Slicely Countably Determined Banach spaces

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Slicely Countably Determined Banach spaces
Introduction

Section 1
Basic notation

- $X$ real or complex Banach space.
- $S_X$ unit sphere, $B_X$ closed unit ball, $\mathbb{T}$ modulus-one scalars.
- $X^*$ dual space, $L(X)$ bounded linear operators from $X$ to $X$.
- $\text{conv}(\cdot)$ convex hull, $\overline{\text{conv}}(\cdot)$ closed convex hull,
- A slice of $A \subset X$ is a (nonempty) subset of the form

$$S(A, x^*, \alpha) = \{x \in A : \text{Re} x^*(x) > \sup \text{Re} x^*(A) - \alpha\} \quad (x^* \in X^*, \alpha > 0)$$
Two classical concepts: Radon-Nikodým property and Asplund spaces

**The Radon-Nikodým property or RNP (1930’s)**
- $X$ has the RNP iff the Radon-Nikodým theorem is valid for $X$-valued measures;
- Equivalently [1960’s], every bcc subset contains a denting point (i.e. a point belonging to slices of arbitrarily small diameter).

$X$ Asplund $\iff X^* \text{ RNP}$

**Asplund spaces (1960’s)**
- $X$ is an Asplund space if every continuous convex real-valued function defined on an open subset of $X$ is F-differentiable on a dense subset;
- Equivalently [1970’s], every separable subspace has separable dual.
Introduction

The road map of the talk

The property

We introduce an isomorphic property for (separable) Banach spaces, the so-called

slicely countably determination (SCD)

such that

- it is satisfied by RNP spaces
  (actually, by strongly regular spaces – PCP in particular–);
- it is satisfied by Asplund spaces
  (actually, by spaces not containing $\ell_1$).

We also present examples and stability properties.

The applications

- We apply SCD to get results for the Daugavet property, the alternative Daugavet property and spaces with numerical index 1.
- We present SCD operators and applications.
Outline

1. Introduction
2. Slicely Countably Determined sets and spaces
3. Applications
4. SCD operators
5. Open problems
Section 2

2 Slicely Countably Determined sets and spaces
   - SCD sets
   - SCD spaces
SCD sets: Definitions and preliminary remarks

$X$ Banach space, $A \subseteq X$ bounded and convex.

**SCD sets**

$A$ is **Slicely Countably Determined (SCD)** if there is a sequence $\{S_n : n \in \mathbb{N}\}$ of slices of $A$ satisfying one of the following equivalent conditions:

- if $B \subseteq A$ satisfies $B \cap S_n \neq \emptyset \ \forall n$, then $A \subseteq \overline{\text{conv}}(B)$,
- given $\{x_n\}_{n \in \mathbb{N}}$ with $x_n \in S_n \ \forall n \in \mathbb{N}$, $A \subseteq \overline{\text{conv}}(\{x_n : n \in \mathbb{N}\})$,
- every slice of $A$ contains one of the $S_n$’s,

**Remarks**

- $A$ is SCD iff $\overline{A}$ is SCD.
- If $A$ is SCD, then it is separable.
SCD sets: Elementary examples I

Example

A separable and $A = \overline{\text{conv}}(\text{dent}(A)) \implies A$ is SCD.

Proof.

- Take $\{a_n : n \in \mathbb{N}\}$ denting points with $A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\})$.
- For every $n, m \in \mathbb{N}$, take a slice $S_{n,m}$ containing $a_n$ and of diameter $1/m$.
- If $B \cap S_{n,m} \neq \emptyset \implies a_n \in \overline{B}$.
- Therefore, $A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\}) \subseteq \overline{\text{conv}}(\overline{B}) = \overline{\text{conv}}(B)$.

Example

In particular, $A$ RNP separable $\implies A$ SCD.

Corollary

- If $X$ is separable LUR $\implies B_X$ is SCD.
- So, every separable space can be renormed such that $B(X, |\cdot|)$ is SCD.
Example

If \( X^* \) is separable \( \implies A \) is SCD.

Proof.

- Take \( \{x_n^* : n \in \mathbb{N}\} \) dense in \( S_{X^*} \).
- For every \( n, m \in \mathbb{N} \), consider \( S_{n,m} = S(A, x_n^*, 1/m) \).
- It is easy to show that any slice of \( A \) contains one of the \( S_{n,m} \).

Example

\( B_{C[0,1]} \) and \( B_{L_1[0,1]} \) are not SCD.
SCD sets: Further examples I

Convex combination of slices

\[ W = \sum_{k=1}^{m} \lambda_k S_k \subset A \] where \( \lambda_k \geq 0, \sum \lambda_k = 1, S_k \) slices.

Proposition

In the definition of SCD we can use a sequence \( \{S_n : n \in \mathbb{N}\} \) of convex combination of slices.

Small combinations of slices

\( A \) has small combinations of slices iff every slice of \( A \) contains convex combinations of slices of \( A \) with arbitrary small diameter.

Example

If \( A \) has small combinations of slices + separable \( \implies \) \( A \) is SCD.

Particular case

\( A \) strongly regular (in particular, PCP) + separable \( \implies \) \( A \) is SCD.
SCD sets: Further examples II

**Bourgain’s lemma**

Every relative weak open subset of $A$ contains a convex combination of slices.

**Corollary**

In the definition of SCD we can use a sequence of relative weak open subsets: the set $A$ is SCD iff there is a sequence $\{V_n : n \in \mathbb{N}\}$ of relative weak open subsets of $A$ such that every slice of $A$ contains one of the $V_n$’s.

**$\pi$-bases**

A $\pi$-base of the weak topology of $A$ is a family $\{V_i : i \in I\}$ of weak open sets of $A$ such that every weak open subset of $A$ contains one of the $V_i$’s.

**Proposition**

If $(A, \sigma(X, X^*))$ has a countable $\pi$-base $\implies A$ is SCD.
Theorem

A separable without \( \ell_1 \)-sequences \( \implies (A, \sigma(X, X^*)) \) has a countable \( \pi \)-base.

Proof.

- We see \((A, \sigma(X, X^*)) \subset C(T)\) where \(T = (B_{X^*}, \sigma(X^*, X))\).
- By Rosenthal \( \ell_1 \) theorem, \((A, \sigma(X, X^*))\) is a relatively compact subset of the space of first Baire class functions on \(T\).
- By a result of Todor\'čević, \((A, \sigma(X, X^*))\) has a \(\sigma\)-disjoint \(\pi\)-base.
- \(\{V_i : i \in I\}\) is \(\sigma\text{-disjoint}\) if \(I = \bigcup_{n \in \mathbb{N}} I_n\) and each \(\{V_i : i \in I_n\}\) is pairwise disjoint.
- A \(\sigma\)-disjoint family of open subsets in a separable space is countable. ✓

Main example

A separable without \( \ell_1 \)-sequences \( \implies A \) is SCD.
SCD spaces: definition and examples

**SCD space**

\( X \) is **Slicely Countably Determined (SCD)** if so are its convex bounded subsets.

**Examples of SCD spaces**

1. \( X \) separable strongly regular. In particular, RNP, PCP spaces.
2. \( X \) separable \( X \not\supset \ell_1 \). In particular, if \( X^* \) is separable.

**Examples of NOT SCD spaces**

1. \( C[0,1], L_1[0,1] \)
2. Actually, every \( X \) containing (an isomorphic copy of) \( C[0,1] \) or \( L_1[0,1] \).
3. There is \( X \) with the Schur property which is not SCD.

**Remark**

- Every subspace of a SCD space is SCD.
- This is false for quotients.
Theorem

\[ Z \subset X. \text{ If } Z \text{ and } X/Z \text{ are SCD } \implies X \text{ is SCD.} \]

Corollary

If \( \ell_1 \cong Y \subset X \), then \( X/Y \) contains a copy of \( \ell_1 \).

If \( \ell_1 \cong Y_1 \subset X \), then there is \( \ell_1 \cong Y_2 \subset X \) with \( Y_1 \cap Y_2 = 0 \).

Corollary

\[ X_1, \ldots, X_m \text{ SCD } \implies X_1 \oplus \cdots \oplus X_m \text{ SCD.} \]
SCD spaces: stability properties II

**Theorem**

$X_1, X_2, \ldots$ SCD, $E$ with unconditional basis.

- $E \not\subset c_0 \implies \left[ \bigoplus_{n \in \mathbb{N}} X_n \right]_E$ SCD.
- $E \not\subset \ell_1 \implies \left[ \bigoplus_{n \in \mathbb{N}} X_n \right]_E$ SCD.

**Examples**

1. $c_0(\ell_1)$ and $\ell_1(c_0)$ are SCD.
2. $c_0 \otimes \varepsilon c_0$, $c_0 \otimes \pi c_0$, $c_0 \otimes \varepsilon \ell_1$, $c_0 \otimes \pi \ell_1$, $\ell_1 \otimes \varepsilon \ell_1$, and $\ell_1 \otimes \pi \ell_1$ are SCD.
3. $K(c_0)$ and $K(c_0, \ell_1)$ are SCD.
4. $\ell_2 \otimes \varepsilon \ell_2 \equiv K(\ell_2)$ and $\ell_2 \oplus \pi \ell_2 \equiv \mathcal{L}_1(\ell_2)$ are SCD.
Applications

Section 3

Applications
- The DPr, the ADP and numerical index 1
- Lush spaces
- From ADP to lushness
The DPr, the ADP and numerical index 1

**Definition of the properties**

1. **Kadets-Shvidkoy-Sirotkin-Werner, 1997:**
   \( X \) has the **Daugavet property (DPr)** if
   \[
   \|\text{Id} + T\| = 1 + \|T\| \quad \text{(DE)}
   \]
   for every rank-one \( T \in L(X) \).
   - Then every \( T \) not fixing copies of \( \ell_1 \) also satisfies (DE).

2. **Lumer, 1968:** \( X \) has **numerical index 1** \( (n(X) = 1) \) if
   \[
   \max_{\theta \in \mathbb{T}} \|\text{Id} + \theta T\| = 1 + \|T\| \quad \text{(aDE)}
   \]
   for EVERY operator on \( X \).
   - Equivalently,
     \[
     \|T\| = \sup\{|x^*(Tx)| : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}
     \]
     for EVERY \( T \in L(X) \).

3. **M.-Oikhberg, 2004:** \( X \) has the **alternative Daugavet property (ADP)** if every rank-one \( T \in L(X) \) satisfies (aDE).
   - Then every weakly compact \( T \) also satisfies (aDE).
Relations between these properties

![Diagram showing the relations between Daugavet property (DPr), ADP, and numerical index 1]

**Examples**
- \( C([0, 1], K(\ell_2)) \) has DPr, but has not numerical index 1
- \( c_0 \) has numerical index 1, but has not DPr
- \( c_0 \oplus \infty C([0, 1], K(\ell_2)) \) has ADP, neither DPr nor numerical index 1

**Remark**

For RNP or Asplund spaces, \( \text{ADP} \implies \text{numerical index 1} \).
For $C^*$-algebras and preduals

Let $V_*$ be the predual of the von Neumann algebra $V$.

The Daugavet property of $V_*$ is equivalent to:
- $V$ has no atomic projections, or
- the unit ball of $V_*$ has no extreme points.

$V_*$ has numerical index 1 iff:
- $V$ is commutative, or
- $|v^*(v)| = 1$ for $v \in \text{ext}(B_V)$ and $v^* \in \text{ext}(B_{V_*})$.

The alternative Daugavet property of $V_*$ is equivalent to:
- the atomic projections of $V$ are central, or
- $|v(v_*)| = 1$ for $v \in \text{ext}(B_V)$ and $v_* \in \text{ext}(B_{V_*})$, or
- $V = C \oplus \infty N$, where $C$ is commutative and $N$ has no atomic projections.
Let $X$ be a $C^*$-algebra.

The Daugavet property of $X$ is equivalent to:

- $X$ does not have any atomic projection, or
- the unit ball of $X^*$ does not have any $w^*$-strongly exposed point.

$X$ has numerical index 1 iff:

- $X$ is commutative, or
- $|x^{**}(x^*)| = 1$ for $x^{**} \in \text{ext}(B_{X^{**}})$ and $x^* \in \text{ext}(B_{X^*})$.

The alternative Daugavet property of $X$ is equivalent to:

- the atomic projections of $X$ are central, or
- $|x^{**}(x^*)| = 1$, for $x^{**} \in \text{ext}(B_{X^{**}})$, and $x^* \in B_{X^*}$ $w^*$-strongly exposed, or
- $\exists$ a commutative ideal $Y$ such that $X/Y$ has the Daugavet property.
A sufficient condition for numerical index 1: lushness

Lushness (Boyko-Kadets-M.-Werner, 2007)

$X$ is **lush** if given $x, y \in S_X$, $\varepsilon > 0$, there is $y^* \in S_{X^*}$ such that

$$x \in S = S(B_X, y^*, \varepsilon) \quad \text{dist} (y, \text{conv}(T S')) < \varepsilon.$$

Theorem (Boyko-Kadets-M.-Werner, 2007)

If $X$ is lush, then $X$ has numerical index 1

Example (Kadets-M.-Merí-Shepelska, 2009)

There is $X$ with numerical index 1 which is not lush.
Applications

From ADP to lushness

Characterization of ADP

$X$ Banach space. TFAE:

- $X$ has ADP (i.e. $\max_{\theta \in \mathbb{T}} \|\text{Id} + \theta T\| = 1 + \|T\|$ for all $T$ rank-one).
- Given $x \in S_X$, a slice $S$ of $B_X$ and $\varepsilon > 0$, there is $y \in S$ with
  \[
  \max_{\theta \in \mathbb{T}} \|x + \theta y\| > 2 - \varepsilon.
  \]
- Given $x \in S_X$, a sequence $\{S_n\}$ of slices of $B_X$, and $\varepsilon > 0$, there is $y^* \in S_{X^*}$ such that $x \in S(B_X, y^*, \varepsilon)$ and
  \[
  \overline{\text{conv}}(\mathbb{T} S(B_X, y^*, \varepsilon)) \cap S_n \neq \emptyset \quad (n \in \mathbb{N}).
  \]

Theorem

$X$ ADP + $B_X$ SCD $\implies$ given $x \in S_X$ and $\varepsilon > 0$, there is $y^* \in S_{X^*}$ such that

- $x \in S(B_X, y^*, \varepsilon)$ and $B_X = \overline{\text{conv}}(\mathbb{T} S(B_X, y^*, \varepsilon))$.
- This clearly implies lushness, and so numerical index 1 (i.e. $\max_{\theta \in \mathbb{T}} \|\text{Id} + \theta T\| = 1 + \|T\|$ for all $T$).
Some consequences

Corollary

- ADP + strongly regular $\implies$ numerical index 1.
- ADP + $X \not\subseteq \ell_1 \implies$ numerical index 1.

Corollary

$X$ real + dim$(X) = \infty$ + ADP $\implies$ $X^* \supseteq \ell_1$.

In particular,

Corollary

$X$ real + dim$(X) = \infty$ + numerical index 1 $\implies$ $X^* \supseteq \ell_1$. 
Section 4

$SCD$ operators
**SCD operators**

**SCD operator**

\[ T \in L(X) \text{ is an SCD-operator if } T(B_X) \text{ is an SCD-set.} \]

**Examples**

\( T \) is an SCD-operator when \( T(B_X) \) is separable and

1. \( T(B_X) \) is RNP,
2. \( T(B_X) \) has no \( \ell_1 \) sequences,
3. \( T \) does not fix copies of \( \ell_1 \)

**Theorem**

- \( X \text{ ADP } + T \text{ SCD-operator } \implies \max_{\theta \in T} \| \text{Id} + \theta T \| = 1 + \| T \| \).
- \( X \text{ DPr } + T \text{ SCD-operator } \implies \| \text{Id} + T \| = 1 + \| T \| \).

**Main corollary**

\( X \text{ ADP } + T \text{ does not fix copies of } \ell_1 \implies \max_{\theta \in T} \| \text{Id} + \theta T \| = 1 + \| T \| \).
HSCD-majorized operators (Kadets-Shepelska, 2010)

HSCD and HSDC-majorized operator

- \( T \in L(X,Y) \) is an **Hereditary-SCD-operator** if every convex subset of \( T(B_X) \) is an SCD-set.
- \( T \in L(X,Y) \) is an **HSCD-majorized operator** if there is \( S \in L(X,Z) \) HSCD-operator such that \( \|Tx\| \leq \|Sx\| \) for every \( x \in X \).

**Proposition**

The class of HSCD-majorized operators is a two-sided operator ideal.

**Theorem**

\( X \text{ DPr } T \in L(X) \text{ HSCD-majorized operator } \implies \|\text{Id} + T\| = 1 + \|T\| \).

**Remark**

The class of operators satisfying (DE) is not even a subspace.
Open problems

Section 5

Open problems
Open questions

1. Find more sufficient conditions for a set to be SCD.

2. Is SCD equivalent to the existence of a countable $\pi$-base for the weak topology?

3. $E$ with $(1)$-unconditional basis. Is $E$ SCD?

4. $E$ with 1-unconditional basis, $\{X_n\}$ a family of SCD spaces. Is $[\bigoplus X_n]_E$ SCD?

5. $X, Y$ SCD. Are $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$ SCD?

6. Find a good extension of the SCD property to the nonseparable case.

7. Clarify the relationship between SCD and the Daugavet property.

8. $X$ ADP, $T \in L(X)$ HSCD-majorized, does $T$ satisfies (aDE)?