The Bishop-Phelps-Bollobás modulus of a Banach space

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M. Chica, V. Kadets, M. Martín, S. Moreno, F. Rambla
The Bishop-Phelps-Bollobás modulus of a Banach space

In preparation
Outline of the talk

1. Introduction
   - Notation
   - The starting point

2. Definition and first properties
   - Definition
   - The upper bound of the modulus
   - Some properties

3. Examples

4. Spaces with the greatest possible value of the modulus

5. Open problems
Introduction

Section 1
**Basic notation**

**$X$** real or complex Banach space.

- $S_X$ unit sphere
- $B_X$ closed unit ball
- $X^*$ dual space
- An element $f \in X^*$ **attains its norm** if

\[
\|f\| = \max\{|f(x)| : x \in B_X\},
\]

that is, there is $x_0 \in B_X$ such that $\|f\| = |f(x_0)|$.

- The above is equivalent to say that $\text{Re} f$ is a **supporting functional** of $B_X$ at $x_0$.
- $\Pi(X) := \{(x, x^*) \in S_X \times S_{X^*} : x^*(x) = 1\}$
## Three theorems and one definition

### James, 1957
Let $X$ be a Banach space. Then

$$X \text{ is reflexive} \iff \text{every element of } X^* \text{ attains its norm}$$

### Bishop-Phelps, 1961
The set of norm-attaining functionals on a Banach space $X$ is dense in $X^*$.

### Bollobás, 1970 (known as Bishop-Phelps-Bollobás theorem)
Let $X$ be a Banach space. Suppose $x \in S_X$ and $x^* \in S_{X^*}$ satisfy

$$|1 - x^*(x)| \leq \frac{\varepsilon^2}{2} \quad (0 < \varepsilon < 1/2).$$

Then there exists $(y, y^*) \in \Pi(X)$ (i.e. $y^*(y) = 1$) such that

$$\|x - y\| < \varepsilon + \frac{\varepsilon^2}{2} \quad \text{and} \quad \|x^* - y^*\| \leq \varepsilon.$$
Three theorems and one definition

Our idea

- Can the result below be improved for concrete Banach spaces?
- That is, for a Banach space $X$, we want to quantify how good or bad is the approximation in Bollobás’ theorem:

**Bishop-Phelps, 1961**

The set of norm-attaining functionals on a Banach space $X$ is dense in $X^*$. 

# Three theorems and one definition

## James, 1957

Let $X$ be a Banach space. Then

$$X \text{ is reflexive } \iff \text{ every element of } X^* \text{ attains its norm}$$

## Bishop-Phelps, 1961

The set of norm-attaining functionals on a Banach space $X$ is dense in $X^*$.

## Bishop-Phelps-Bollobás modulus

Let $X$ be a Banach space. For every $\delta \in (0, 2)$ find the smaller $\varepsilon > 0$ such that whenever $x \in B_X$ and $x^* \in B_{X^*}$ satisfy

$$\Re x^*(x) > 1 - \delta,$$

there exists $(y, y^*) \in \Pi(X)$ (i.e. $y^*(y) = 1$) such that

$$\|x - y\| < \varepsilon \quad \text{and} \quad \|x^* - y^*\| < \varepsilon.$$
Section 2
Definition of the Bishop-Phelps-Bollobás modulus

Bishop-Phelps-Bollobás modulus of a Banach space $X$

It is the function $\Phi_X : (0, 2) \to \mathbb{R}$ defined as

$$\Phi_X(\delta) := \inf \{ \varepsilon > 0 : \forall (x, x^*) \in B_X \times B_{X^*} \text{ with } \text{Re} x^*(x) > 1 - \delta, \exists (y, y^*) \in \Pi(X) \text{ with } \|x - y\| < \varepsilon \text{ and } \|x^* - y^*\| < \varepsilon \}$$

- In other words: if for $\delta \in (0, 2)$ we write
  $$A_X(\delta) := \{(x, x^*) \in B_X \times B_{X^*} : \text{Re} x^*(x) > 1 - \delta\},$$
  it is clear that
  $$\Phi_X(\delta) = \sup_{(x, x^*) \in A_X(\delta)} \inf_{(y, y^*) \in \Pi(X)} \max\{\|x - y\|, \|x^* - y^*\|\}.$$
- Therefore,
  $$\Phi_X(\delta) = d_H\left(A_X(\delta), \Pi(X)\right) \quad (0 < \delta < 2)$$
  where $d_H$ is the Hausdorff distance in $X \oplus_{\infty} X^*$. 

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A remark

\[ \Phi_X(\delta) = \inf \left\{ \varepsilon > 0 : \forall (x, x^*) \in B_X \times B_{X^*} \text{ with } \Re x^*(x) > 1 - \delta, \right. \]
\[ \left. \exists (y, y^*) \in \Pi(X) \text{ with } \|x - y\| < \varepsilon \text{ and } \|x^* - y^*\| < \varepsilon \right\} \]

\[ = \inf \left\{ \varepsilon > 0 : \forall (x, x^*) \in B_X \times B_{X^*} \text{ with } \Re x^*(x) \geq 1 - \delta, \right. \]
\[ \left. \exists (y, y^*) \in \Pi(X) \text{ with } \|x - y\| < \varepsilon \text{ and } \|x^* - y^*\| < \varepsilon \right\} \]

\[ = \inf \left\{ \varepsilon > 0 : \forall (x, x^*) \in B_X \times B_{X^*} \text{ with } \Re x^*(x) > 1 - \delta, \right. \]
\[ \left. \exists (y, y^*) \in \Pi(X) \text{ with } \|x - y\| \leq \varepsilon \text{ and } \|x^* - y^*\| \leq \varepsilon \right\} \]

\[ = \inf \left\{ \varepsilon > 0 : \forall (x, x^*) \in B_X \times B_{X^*} \text{ with } \Re x^*(x) \geq 1 - \delta, \right. \]
\[ \left. \exists (y, y^*) \in \Pi(X) \text{ with } \|x - y\| \leq \varepsilon \text{ and } \|x^* - y^*\| \leq \varepsilon \right\} \]
Three observations

Observation 1
\[ \Phi_X(\delta) \text{ is increasing in } \delta. \]

Observation 2
As a consequence of the Bishop-Phelps-Bollobás theorem, we have
\[ \lim_{\delta \downarrow 0} \Phi_X(\delta) = 0 \]

Observation 3
The smaller is \( \Phi_X(\cdot) \), the better is the approximation in the space \( X \).
The upper bound of the modulus

Theorem

For every Banach space $X$ and every $\delta \in (0, 2)$,

$$\Phi_X(\delta) \leq \sqrt{2\delta}$$

Some comments:

- We prove the result using a lemma by Phelps from 1974.
- Most of the technical main difficulties come from the fact that we approximate elements from $B_X$ and functional from $B_{X^*}$.
- But, on the other hand, this gives a slightly improved version of Bollobás theorem:
The Bishop-Phelps-Bollobás revisited

**Corollary**

Let $X$ be a Banach space.

- Let $0 < \varepsilon < 2$ and suppose that $x \in B_X$ and $x^* \in B_{X^*}$ satisfy
  \[ \text{Re } x^*(x) > 1 - \varepsilon^2 / 2. \]

  Then, there exists $(y, y^*) \in \Pi(X)$ such that
  \[ \|x - y\| < \varepsilon \quad \text{and} \quad \|x^* - y^*\| < \varepsilon. \]

- Let $0 < \delta < 2$ and suppose that $x \in B_X$ and $x^* \in B_{X^*}$ satisfy
  \[ \text{Re } x^*(x) > 1 - \delta. \]

  Then, there exists $(y, y^*) \in \Pi(X)$ such that
  \[ \|x - y\| < \sqrt{2\delta} \quad \text{and} \quad \|x^* - y^*\| < \sqrt{2\delta}. \]
**Some properties**

**Proposition**

The function $\delta \mapsto \Phi_X(\delta)$ is continuous in $(0, 2)$

**Proposition**

\[
\Phi_X(\delta) \leq \Phi_{X^*}(\delta)
\]

- We do not know whether equality holds or not

**Corollary**

If $X$ is reflexive, then $\Phi_X(\delta) = \Phi_{X^*}(\delta)$. 
Examples

Section 3
The one dimensional case

Example

\[ \Phi_{\mathbb{R}}(\delta) = \begin{cases} \delta & \text{if } 0 < \delta \leq 1 \\ \sqrt{\delta - 1} + 1 & \text{if } 1 < \delta < 2 \end{cases} \]
Example

Let $H$ be a Hilbert space, $\dim(H) > 1$,

$$\Phi_H(\delta) \leq \sqrt{\delta} \quad \text{for } 0 < \delta < 2,$$

$$\Phi_H(\delta) = \sqrt{\delta} \quad \text{for } 1 \leq \delta < 2$$
Catching the maximum value of the modulus

**Proposition**

Suppose $X = Y \oplus_1 Z$. Then

$$\Phi_X(\delta) = \sqrt{2\delta} \quad (0 < \delta < 1/2)$$

**Proposition**

Suppose $X = Y \oplus_\infty Z$. Then

$$\Phi_X(\delta) = \sqrt{2\delta} \quad (0 < \delta < 1/2)$$

**Examples**

$$\Phi_X(\delta) = \sqrt{2\delta} \quad (0 < \delta < 1/2)$$

for $X$ equals $c_0$, $l_1$, $l_\infty$, $L_1[0,1]$, $L_\infty[0,1]$...
Catching the maximum value of the modulus II

**Proposition**

Suppose $X^* = Y \oplus_1 Z$ and $Y, Z$ are NOT $w^*$-dense in $X^*$. Then

$$\Phi_X(\delta) = \sqrt{2\delta} \quad (0 < \delta < 1/2)$$

**Corollary**

Suppose $X$ contains two $M$-ideals $J_1$ and $J_2$ with $J_1 \cap J_2 = \{0\}$. Then

$$\Phi_X(\delta) = \sqrt{2\delta} \quad (0 < \delta < 1/2)$$

**Examples**

$$\Phi_X(\delta) = \sqrt{2\delta} \quad (0 < \delta < 1/2)$$

for $X$ equals $C[0,1]$, $C_0(\mathbb{R})$, $C_b(\mathbb{R}^N)$...
A picture of the values of the modulus for some examples
Section 4
A necessary condition...

**Theorem**

Let $X$ be a Banach space. Suppose there is $\delta_0 \in (0, 2)$ such that $\Phi_X(\delta_0) = \sqrt{2}\delta_0$. Then $X^*$ contains an almost isometric copy of the real two-dimensional $\ell_\infty$.

Some comments:

- What we show: $\forall \varepsilon > 0$, $\exists x^*_\varepsilon, y^*_\varepsilon \in S_{X^*}$ with
  \[ \|x^*_\varepsilon + y^*_\varepsilon\| = 2 \quad \text{and} \quad \|x^*_\varepsilon - y^*_\varepsilon\| \geq 2 - \varepsilon. \]

- The proof is rather technical. It is actually an analysis of techniques used in the proof of the Bishop-Phelps theorem, but studying what happens when they give the “worst” possible value.

- In the complex case, it is not possible to get an almost isometric copy of either $\ell_1^2$ or $\ell_\infty^2$, since they are not isometric and both have the greatest possible Bishop-Phelps-Bollobás modulus.
Spaces with the greatest possible value of the modulus

...which is not sufficient

Example

There is a real three-dimensional space $X$ whose dual contains an isometric copy of the two-dimensional $\ell_\infty$ space, but for which

$$\Phi_X(\delta) < \sqrt{2}\delta$$

for every $\delta \in (0, 2)$. 
Open problems

Section 5
Open problems

Problem 1
Is $\Phi_X(\delta)$ equal to $\Phi_{X^*}(\delta)$ for every Banach space?

Problem 2
Calculate $\Phi_H(\delta)$ for a Hilbert space $H$ of dimension greater than one. In particular, is $\Phi_H(\delta) = \sqrt{\delta}$?

Problem 3
Is $\Phi_X(\delta) \geq \sqrt{\delta}$ when $\dim(X) \geq 2$?

Problem 4
Characterize those Banach spaces for which $\Phi_X(\delta) = \sqrt{2\delta}$. 