm z\| \leq 1. \)

Main consequence

\[ X \leq c_0, \text{ } Y \text{ strictly convex. Then } \text{NA}(X, Y) \subseteq \mathcal{F}(X, Y). \]

Question

What’s next? How to use this result?
Grothendieck’s approximation property

**Definition (Grothendieck, 1950’s)**

$Z$ has the **approximation property (AP)** if for every $K \subset Z$ compact and every $\varepsilon > 0$, there exists $F \in \mathcal{F}(Z)$ such that $\|Fz - z\| < \varepsilon$ for all $z \in K$.

**Basic results**

$X, Y$ Banach spaces.

- (Grothendieck) $Y$ has AP $\iff \overline{\mathcal{F}(Z,Y)} = \mathcal{K}(Z,Y)$ for all $Z$.
- (Grothendieck) $X^*$ has AP $\iff \overline{\mathcal{F}(X,Z)} = \mathcal{K}(X,Z)$ for all $Z$.
- (Grothendieck) $X^* \text{ AP } \implies X \text{ AP}$.
- (Enflo, 1973) There exists $X \leq c_0$ without AP.
The first example

**Theorem**

There exists a **compact** operator which cannot be approximated by norm attaining operators.

**Proof:**

- consider $X \leq c_0$ without AP (Enflo);
- $X^*$ does not has AP
  \[ \implies \text{there exists } Y \text{ and } T \in \mathcal{K}(X, Y) \text{ such that } T \not\in \overline{\mathcal{F}(X, Y)}; \]
- we may suppose $Y = \overline{T(X)}$, which is separable;
- so $Y$ admits an equivalent strictly convex renorming (Klee);
- we apply the extension of Lindenstrauss result: $\overline{\text{NA}(X, Y)} \subseteq \overline{\mathcal{F}(X, Y)}$;
- therefore, $T \not\in \overline{\text{NA}(X, Y)}$. 
Two useful definitions

Definitions

$X$ and $Y$ Banach spaces.

- $X$ has property AK when $\overline{\text{NA}(X, Z) \cap \mathcal{K}(X, Z)} = \mathcal{K}(X, Z)$ $\forall Z$;
- $Y$ has property BK when $\overline{\text{NA}(Z, Y) \cap \mathcal{K}(Z, Y)} = \mathcal{K}(Z, Y)$ $\forall Z$.

Some basic results

- Finite-dimensional spaces have property AK;
- $Y = \mathbb{K}$ has property BK;
- Real finite-dimensional polyhedral spaces have property BK.

Our negative example (recalling)

There exists $X \leq c_0$ failing AK and there exits $Y$ failing BK.
Section 3

3 Norm attaining compact operators
- Posing the problem for compact operators
- The easiest negative example
- More negative examples
- Positive results on property AK
- Positive results on property BK
- Open Problems
More examples: Domain space

Proposition (what we have proved so far...)

\[ X \leq c_0 \text{ such that } X^* \text{ fails AP } \implies X \text{ does not have AK.} \]

Example by Johnson-Schechtman, 2001

Exists \( X \) subspace of \( c_0 \) with Schauder basis such that \( X^* \) fails the AP.

Corollary

There exists a Banach space \( X \) with Schauder basis failing property AK.
More examples: Range space

**Strictly convex spaces**

\[ Y \text{ strictly convex without AP} \implies Y \text{ fails BK}. \]

**Lemma (Grothendieck)**

\[ Y \text{ has AP iff } \mathcal{F}(X, Y) \text{ is dense in } \mathcal{K}(X, Y) \text{ for every } X \leq c_0. \]

**Subspaces of** \( L_1(\mu) \)

\[ Y \leq L_1(\mu) \text{ (complex case) without AP} \implies Y \text{ fails BK}. \]

**Observation (Globevnik, 1975)**

Complex \( L_1(\mu) \) spaces are complex strictly convex:

\[ f, g \in L_1(\mu), \|f\| = 1 \text{ and } \|f + \theta g\| \leq 1 \forall \theta \in B_{\mathbb{C}} \implies g = 0. \]
More examples: Domain=Range

**Theorem**

There exists a Banach space $Z$ and a compact operator from $Z$ to $Z$ which cannot be approximated by norm attaining operators.

**Proposition**

$X$ and $Y$ Banach spaces, $Z = X \oplus_1 Y$ or $Z = X \oplus_\infty Y$. 
$\text{NA}(Z, Z) \cap K(Z, Z)$ dense in $K(Z, Z) \implies \text{NA}(X, Y) \cap K(X, Y)$ dense in $K(X, Y)$. 

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Section 3

3 Norm attaining compact operators

- Posing the problem for compact operators
- The easiest negative example
- More negative examples
- Positive results on property AK
- Positive results on property BK
- Open Problems
Positive results on property AK

Problem

\[ X^* \text{ AP } \implies X \text{ AK?} \]

Partial answer:

(Johnson-Wolfe) With a strong approximation property of the dual...

Suppose there exists a net of contractive projections \((P_\alpha)_{\alpha}\) in \(X\) with finite rank such that \(\lim_\alpha P_\alpha^* = \text{Id}_{X^*}\) in SOT. Then, \(X\) has AK.

Consequences

- (Diestel-Uhl) \(L_1(\mu)\) has AK.
- (Johnson-Wolfe) \(C_0(L)\) has AK.
- \(X\) with monotone and shrinking basis \(\implies X\) has AK.
- \(X\) with monotone unconditional basis, \(X \not\cong \ell_1 \implies X\) has AK.
- \(X^* \equiv \ell_1 \implies X\) has AK (using a result by Gasparis).
- \(X \leq c_0\) with monotone basis \(\implies X\) has AK (using a result by Godefroy–Saphar).
Section 3

Norm attaining compact operators

- Posing the problem for compact operators
- The easiest negative example
- More negative examples
- Positive results on property AK
- Positive results on property BK
- Open Problems
Positive results on property BK I

Main open question

\[ \text{AP } \implies \text{BK?} \]

A partial answer (Johnson-Wolfe)

- If \( Y \) is polyhedral (real) and has \( \text{AP} \implies Y \) has BK.
- \( X \) (complex) space with \( \text{AP} \) such that the norm of every finite-dimensional subspace can be calculated as the maximum of a finite set of functionals \( \implies Y \) has BK.

Example (Johnson-Wolfe)

\[ Y \leq c_0 \text{ (real or complex) with } \text{AP } \implies Y \text{ has BK.} \]

A somehow reciprocal to the problem...

\( Y \) separable with BK for every equivalent norm \( \implies Y \) has AP.
Positive results on property BK II

Main open question

$$\text{AP} \implies \text{BK}$$

Another partial answer (Johnson-Wolfe)

$Y$ Banach space. Suppose there exists a uniformly bounded net of projections $(Q_\alpha)_\alpha$ in $Y$ such that $\lim_\alpha Q_\alpha = \text{Id}_Y$ in SOT and $Q_\alpha(Y)$ has property BK. Then, $Y$ has property BK.

Examples (Johnson-Wolfe)

- $Y$ predual of $L_1(\mu)$ (real or complex) $\implies Y$ has BK;
- in particular, real or complex $C_0(L)$ spaces have property BK;
- real $L_1(\mu)$ spaces have property BK.
Section 3

3 Norm attaining compact operators
  - Posing the problem for compact operators
  - The easiest negative example
  - More negative examples
  - Positive results on property AK
  - Positive results on property BK
  - Open Problems
### Some open problems

#### Main open problem

- Can every finite-rank operator be approximated by norm-attaining operators?

#### Open problem

- $X$ Banach space, does there exist a norm-attaining rank-two operator from $X$ to a Hilbert space?

#### Another main open problem

- $X^* \text{ AP} \implies X \text{ AK}$?

#### Open problem

- $X \leq c_0$ with the metric AP, does it have AK?

#### Open problem

- $X$ such that $X^* \equiv L_1(\mu)$, does $X$ have AK?

#### Open problem

- $Y$ subspace of the real $L_1(\mu)$ without the AP, does $Y$ fail property BK?
Further developments

Section 4

4 Further developments
  ■ Bishop-Phelps-Bollobás property for compact operators
  ■ Numerical radius attaining operators
Further developments

Section 4

4 Further developments

- Bishop-Phelps-Bollobás property for compact operators
- Numerical radius attaining operators
Bishop-Phelps-Bollobás property (Acosta, Aron, García, Maestre, 2008)

A pair of Banach spaces \((X, Y)\) has the Bishop-Phelps-Bollobás property (BPBp) if given \(\varepsilon \in (0, 1)\) there is \(\eta(\varepsilon) > 0\) such that whenever

\[
T_0 \in S_{\mathcal{L}(X,Y)}, \quad x_0 \in S_X, \quad \|T_0 x_0\| > 1 - \eta(\varepsilon),
\]

there exist \(S \in \mathcal{L}(X, Y)\) and \(x \in S_X\) such that

\[
1 = \|S\| = \|Sx\|, \quad \|x_0 - x\| < \varepsilon, \quad \|T_0 - S\| < \varepsilon.
\]

Some results

- Bollobás, 1970: \((X, K)\) has the BPBp for every \(X\),
- of course, if \((X, Y)\) has the BPBp, then \(\text{NA}(X, Y)\) is dense in \(\mathcal{L}(X, Y)\),
- but there is \(Y\) with Lindenstrauss property B such that \((\ell_2^2, Y)\) fails BPBp.
- Kim-Lee, 2014; Acosta-Becerra-García-Maestre, 2014:
  \(X\) uniformly convex \(\implies (X, Y)\) has BPBp for every \(Y\),
- Aron-Choi-Kim-Lee-M., 2015:
  \(\dim(X) = 2, (X, Y)\) BPBp for every \(Y\) \(\implies X\) is uniformly convex.
Bishop-Phelps-Bollobás property for compact operators

A pair of Banach spaces \((X, Y)\) has the **Bishop-Phelps-Bollobás property for compact operators** (BPBp for compact) if given \(\varepsilon \in (0, 1)\) there is \(\eta(\varepsilon) > 0\) such that whenever

\[
T_0 \in S_{\mathcal{K}(X,Y)}, \quad x_0 \in S_X, \quad \|T_0 x_0\| > 1 - \eta(\varepsilon),
\]

there exist \(S \in \mathcal{K}(X,Y)\) and \(x \in S_X\) such that

\[
1 = \|S\| = \|Sx\|, \quad \|x_0 - x\| < \varepsilon, \quad \|T_0 - S\| < \varepsilon.
\]

**Remarks**
- Most of the results for BPBp are also true for BPBp for compact,
- also, many results about the density of norm attaining compact operators can be actually extended to the BPBp for compact.

**Open problem**
There is a wide line of research here...
Further developments

Section 4

4 Further developments

- Bishop-Phelps-Bollobás property for compact operators
- Numerical radius attaining operators
**Numerical radius attaining operators**

**Numerical radius attaining operators**

$X$ Banach space, $T \in \mathcal{L}(X)$ attains its numerical radius when

$$\exists (x, x^*) \in \Pi(X) : |x^*T(x)| = \sup \{|y^*(Ty)| : (y, y^*) \in \Pi(X)\}$$

where $\Pi(X) := \{(x, x^*) : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}$.

**Some positive results**

The set of numerical radius attaining operators is dense for:

- (Cardasi, 1985) $C(K)$ and $L_1(\mu)$ (real case),
- (Acosta-Payá, 1993) spaces with the RNP.

**Negative examples**

The set of numerical radius attaining operators is NOT dense in some examples:

- Payá, 1992: $c_0 \oplus_{\infty} Y$ ($Y$ strictly convex renorming of $c_0$),
- Acosta-Aguirre-Payá, 1992: $\ell_2 \oplus_{\infty} d_{\ast}(w)$,
- Capel-M.-Merí, 2017: $C[0, 1] \oplus_{\infty} L_1[0, 1]$. 
In none of the previous examples it is produced a **compact** operator which cannot be approximated by numerical radius attaining operators.

**Example (Capel-M.-Merí, 2017)**

Given $1 < p < 2$, there are a subspace $X$ of $c_0$ and a quotient $Y$ of $\ell_p$ such that $\mathcal{K}(X \oplus_\infty Y)$ is not contained in the closure of the set of numerical radius attaining operators.

**Note**

The proof is involved and needs a careful adaptation of many ideas from previous proofs.

**Open problem**

We know only few positive results about numerical radius attaining compact operators.
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Section 5

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