Numerical index theory

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Mini-course

Kent State University, Spring 2012
Schedule of the talk

1. Basic notation
2. Numerical range of operators
3. Two results on surjective isometries
4. Numerical index of Banach spaces
5. The alternative Daugavet property
6. Lush spaces
7. Slicely countably determined spaces
8. Remarks on the containment of $c_0$ and $\ell_1$
9. Numerical index of $L_p$-spaces
10. Extremely non-complex Banach spaces
Basic notation

### Basic notation I

- **$K$** base field ($\mathbb{R}$ or $\mathbb{C}$):
  - $T$ modulus-one scalars,
  - $\text{Re } z$ real part of $z$ ($\text{Re } z = z$ if $K = \mathbb{R}$).
- **$H$** Hilbert space: $(\cdot | \cdot)$ denotes the inner product.
- **$X$** Banach space:
  - $S_X$ unit sphere, $B_X$ unit ball,
  - $X^*$ dual space,
  - $L(X)$ bounded linear operators,
  - $W(X)$ weakly compact linear operators,
  - $\text{Iso}(X)$ surjective linear isometries,
- **$X$** Banach space, $T \in L(X)$:
  - $\text{Sp}(T)$ spectrum of $T$.
  - $T^* \in L(X^*)$ adjoint operator of $T$. 
Basic notation (II)

$X$ Banach space, $B \subset X$, $C$ convex subset of $X$:

- $B$ is *rounded* if $\mathbb{T}B = B$,
- $\text{co}(B)$ convex hull of $B$,
- $\overline{\text{co}}(B)$ closed convex hull of $B$,
- $\text{aconv}(B) = \text{co}(\mathbb{T}B)$ absolutely convex hull of $B$,
- $\overline{\text{aconv}}(B) = \overline{\text{co}}(\mathbb{T}B)$ absolutely convex hull of $B$,
- $\text{ext}(C)$ extreme points of $C$,
- *slice* of $C$:

$$S(C, x^*, \alpha) = \{x \in C : \text{Re } x^*(x) > \sup \text{Re } x^*(C) - \alpha\}$$

where $x^* \in X^*$ and $0 < \alpha < \sup \text{Re } x^*(C)$. 

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Notation
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where \( x^* \in X^* \) and \( 0 < \alpha < \sup \text{Re} \, x^*(C) \).
Numerical range of operators

2 Numerical range of operators
  • Definitions and first properties
    • Numerical range
    • Numerical radius
    • The Bohnenblust-Karlin theorem
    • The numerical index

F. F. Bonsall and J. Duncan

*Numerical Ranges. Vol I and II.*

Hilbert space numerical range (Toeplitz, 1918)

- A \( n \times n \) real or complex matrix
  \[
  W(A) = \{(Ax \mid x) : x \in \mathbb{K}^n, (x \mid x) = 1\}.
  \]

- \( H \) real or complex Hilbert space, \( T \in L(H) \),
  \[
  W(T) = \{(Tx \mid x) : x \in H, \|x\| = 1\}.
  \]
Numerical range: Hilbert spaces

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Remark

- Given $T \in L(H)$ we associate
  - a sesquilinear form $\varphi_T(x, y) = (Tx \mid y)$ $(x, y \in H)$,
  - a quadratic form $\widehat{\varphi}_T(x) = \varphi_T(x, x) = (Tx \mid x)$ $(x \in H)$.
- Then, $W(T) = \widehat{\varphi}_T(S_H)$. 
Numerical range: Hilbert spaces

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★ Given \( T \in L(H) \) we associate

- a sesquilinear form \( \varphi_T(x,y) = (Tx \mid y) \) \( (x,y \in H) \),
- a quadratic form \( \widehat{\varphi}_T(x) = \varphi_T(x,x) = (Tx \mid x) \) \( (x \in H) \).

★ Then, \( W(T) = \widehat{\varphi}_T(S_H) \). Therefore:

- \( \widehat{\varphi}_T(B_H) = [0,1]W(T) \),
- \( \widehat{\varphi}_T(H) = \mathbb{R}^+W(T) \).
- But we cannot get \( W(T) \) from \( \widehat{\varphi}_T(B_H) \)!

Some properties

H \in \text{Hilbert space}, T \in L(H): (Toeplitz-Hausdorff) \quad W(T) is convex.

T, S \in L(H), \alpha, \beta \in K:

W(\alpha T + \beta S) \subseteq \alpha W(T) + \beta W(S)

W(U^* T U) = W(T) for every T \in L(H) and every U unitary.

Sp(T) \subseteq W(T).

If T is normal, then W(T) = \text{co Sp}(T).

In the real case (\dim(H) > 1), there is T \in L(H), T \neq 0 with W(T) = \{0\}.

In the complex case, \sup\{|(Tx)|: x \in \mathcal{S}H\} \geq \frac{1}{2} \|T\|.

If T is actually self-adjoint, then \sup\{|(Tx)|: x \in \mathcal{S}H\} = \|T\|.
Some properties

$H$ Hilbert space, $T \in L(H)$:

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**H** Hilbert space, **T** ∈ **L**(H):

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**H** Hilbert space, \( T \in L(H) \):
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**H** Hilbert space, **T** ∈ **L**(H):
- (Toeplitz-Hausdorff) **W**(T) is convex.
- **T**, **S** ∈ **L**(H), **α**, **β** ∈ **I**:K:
  - **W**(α**T** + β**S**) ⊆ α**W**(T) + β**W**(S);
  - **W**(α**Id** + β**S**) = α + β**W**(S).
**Some properties**

*H* Hilbert space, *T* ∈ *L(H)*:

- (Toeplitz-Hausdorff) *W(T)* is convex.
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- \( W(U^*TU) = W(T) \) for every \( T \in L(H) \) and every \( U \) unitary.
- \( \text{Sp}(T) \subseteq \overline{W(T)} \).

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6. In the real case (\( \dim(H) > 1 \)), there is \( T \in L(H), T \neq 0 \) with \( W(T) = \{0\} \).
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- $W(U^* TU) = W(T)$ for every $T \in L(H)$ and every $U$ unitary.
- $\text{Sp}(T) \subseteq \overline{W(T)}$.
- If $T$ is normal, then $\overline{W(T)} = \overline{\text{coSp}(T)}$.
- In the real case ($\dim(H) > 1$), there is $T \in L(H)$, $T \neq 0$ with $W(T) = \{0\}$.
- In the complex case,
  \[
  \sup\{|(Tx \mid x)| : x \in S_H\} \geq \frac{1}{2} \|T\|.
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$H$ complex Hilbert space, $T \in L(H)$, then

$$v(T) := \sup \{|(Tx \mid x)| : x \in S_H\} \geq \frac{1}{2} \|T\|.$$
Proving a result

For \( x, y \in S_H \) fixed, use the polarization formula:

\[
(Tx \mid y) = \frac{1}{4} \left[ (T(x + y) \mid x + y) - (T(x - y) \mid x - y) + i(T(x + iy) \mid x + iy) - i(T(x - iy) \mid x - iy) \right].
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- $|(Tx \mid y)| \leq \frac{1}{4} v(T) [\|x + y\|^2 + \|x - y\|^2 + \|x + iy\|^2 + \|x - iy\|^2].$
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By the parallelogram’s law:

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| (Tx \mid y) | \leq \frac{1}{4} v(T) \left[ 2\| x \|^2 + 2\| y \|^2 + 2\| x \|^2 + 2\| iy \|^2 \right] = 2v(T).
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**Proving a result**

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- We just take supremum on $x, y \in S_H$
Some reasons to study numerical ranges

- It gives a "picture" of the matrix/operator which allows to "see" many properties (algebraic or geometrical) of the matrix/operator.
- It is a comfortable way to study the spectrum.
- It is useful to estimate spectral radii of small perturbations of matrices.
- It is useful to work with some concepts like hermitian operator, skew-hermitian operator, dissipative operator,...

Example

Consider $A = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ \varepsilon & 0 \end{pmatrix}$.

$\text{Sp}(A) = \{0\}$,

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$\text{Sp}(A + B) = \{\pm \sqrt{M \varepsilon}\} \subseteq W(A + B) \subseteq W(A) + W(B)$,

so the spectral radius of $A + B$ is bounded above by $\frac{1}{2}(|M| + |\varepsilon|)$. 


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Numerical range: Banach spaces (I)

Banach spaces numerical range (Bauer 1962; Lumer, 1961)

$X$ Banach space, $T \in L(X)$,

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Sp$(T) \subseteq V(T)$.

Actually, $\text{co} \ Sp(T) \subseteq V(T)$.

$\text{co} \ Sp(T) = \bigcap \{ V_p(T) : p \text{ equivalent norm} \}$

where $V_p(T)$ is the numerical range of $T$ in the Banach space $(X, p)$.

$V(U^{-1}TU) = V(T)$ for every $T \in L(X)$ and every $U \in \text{Iso}(X)$. 

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- \( \overline{\text{co}} \text{Sp}(T) = \bigcap \{ \overline{V_p(T)} : p \text{ equivalent norm} \} \)
  where \( V_p(T) \) is the numerical range of \( T \) in the Banach space \( (X, p) \).
Numerical range of operators
Definitions and first properties

Numerical range: Banach spaces (I)

Banach spaces numerical range (Bauer 1962; Lumer, 1961)

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- \( V(U^{-1}TU) = V(T) \) for every \( T \in L(X) \) and every \( U \in \text{Iso}(X) \).
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The numerical range as a derivative

$X$ Banach space, $T \in L(X)$. Then

$$\sup \Re V(T) = \lim_{\alpha \to 0^*} \frac{\|\text{Id} + \alpha T\| - 1}{\alpha}$$

i.e. $\sup \Re V(T)$ is the derivative of the norm at $\text{Id}$ in the direction of $T$. 
Numerical range of operators
Definitions and first properties

**Numerical range: Banach spaces (II)**

**Banach spaces numerical range (Bauer 1962; Lumer, 1961)**

\[ X \text{ Banach space, } T \in L(X), \]
\[ V(T) = \{ x^*(Tx) : x^* \in S_{X^*}, x \in S_X, x^*(x) = 1 \} \]

**The numerical range as a derivative**

\[ X \text{ Banach space, } T \in L(X). \text{ Then} \]
\[ \sup \operatorname{Re} V(T) = \lim_{\alpha \to 0^*} \frac{\| \text{Id} + \alpha T \| - 1}{\alpha} \]

\[ \text{i.e. } \sup \operatorname{Re} V(T) \text{ is the derivative of the norm at Id in the direction of } T. \]

**Consequence**

\[ X \text{ Banach space, } T \in L(X). \text{ Then } \overline{\operatorname{co}}(V(T)) = \overline{\operatorname{co}}(V(T^*)). \]
Numerical range: Banach spaces (II)

Banach spaces numerical range (Bauer 1962; Lumer, 1961)

Let $X$ be a Banach space and $T \in L(X)$. Then

$$V(T) = \{x^*(Tx) : x^* \in S_{X^*}, \ x \in S_X, \ x^*(x) = 1\}$$

The numerical range as a derivative

Let $X$ be a Banach space and $T \in L(X)$. Then

$$\sup \text{Re} \ V(T) = \lim_{\alpha \to 0^*} \frac{\|\text{Id} + \alpha T\| - 1}{\alpha}$$

i.e. $\sup \text{Re} \ V(T)$ is the derivative of the norm at $\text{Id}$ in the direction of $T$.

Consequence

Let $X$ be a Banach space and $T \in L(X)$. Then

$$\overline{\text{co}}(V(T)) = \overline{\text{co}}(V(T^*))$$

Stronger result (Bollobás, 1970)

Let $X$ be a Banach space and $T \in L(X)$. Then

$$V(T) \subseteq V(T^*) \subseteq \overline{V(T)}.$$
Observation

The numerical range depends on the base field:

$X$ complex Banach space $\implies X_{\mathbb{R}}$ real space underlying $X$.

$T \in \mathcal{L}(X) \implies T_{\mathbb{R}} \in \mathcal{L}(X_{\mathbb{R}})$ is $T$ viewed as a real operator.

Then $V(T_{\mathbb{R}}) = \Re V(T)$.

Consequence:

$X$ complex, then there is $S \in \mathcal{L}(X_{\mathbb{R}})$ with $\|S\| = 1$ and $V(S) = \{0\}$. 
Observation

The numerical range depends on the base field:

- $X$ complex Banach space $\Rightarrow X_\mathbb{R}$ real space underlying $X$. 
Observation

The numerical range depends on the base field:

- $X$ complex Banach space $\implies X_\mathbb{R}$ real space underlying $X$.
- $T \in L(X) \implies T_\mathbb{R} \in L(X_\mathbb{R})$ is $T$ view as a real operator.
Observation

The numerical range depends on the base field:

- $X$ complex Banach space $\implies X_\mathbb{R}$ real space underlying $X$.
- $T \in L(X) \implies T_\mathbb{R} \in L(X_\mathbb{R})$ is $T$ view as a real operator.
- Then $V(T_\mathbb{R}) = \text{Re} V(T)$. 
Observation

The numerical range depends on the base field:

- $X$ complex Banach space $\implies X_\mathbb{R}$ real space underlying $X$.
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- Then $V(T_\mathbb{R}) = \text{Re} V(T)$.

Consequence:

$X$ complex, then there is $S \in L(X_\mathbb{R})$ with $\|S\| = 1$ and $V(S) = \{0\}$. 
Some motivation for the numerical range

It allows to carry to the general case the concepts of hermitian operator, skew-hermitian operator, dissipative operators, etc.

It gives a description of the Lie algebra corresponding to the Lie group of all onto isometries on the space.

It gives an easy and quantitative proof of the fact that \( \text{Id} \) is an strongly extreme point of \( B(X) \) (MLUR point).
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Some motivation for the numerical range

- It allows to carry to the general case the concepts of hermitian operator, skew-hermitian operator, dissipative operators...
- It gives a description of the Lie algebra corresponding to the Lie group of all onto isometries on the space.
- It gives an easy and quantitative proof of the fact that \( \text{Id} \) is an strongly extreme point of \( B_{L(X)} \) (MLUR point).
Numerical radius: definition and properties

Let $X$ be a real or complex Banach space, $T \in \mathcal{L}(X)$, and $v(T) = \sup \{ |\lambda| : \lambda \in \sigma(T) \}$,
where $\sigma(T)$ is the spectrum of $T$.

**Elementary properties**

- For every $T, S \in \mathcal{L}(X)$,
  $$v(T + S) \leq v(T) + v(S).$$
- For every $\lambda \in K$ and $T \in \mathcal{L}(X)$,
  $$v(\lambda T) = |\lambda| v(T).$$
- For every $U \in \text{Iso}(X)$,
  $$v(U^{-1} T U) = v(T).$$
- For every $T \in \mathcal{L}(X)$,
  $$v(T^*) = v(T).$$
Numerical radius: definition and properties

**Numerical radius**

Let $X$ be a real or complex Banach space, $T \in L(X)$,

\[ v(T) = \sup \{|\lambda| : \lambda \in V(T)\} \]

\[ = \sup \{|x^*(Tx)| : x^* \in S_{X^*}, x \in S_X, x^*(x) = 1\} \]
# Numerical radius: definition and properties

## Numerical radius

\( X \) real or complex Banach space, \( T \in L(X) \),

\[
\nu(T) = \sup \{ |\lambda| : \lambda \in V(T) \} \\
= \sup \{ |x^*(Tx)| : x^* \in S_{X^*}, x \in S_X, x^*(x) = 1 \}
\]

## Elementary properties

\( X \) Banach space, \( T \in L(X) \)

- \( \nu(\cdot) \) is a seminorm, i.e.
  - \( \nu(T + S) \leq \nu(T) + \nu(S) \) for every \( T, S \in L(X) \).
Numerical radius: definition and properties

**Numerical radius**

Let $X$ be a real or complex Banach space, $T \in L(X)$,

$$v(T) = \sup \{|\lambda| : \lambda \in V(T)\}$$

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**Elementary properties**

Let $X$ be a Banach space, $T \in L(X)$

- $v(\cdot)$ is a seminorm, i.e.
  - $v(T + S) \leq v(T) + v(S)$ for every $T, S \in L(X)$.
  - $v(\lambda T) = |\lambda| v(T)$ for every $\lambda \in \mathbb{K}$, $T \in L(X)$. 
Numerical radius: definition and properties

**Numerical radius**

Let $X$ be a real or complex Banach space, and let $T \in L(X)$. The numerical radius of $T$ is defined as

$$v(T) = \sup \{ |\lambda| : \lambda \in V(T) \}$$

$$= \sup \{ |x^* (Tx)| : x^* \in S_{X^*}, \ x \in S_X, \ x^*(x) = 1 \}$$

**Elementary properties**

Let $X$ be a Banach space, and let $T \in L(X)$. The numerical radius $v(\cdot)$ satisfies the following properties:

- $v(\cdot)$ is a seminorm, i.e.
  - $v(T + S) \leq v(T) + v(S)$ for every $T, S \in L(X)$.
  - $v(\lambda T) = |\lambda| v(T)$ for every $\lambda \in \mathbb{K}$, $T \in L(X)$.
- $\sup |\text{Sp}(T)| \leq v(T)$.
Numerical radius: definition and properties

Numerical radius

$X$ real or complex Banach space, $T \in L(X)$,

$$v(T) = \sup \{ |\lambda| : \lambda \in V(T) \}$$

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Elementary properties

$X$ Banach space, $T \in L(X)$

- $v(\cdot)$ is a seminorm, i.e.
  - $v(T + S) \leq v(T) + v(S)$ for every $T, S \in L(X)$.
  - $v(\lambda T) = |\lambda| v(T)$ for every $\lambda \in \mathbb{K}, T \in L(X)$.

- $\sup \left| \text{Sp}(T) \right| \leq v(T)$.

- $v(U^{-1}TU) = v(T)$ for every $U \in \text{Iso}(X)$.  


### Numerical radius

**X** real or complex Banach space, **T** ∈ **L**(**X**),

\[ v(T) = \sup \{ |\lambda| : \lambda \in V(T) \} \]

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### Elementary properties

**X** Banach space, **T** ∈ **L**(**X**)

- *v(·)* is a seminorm, i.e.
  - \( v(T + S) \leq v(T) + v(S) \) for every **T**, **S** ∈ **L**(**X**).
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- \( \sup |\text{Sp}(T)| \leq v(T) \).
- \( v(U^{-1}TU) = v(T) \) for every **U** ∈ Iso(**X**).
- \( v(T^*) = v(T) \).
Numerical radius: examples

Some examples

1. Let $H$ be a real Hilbert space with dimension $\dim(H) > 1$. Then there exists $T \in L(X)$ with $v(T) = 0$ and $\|T\| = 1$.

2. Let $H$ be a complex Hilbert space with dimension $\dim(H) > 1$. Then $v(T) \geq \frac{1}{2}\|T\|$, where the constant $\frac{1}{2}$ is optimal.

3. Let $X = L^1(\mu)$, then $v(T) = \|T\|$ for every $T \in L(X)$.

4. Let $X^* \equiv L^1(\mu)$, then $v(T) = \|T\|$ for every $T \in L(X)$.

5. In particular, this is the case for $X = C(K)$. 

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Numerical range of operators  Definitions and first properties

Numerical radius: examples

Some examples

1. $H$ real Hilbert space $\dim(H) > 1$ implies that there exists $T \in L(X)$ with $v(T) = 0$ and $\|T\| = 1$. 

2. For $X = L^1(\mu)$, we have $v(T) = \|T\|$ for every $T \in L(X)$. 

3. In particular, this is the case for $X = C(K)$. 

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Numerical range of operators  Definitions and first properties

Numerical radius: examples

Some examples

1. \( H \) real Hilbert space \( \dim(H) > 1 \)
   \[ \implies \exists T \in L(X) \text{ with } v(T) = 0 \text{ and } \|T\| = 1. \]

2. \( H \) complex Hilbert space \( \dim(H) > 1 \)
Some examples

1. $H$ real Hilbert space $\dim(H) > 1$
   $\implies$ exist $T \in L(X)$ with $v(T) = 0$ and $\|T\| = 1$.

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Numerical range of operators
Definitions and first properties

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Numerical range of operators
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Numerical range of operators
Definitions and first properties

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Numerical range of operators  Definitions and first properties

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4. $X^* \equiv L_1(\mu) \implies v(T) = \|T\|$ for every $T \in L(X)$.

5. In particular, this is the case for $X = C(K)$. 
Proving a result

\[ X = C(K) \implies \nu(T) = \|T\| \text{ for every } T \in L(X). \]
Proving a result

\[ X = \mathcal{C}(K) \quad \Rightarrow \quad v(T) = \|T\| \text{ for every } T \in L(X). \]

- Fix \( T \in L(C(K)) \). Find \( f_0 \in S_{\mathcal{C}(K)} \) and \( \xi_0 \in K \) such that \( |[Tf_0](\xi_0)| \sim \|T\| \).
Proving a result

\[ X = C(K) \implies \nu(T) = \|T\| \text{ for every } T \in L(X). \]

- Fix \( T \in L(C(K)). \) Find \( f_0 \in S_{C(K)} \) and \( \xi_0 \in K \) such that \( |[Tf_0](\xi_0)| \sim \|T\|. \)

If \( f_0(\xi_0) \sim 1 \), then we were done. This our goal.
Proving a result

\[ X = C(K) \implies v(T) = \|T\| \text{ for every } T \in L(X). \]

- Fix \( T \in L(C(K)) \). Find \( f_0 \in S_{C(K)} \) and \( \xi_0 \in K \) such that \( |[Tf_0](\xi_0)| \sim \|T\| \).

- Consider the non-empty open set
  \[ V = \{ \xi \in K : f_0(\xi) \sim f_0(\xi_0) \} \]
  and find \( \varphi : K \rightarrow [0, 1] \) continuous with \( \text{supp}(\varphi) \subset V \) and \( \varphi(\xi_0) = 1 \).
Proving a result

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- Write \( f_0(\xi_0) = \lambda \omega_1 + (1 - \lambda) \omega_2 \) with \( |\omega_i| = 1 \), and consider the functions
  \[ f_i = (1 - \varphi)f_0 + \varphi \omega_i \text{ for } i = 1, 2. \]
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  \[ f_i = (1 - \varphi)f_0 + \varphi \omega_i \text{ for } i = 1, 2. \]

- Then, \( f_i \in C(K), \|f_i\| \leq 1 \), and
  \[ \|f_0 - (\lambda f_1 + (1 - \lambda)f_2)\| = \|\varphi f_0 - \varphi f_0(\xi_0)\| \sim 0. \]
Proving a result

\[ X = C(K) \implies v(T) = \|T\| \text{ for every } T \in L(X). \]

- Fix \( T \in L(C(K)). \) Find \( f_0 \in S_{C(K)} \) and \( \xi_0 \in K \) such that \( |[Tf_0](\xi_0)| \sim \|T\|. \)

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- Then, \( f_i \in C(K), \|f_i\| \leq 1, \) and
  \[ \|f_0 - (\lambda f_1 + (1 - \lambda)f_2)\| = \|\varphi f_0 - \varphi f_0(\xi_0)\| \sim 0. \]

- Therefore, there is \( i \in \{1, 2\} \) such that \( |[T(f_i)](\xi_0)| \sim \|T\|, \) but now \( |f_i(\xi_0)| = 1. \)
Proving a result

\[ X = C(K) \implies v(T) = \|T\| \text{ for every } T \in L(X). \]

- Fix \( T \in L(C(K)). \) Find \( f_0 \in S_{C(K)} \) and \( \xi_0 \in K \) such that \( |Tf_0(\xi_0)| \sim \|T\|. \)

- Consider the non-empty open set
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- Write \( f_0(\xi_0) = \lambda \omega_1 + (1 - \lambda) \omega_2 \) with \( |\omega_i| = 1 \), and consider the functions
  \[ f_i = (1 - \varphi)f_0 + \varphi \omega_i \text{ for } i = 1, 2. \]

- Then, \( f_i \in C(K), \|f_i\| \leq 1, \text{ and} \)
  \[ \|f_0 - (\lambda f_1 + (1 - \lambda) f_2)\| = \|\varphi f_0 - \varphi f_0(\xi_0)\| \sim 0. \]

- Therefore, there is \( i \in \{1, 2\} \) such that \( |T(f_i)(\xi_0)| \sim \|T\|, \) but now \( |f_i(\xi_0)| = 1. \)

- Equivalently,
  \[ |\delta_{\xi_0}(T(f_i))| \sim \|T\| \quad \text{and} \quad |\delta_{\xi_0}(f_i)| = 1, \]
  meaning that \( v(T) \sim \|T\|. \) \( \checkmark \)

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If \( X = L_1(\mu) \), then \( X^* \cong C(\mathbb{R}^+). \) Therefore, \( v(T) = v(T^*) = \|T^*\| = \|T\|. \) Therefore, \( v(T) \sim \|T\|. \) \( \checkmark \)
Proving a result

\[ X = C(K) \implies \nu(T) = \|T\| \text{ for every } T \in L(X). \]

- Fix \( T \in L(C(K)). \) Find \( f_0 \in S_{C(K)} \) and \( \xi_0 \in K \) such that \( |[T f_0](\xi_0)| \sim \|T\|. \)

- Consider the non-empty open set
  \[ V = \{ \xi \in K : f_0(\xi) \sim f_0(\xi_0) \} \]
  and find \( \varphi : K \to [0, 1] \) continuous with \( \text{supp}(\varphi) \subset V \) and \( \varphi(\xi_0) = 1. \)

- Write \( f_0(\xi_0) = \lambda \omega_1 + (1 - \lambda) \omega_2 \) with \( |\omega_i| = 1 \), and consider the functions
  \[ f_i = (1 - \varphi) f_0 + \varphi \omega_i \text{ for } i = 1, 2. \]

- Then, \( f_i \in C(K), \|f_i\| \leq 1 \), and
  \[ \|f_0 - (\lambda f_1 + (1 - \lambda) f_2)\| = \|\varphi f_0 - \varphi f_0(\xi_0)\| \sim 0. \]

- Therefore, there is \( i \in \{1, 2\} \) such that \( |[T(f_i)](\xi_0)| \sim \|T\|, \) but now \( f_i(\xi_0) = 1. \)

- Equivalently,
  \[ |\delta_{\xi_0}(T(f_i))| \sim \|T\| \quad \text{and} \quad |\delta_{\xi_0}(f_i)| = 1, \]
  meaning that \( \nu(T) \sim \|T\|. \)

\[ \text{If } X = L_1(\mu), \text{ then } X^* \equiv C(K_\mu). \text{ Therefore, } \nu(T) = \nu(T^*) = \|T^*\| = \|T\|. \]

Differences between real and complex spaces

Example

$X$ complex Banach space, define $T \in L(X)$ by $T(x) = ix$ ($x \in X$).

$\|T\| = 1$ and $v(T) = \emptyset$ if viewed in $X_R$.

$\|T\| = 1$ and $V(T) = \{i\}$, so $v(T) = 1$ if viewed in (complex) $X$.

Theorem (Bohnenblust-Karlin, 1955; Glickfeld, 1970)

$X$ complex Banach space, $T \in L(X)$:

$v(T) \geq 1$ $e^{\|T\|}$.

The constant $1$ $e$ is optimal: $\exists X$ two-dimensional complex, $\exists T \in L(X)$ with $\|T\| = e$ and $v(T) = 1$. 
Example

$X$ complex Banach space, define $T \in L(X_{\mathbb{R}})$ by

$$T(x) = i x \quad (x \in X).$$

- $\|T\| = 1$ and $v(T) = 0$ if viewed in $X_{\mathbb{R}}$.
- $\|T\| = 1$ and $V(T) = \{i\}$, so $v(T) = 1$ if viewed in (complex) $X$. 

Theorem (Bohnenblust-Karlin, 1955; Glickfeld, 1970)

$X$ complex Banach space, $T \in L(X_{\mathbb{R}})$:

$$v(T) \geq 1 e^{\|T\|}.$$ 

The constant $1 e^{\|T\|}$ is optimal:

$\exists X$ two-dimensional complex, $\exists T \in L(X_{\mathbb{R}})$ with $\|T\| = e$ and $v(T) = 1$. 

Differences between real and complex spaces

Example

\( X \) complex Banach space, define \( T \in L(X_R) \) by

\[ T(x) = ix \quad (x \in X). \]

- \( \|T\| = 1 \) and \( v(T) = 0 \) if viewed in \( X_R \).
- \( \|T\| = 1 \) and \( V(T) = \{i\} \), so \( v(T) = 1 \) if viewed in (complex) \( X \).

Theorem (Bohnenblust-Karlin, 1955; Glickfeld, 1970)

\( X \) complex Banach space, \( T \in L(X) \):

\[ v(T) \geq \frac{1}{e} \|T\|. \]

The constant \( \frac{1}{e} \) is optimal:

\[ \exists X \text{ two-dimensional complex}, \exists T \in L(X) \text{ with } \|T\| = e \text{ and } v(T) = 1. \]
The exponential function

Let $X$ be a Banach space, $T \in L(X)$, define

$$\exp(T) = \sum_{n=0}^{\infty} \frac{1}{n!} T^n.$$
Proof of Bohnenblust-Karlin’s theorem. Preliminaries

The exponential function

$X$ Banach space, $T \in L(X)$, define $\exp(T) = \sum_{n=0}^{\infty} \frac{1}{n!} T^n$.

First properties

$X$ Banach space, $T, S \in L(X)$.

- $TS = ST \implies \exp(T + S) = \exp(T) \exp(S)$.
- $\exp(T) \exp(-T) = \exp(0) = \text{Id} \implies \exp(T)$ surjective isomorphism.
- $\{\exp(\rho T) : \rho \in \mathbb{R}_0^+\}$ one-parameter semigroup generated by $T$.
- $\|\exp(T)\| \leq e^{\|T\|}$ (we will improve this inequality in the sequel).
Proof of Bohnenblust-Karlin’s theorem. Preliminaries

The exponential function

$X$ Banach space, $T \in L(X)$, define $\exp(T) = \sum_{n=0}^{\infty} \frac{1}{n!} T^n$.

First properties

$X$ Banach space, $T, S \in L(X)$.

- $TS = ST \implies \exp(T + S) = \exp(T) \exp(S)$.
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- $\{\exp(\rho T) : \rho \in \mathbb{R}_0^+\}$ one-parameter semigroup generated by $T$.
- $\|\exp(T)\| \leq e^\|T\|$ (we will improve this inequality in the sequel).

Exponential formula

$X$ Banach, $T \in L(X)$, then $\|\exp(\zeta T)\| \leq e^{\|\zeta\|v(T)}$ for every $\zeta \in \mathbb{K}$. 
Proof of Bohnenblust-Karlin’s theorem. Preliminaries

The exponential function

\[ \exp(T) = \sum_{n=0}^{\infty} \frac{1}{n!} T^n. \]

First properties

\[ X \text{ Banach, } T, S \in L(X), \quad TS = ST \Rightarrow \exp(T+S) = \exp(T) \exp(S). \]

\[ \exp(T) \exp(-T) = \exp(0) = \text{Id} \Rightarrow \exp(T) \text{ surjective isomorphism.} \]

\[ \{ \exp(\rho T) : \rho \in \mathbb{R}^+ \} \text{ one-parameter semigroup generated by } T. \]

\[ \|\exp(T)\| \leq e\|T\| (\text{we will improve this inequality in the sequel}). \]

Exponential formula

\[ X \text{ Banach, } T \in L(X), \text{ then } \|\exp(\zeta T)\| \leq e^{\|\zeta\|v(T)} \text{ for every } \zeta \in \mathbb{K}. \]
Proof of Bohnenblust-Karlin’s theorem. Preliminaries

For $\alpha > 0$ and $T \in L(X)$,

$$ e^{1/\alpha} \| \exp(T) \| = \left\| \exp \left( \frac{1}{\alpha} \text{Id} + T \right) \right\| \leq \exp \left( \left\| \frac{1}{\alpha} \text{Id} + T \right\| \right). $$

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$X$ Banach, $T \in L(X)$, then $\| \exp(\zeta T) \| \leq e^{\| \zeta \| v(T)}$ for every $\zeta \in \mathbb{K}$. 
Proof of Bohnenblust-Karlin’s theorem. Preliminaries

**Proof**

- For \( \alpha > 0 \) and \( T \in L(X) \),
  
  \[
  e^{1/\alpha} \| \exp(T) \| = \left\| \exp \left( \frac{1}{\alpha} \Id + T \right) \right\| \leq \exp \left( \left\| \frac{1}{\alpha} \Id + T \right\| \right).
  \]

- Therefore,
  
  \[
  \| \exp(T) \| \leq \exp \left( \frac{\| \Id + \alpha T \| - 1}{\alpha} \right).
  \]

**Exponential formula**

- For every \( \zeta \in \mathbb{K} \) and \( T \in L(X) \),
  
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  \| \exp(\zeta T) \| \leq e^{\| \zeta \| \nu(T)} \]
Proof of Bohnenblust-Karlin’s theorem. Preliminaries

**Proof**

- For $\alpha > 0$ and $T \in L(X)$,
  
  \[ e^{1/\alpha} \| \exp(T) \| = \left\| \exp \left( \frac{1}{\alpha} \operatorname{Id} + T \right) \right\| \leq \exp \left( \left\| \frac{1}{\alpha} \operatorname{Id} + T \right\| \right). \]

- Therefore,
  
  \[ \| \exp(T) \| \leq \exp \left( \frac{\| \operatorname{Id} + \alpha T \| - 1}{\alpha} \right). \]

- Taking limit with $\alpha \to 0^+$, we get
  
  \[ \| \exp(T) \| \leq \exp \left( \sup \Re V(T) \right) \leq e^{v(T)} \]

  and the result follows.

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Proof of Bohnenblust-Karlin’s theorem. Preliminaries

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- Actually, $\| \exp(T) \| \leq e^{\sup \Re V(T)} \leq e^{\nu(T)}$. 
Proof of Bohnenblust-Karlin’s theorem

**Theorem**

X complex Banach space, \( T \in L(X) \). Then \( \|T\| \leq e^{v(T)} \).
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Consider $f(\zeta) = \exp(\zeta T)$ ($\zeta \in \mathbb{C}$) which is an entire function.
Proof of Bohnenblust-Karlin’s theorem

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**Proof.**

Consider \(f(\zeta) = \exp(\zeta T)\) (\(\zeta \in \mathbb{C}\)) which is an entire function.

- If \(v(T) = 0\), then \(\|f(\zeta)\| \leq \exp(|\zeta|v(T)) \leq 1\)
  
  [Liouville’s theorem] \(\implies f\) is constant, so \(T = f'(0) = 0\).
Proof of Bohnenblust-Karlin’s theorem

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Consider $f(\zeta) = \exp(\zeta T)$ ($\zeta \in \mathbb{C}$) which is an entire function.

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- Now, it is enough to show that $v(T) = 1$ implies $\|T\| \leq e$.
- Indeed, by Cauchy integral formula

$$T = f'(0) = \frac{1}{2\pi i} \int_{C(0,1)} \frac{f(\zeta)}{\zeta^2} d\zeta.$$
Proof of Bohnenblust-Karlin’s theorem

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- Therefore,

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\|T\| \leq \frac{1}{2\pi} \int_{C(0,1)} \|\exp(\zeta T)\| d\zeta \leq \frac{1}{2\pi} \int_{C(0,1)} e^{|\zeta|v(T)} d\zeta = e
\]

and we are done.
Numerical index: definition and properties

Let $X$ be a real or complex Banach space. The numerical index $n(X)$ is defined by:

$$n(X) = \max \{ k \geq 0 : k \|T\| \leq v(T) \forall T \in \mathcal{L}(X) \} = \inf \{ v(T) : T \in \mathcal{L}(X), \|T\| = 1 \}.$$

**Elementary properties**

- In the real case, $0 \leq n(X) \leq 1$.
- In the complex case, $\frac{1}{e} \leq n(X) \leq 1$.
- Actually, the above inequalities are best possible:
  - $\{ n(X) : X$ complex Banach space $\} = \left[ e^{-1}, 1 \right]$,
  - $\{ n(X) : X$ real Banach space $\} = \left[ 0, 1 \right]$.

**Equivalent norms**

- $v$ norm on $\mathcal{L}(X)$ equivalent to the given norm $\iff n(X) > 0$.
- $v(T) = \|T\|$ for every $T \in \mathcal{L}(X) \iff n(X) = 1$.

- $n(X^*) \leq n(X)$. 


# Numerical index: definition and properties

## Numerical index

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### Numerical index: definition and properties

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Numerical index: definition and properties

Numerical index

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Numerical index: definition and properties

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Numerical range of operators  Definitions and first properties

Numerical index: definition and properties

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Numerical index: examples

Some examples

1. Hilbert, \( \dim(H) > 1 \):
   - \( n(H) = \begin{cases} 0 & \text{real case}, \\ 1 & \text{complex case}. \end{cases} \)

2. Complex space \( X \):
   - \( n(X) \mathbb{R} = 0 \).

3. \( n(L^1(\mu)) = 1 \), \( \mu \) positive measure.

4. \( X^* \equiv L^1(\mu) \Rightarrow n(X) = 1 \).

5. In particular,
   - \( n(C(K)) = 1 \),
   - \( n(C_0(L)) = 1 \),
   - \( n(L^\infty(\mu)) = 1 \).

6. \( n(A(D)) = 1 \) and \( n(H_\infty) = 1 \).
Some examples

1. $H$ Hilbert, $\dim(H) > 1$:

$$n(H) = \begin{cases} 
0 & \text{real case,} \\
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Numerical range of operators
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Some examples

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Some examples

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Numerical index: examples
Two results on surjective isometries

- Numerical ranges and isometries
- Isometries on finite-dimensional spaces
- Isometries and duality

M. Martín
The group of isometries of a Banach space and duality.

M. Martín, J. Merí, and A. Rodríguez-Palacios.
Finite-dimensional spaces with numerical index zero.

H. P. Rosenthal
The Lie algebra of a Banach space.
Semigroups of isometries: motivating example

A motivating example

A real or complex $n \times n$ matrix. TFAE:

- $A$ is skew-adjoint (i.e. $A^* = -A$).

- $B = \exp(\rho A)$ is unitary for every $\rho \in \mathbb{R}$ (i.e. $B^*B = BB^* = \text{Id}$).
A motivating example

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- $\text{Re}(Ax \mid x) = 0$ for every $x \in H$.
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In term of Hilbert spaces

$H$ ($n$-dimensional) Hilbert space, $T \in L(H)$. TFAE:
- $\text{Re} \, W(T) = \{0\}$.
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Semigroups of isometries: motivating example

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For general Banach spaces

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Semigroups of isometries: characterization

**Theorem (Bonsall-Duncan, 1970’s; Rosenthal, 1984)**

Let $X$ be a real or complex Banach space, $T \in L(X)$. TFAE:

- $\text{Re} \ V(T) = \{0\}$ (\textit{T is skew-hermitian}).
- $\| \exp(\rho T) \| \leq 1$ for every $\rho \in \mathbb{R}$.
- $\{ \exp(\rho T) : \rho \in \mathbb{R}_0^+ \} \subset \text{Iso}(X)$.
- $T$ belongs to the tangent space to $\text{Iso}(X)$ at $\text{Id}$.
- $\lim_{\rho \to 0} \frac{\| \text{Id} + \rho T \| - 1}{\rho} = 0$. 
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This follows from the exponential formula

$$
\sup \text{Re } V(T) = \lim_{\beta \downarrow 0} \frac{\| \text{Id} + \beta T \| - 1}{\beta} = \sup_{\alpha > 0} \frac{\log \| \exp(\alpha T) \|}{\alpha}.
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**Remark**

If $X$ is complex, there always exists exponential one-parameter semigroups of surjective isometries:

$$t \mapsto e^{it} \text{Id} \quad \text{generator: } i \text{Id}.$$
Semigroups of isometries: characterization

Theorem (Bonsall-Duncan, 1970's; Rosenthal, 1984)

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- \( \{ \exp(\rho T) : \rho \in \mathbb{R}_0^+ \} \subset \text{Iso}(X) \).
- \( T \) belongs to the tangent space to \( \text{Iso}(X) \) at \( \text{Id} \).
- \( \lim_{\rho \to 0} \frac{\|\text{Id} + \rho T\| - 1}{\rho} = 0 \).

Main consequence

If \( X \) is a real Banach space such that

\[ V(T) = \{0\} \quad \implies \quad T = 0, \]

then \( \text{Iso}(X) \) is “small”:

- it does not contain any exponential one-parameter semigroup,
- the tangent space of \( \text{Iso}(X) \) at \( \text{Id} \) is zero.
Theorem

$X$ finite-dimensional real space. TFAE:

1. $\text{Iso}(X)$ is infinite.
2. $n(X) = 0$.
3. There is $T \in \mathcal{L}(X)$, $T \neq 0$, with $v(T) = 0$.

Examples of spaces of this kind

2. $X \mathbb{R}$, the real space subjacent to any complex space $X$.
3. An absolute sum of any real space and one of the above.
4. Moreover, if $X = X_0 \oplus X_1$ where $X_1$ is complex and $\|x_0 + e^{i\theta}x_1\| = \|x_0 + x_1\|$ ($x_0 \in X_0$, $x_1 \in X_1$, $\theta \in \mathbb{R}$).

(Note that the other 3 cases are included here)

Question

Can every Banach space $X$ with $n(X) = 0$ be decomposed as in 2?
Isometries in finite-dimensional spaces

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Let $X$ be a finite-dimensional real space. TFAE:

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\[
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**Examples of spaces of this kind**

## Isometries in finite-dimensional spaces

### Theorem

Let $X$ be a finite-dimensional real space. Then the following are equivalent (TFAE):

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Isometries in finite-dimensional spaces

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- There is $T \in L(X)$, $T \neq 0$, with $\nu(T) = 0$.

**Examples of spaces of this kind**

2. $X_\mathbb{R}$, the real space subjacent to any complex space $X$.
3. An absolute sum of any real space and one of the above.
4. Moreover, if $X = X_0 \oplus X_1$ where $X_1$ is complex and

   $$\left\| x_0 + e^{i\theta} x_1 \right\| = \left\| x_0 + x_1 \right\| \quad (x_0 \in X_0, \ x_1 \in X_1, \ \theta \in \mathbb{R}).$$

(Note that the other 3 cases are included here)
Isometries in finite-dimensional spaces

**Theorem**

$X$ finite-dimensional real space. TFAE:
- $\text{Iso}(X)$ is infinite.
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(Note that the other 3 cases are included here)

**Question**

Can every Banach space $X$ with $n(X) = 0$ be decomposed as in 4?
Infinite-dimensional case

There is an infinite-dimensional real Banach space $X$ with $n(X) = 0$ but $X$ is polyhedral. In particular, $X$ does not contain $C$ isometrically.

An easy example is $X = \bigoplus_{n \geq 2} X_n$ where $X_n$ is the two-dimensional space whose unit ball is the regular polygon of $2^n$ vertices.

Note such an example is not possible in the finite-dimensional case.
Negative answer

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Such an example is not possible in the finite-dimensional case.
Quasi affirmative answer
Quasi affirmative answer

Finite-dimensional case

Let $X$ be a finite-dimensional real space. TFAE:

1. $n(X) = 0$.
2. $X = X_0 \oplus X_1 \oplus \cdots \oplus X_n$ such that
   - $X_0$ is a (possible null) real space,
   - $X_1, \ldots, X_n$ are non-null complex spaces,
   - there are $\rho_1, \ldots, \rho_n$ rational numbers, such that
     \[
     \|x_0 + e^{i\rho_1 \theta} x_1 + \cdots + e^{i\rho_n \theta} x_n\| = \|x_0 + x_1 + \cdots + x_n\|
     \]
     for every $x_i \in X_i$ and every $\theta \in \mathbb{R}$.

Remark: The theorem is due to Rosenthal, but with real $\rho$'s. The fact that the $\rho$'s may be chosen as rational numbers is due to M. Merí–Rodríguez-Palacios.
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- The theorem is due to Rosenthal, but with real $\rho$’s.
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Sketch of the proof

Fix $T \in L(X)$ with $\|T\| = 1$ and $v(T) = 0$. We get that $\|\exp(\rho T)\| = 1$ for every $\rho \in \mathbb{R}$.

A Theorem by Auerbach: there exists a Hilbert space $H$ with $\dim(H) = \dim(X)$ such that every surjective isometry in $L(X)$ remains isometry in $L(H)$.

Apply the above to $\exp(\rho T)$ for every $\rho \in \mathbb{R}$. You get that $T$ is skew-hermitian in $L(H)$, so $T^* = -T$ and $T^2$ is self-adjoint. The $X_j$'s are the eigenspaces of $T^2$.

Use Kronecker's Approximation Theorem to change the eigenvalues of $T^2$ by rational numbers.
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- Use Kronecker’s Approximation Theorem to change the eigenvalues of $T^2$ by rational numbers.✓
A simple case of getting rational numbers

Let $X = X_0 \oplus X_1 \oplus X_2$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ s.t.

$$\|x_0 + e^{i\rho}x_1 + e^{i\alpha \rho}x_2\| = \|x_0 + x_1 + x_2\| \quad \forall \rho, \forall x_0, x_1, x_2.$$

Then

$$\|x_0 + x_1 + x_2\| = \|x_0 + \exp(i2\pi k(\alpha - 1)\rho)x_2\| \quad \forall \rho.$$ 

But \{\exp(i2\pi k(\alpha - 1)\rho) : k \in \mathbb{Z}\} is dense in $\mathbb{T}$, so

$$\|x_0 + (x_1 + x_2)\| = \|x_0 + e^{i\rho(x_1 + x_2)}\| \quad \forall \rho \in \mathbb{R}$$

and $X = X_0 \oplus \mathbb{Z}$ where $\mathbb{Z} = X_1 \oplus X_2$ is a complex space.
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- Then $\|x_0 + x_1 + x_2\| = \|x_0 + e^{i\rho} \left(x_1 + e^{i(\alpha - 1)\rho} x_2\right)\| \quad \forall \rho.$
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- Then $\left\| x_0 + (x_1 + x_2) \right\| = \left\| x_0 + e^{i\frac{2\pi k}{\alpha - 1}} (x_1 + x_2) \right\| \quad \forall k \in \mathbb{Z}$
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and \( X = X_0 \oplus Z \) where \( Z = X_1 \oplus X_2 \) is a complex space.
Consequences

If \( \dim(X) = 2 \), then \( X \equiv \mathbb{C} \).

If \( \dim(X) = 3 \), then \( X \equiv \mathbb{R} \oplus \mathbb{C} \) (absolute sum).

Natural question

Are all finite-dimensional \( X \)'s with \( n(X) = 0 \) of the form \( X = X_0 \oplus X_1 \)?

Answer

No.

Example

\[ X = (\mathbb{R}^4, \|\cdot\|), \] \[ \| (a, b, c, d) \| = \frac{1}{4} \int_0^{2\pi} |Re\left(e^{2it}(a+ib) + e^{it}(c+id)\right)| dt. \]

Then \( n(X) = 0 \) but the unique possible decomposition is \( X = \mathbb{C} \oplus \mathbb{C} \).
Consequences

**Corollary**

Let $X$ be a real space with $n(X) = 0$.

- If $\dim(X) = 2$, then $X \cong \mathbb{C}$.
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$$\left\| e^{it} x_1 + e^{2it} x_2 \right\| = \left\| x_1 + x_2 \right\|.$$
The Lie-algebra of a Banach space

For a real Banach space $X$, the Lie-algebra $Z(X)$ is defined as:

$$Z(X) = \{ T \in \mathcal{L}(X) : v(T) = 0 \}.$$ 

When $X$ is finite-dimensional, $\text{Iso}(X)$ is a Lie-group and $Z(X)$ is the tangent space (i.e. its Lie-algebra).

**Remark**

$$\dim(X) = n \implies \dim(Z(X)) \leq n(n-1)/2.$$ 

Equality holds if and only if $X$ is a Hilbert space.

An open problem

Given $n \geq 3$, which are the possible $\dim(Z(X))$ over all $n$-dimensional $X$'s?

**Observation (Javier Merí, PhD)**

When $\dim(X) = 3$, $\dim(Z(X))$ cannot be 2.
The Lie-algebra of a Banach space

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**Remark**

- \( \dim(X) = n \implies \dim(\mathcal{Z}(X)) \leq \frac{n(n-1)}{2} \).
- Equality holds \( \iff \) \( H \) Hilbert space.
### The Lie-algebra of a Banach space

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Two results on surjective isometries

Isometries on finite-dimensional spaces

The Lie-algebra of a Banach space

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Proof

If $\dim(X) = 3$, $n(X) = 0$, then $X = \mathbb{C} \oplus \mathbb{R}$ (absolute sum).

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**Proof**

If \( \text{dim}(X) = 3 \), \( n(X) = 0 \), then \( X = \mathbb{C} \oplus \mathbb{R} \) (absolute sum).
- If \( \oplus = \oplus_2 \), then \( X \) is a Hilbert space and \( \text{dim}(\mathcal{Z}(X)) = 3 \). ✓

**Remark**

- If \( \oplus \neq \oplus_2 \), then isometries respect summands and \( \text{dim}(\mathcal{Z}(X)) = 1 \). ✓

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**Proof**

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When $\dim(X) = 3$, $\dim(\mathcal{Z}(X))$ cannot be 2.
Semigroups of surjective isometries and duality

Remark

The problem

How much bigger can be $\text{Iso}(X^*)$ than $\text{Iso}(X)$?

Is it possible that $\mathcal{Z}(\text{Iso}(X^*))$ is big while $\mathcal{Z}(\text{Iso}(X))$ is trivial?

The answer is yes. This is what we are going to present next.
Remark

$X$ Banach space.

- $T \in \text{Iso}(X) \implies T^* \in \text{Iso}(X^*)$.
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Semigroups of surjective isometries and duality

Two results on surjective isometries

Isometries and duality

Spaces $C_{E}(K, L)$

$K$ compact, $L \subset K$ closed nowhere dense, $E \subset C(L)$.

$C_{E}(K, L) = \{ f \in C(K) : f|_{L} \in E \}$.

Theorem $C_{E}(K, L)^{*} \equiv E^{*} \oplus \mathbb{C}0(K, L)^{*}$.

$\forall (C_{E}(K, L))^{*} = \mathbb{C}$. 33 / 152
Spaces $C_E(K\parallel L)$

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Semigroups of surjective isometries and duality

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Theorem

$$C_E(K\|L)^* \equiv E^* \oplus_1 C_0(K\|L)^* \quad \& \quad n(C_E(K\|L)) = 1.$$
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Theorem

$$C_E(K\|L)^* \equiv E^* \oplus_1 C_0(K\|L)^* \quad \& \quad n(C_E(K\|L)) = 1.$$ 

Proof.

- $C_0(K\|L)$ is an $M$-ideal of $C(K)$
  
  $\implies C_0(K\|L)$ is an $M$-ideal of $C_E(K\|L)$.
Semigroups of surjective isometries and duality

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**Proof.**

- $C_0(K∥L)$ is an $M$-ideal of $C(K)$
  $\implies C_0(K∥L)$ is an $M$-ideal of $C_E(K∥L)$.

- Meaning that $C_E(K∥L)^* \equiv C_0(K∥L)\perp \oplus_1 C_0(K∥L)^*$. 
Semigroups of surjective isometries and duality

Spaces \( C_E(K\parallel L) \)

- \( K \) compact, \( L \subset K \) closed nowhere dense, \( E \subset C(L) \).
  
\[
C_E(K\parallel L) = \{ f \in C(K) : f|_L \in E \}.
\]

Theorem

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C_E(K\parallel L)^* \equiv E^* \oplus_1 C_0(K\parallel L)^* \quad \& \quad n(C_E(K\parallel L)) = 1.
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Proof.

- \( C_0(K\parallel L) \) is an \( M \)-ideal of \( C(K) \)
  
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- Meaning that \( C_E(K\parallel L)^* \equiv C_0(K\parallel L)^\perp \oplus_1 C_0(K\parallel L)^* \).

- \( C_0(K\parallel L)^\perp \equiv (C_E(K\parallel L)/C_0(K\parallel L))^* \equiv E^* \).
Semigroups of surjective isometries and duality

Spaces $C_E(K\|L)$

$K$ compact, $L \subset K$ closed nowhere dense, $E \subset C(L)$.

$$C_E(K\|L) = \{ f \in C(K) : f|_L \in E \}. $$

Theorem

$$C_E(K\|L)^* \equiv E^* \oplus_1 C_0(K\|L)^* \quad \& \quad n(C_E(K\|L)) = 1.$$ 

Proof.

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- Meaning that $C_E(K\|L)^* \equiv C_0(K\|L)^\perp \oplus_1 C_0(K\|L)^*.$

- $C_0(K\|L)^\perp \equiv (C_E(K\|L)/C_0(K\|L))^* \equiv E^*$:

- $\Phi : C_E(K\|L) \longrightarrow E$, $\Phi(f) = f|_L$.
  $$\|\Phi\| \leq 1 \quad \text{and} \quad \ker \Phi = C_0(K\|L).$$

- $\tilde{\Phi} : C_E(K\|L)/C_0(K\|E) \longrightarrow E \text{ onto isometry}$:
  
- $\{ g \in E : \|g\| < 1 \} \subseteq \Phi(\{ f \in C_E(K\|L) : \|f\| < 1 \}). \checkmark$
Semigroups of surjective isometries and duality

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$K$ compact, $L \subset K$ closed nowhere dense, $E \subset C(L)$.

$$C_E(K\|L) = \{ f \in C(K) : f|_L \in E \}.$$ 

Theorem

$$C_E(K\|L)^* \equiv E^* \oplus_1 C_0(K\|L)^*$$ 

and 

$$n(C_E(K\|L)) = 1.$$ 

Proof.

- $C_0(K\|L)$ is an $M$-ideal of $C(K)$
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Semigroups of surjective isometries and duality

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$$C_E(K\|L)^* \equiv E^* \oplus_1 C_0(K\|L)^* \quad \& \quad n(C_E(K\|L)) = 1.$$ 

Proof.
Semigroups of surjective isometries and duality

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**Proof.**
- Fix $T \in L(C_E(K\|L))$. Take $f_0 \in S_{C_E(K\|L)}$ and $\xi_0 \in K \setminus L$ with $|[Tf_0](\xi_0)| \sim \|T\|$. 


Two results on surjective isometries

Semigroups of surjective isometries and duality

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$K$ compact, $L \subset K$ closed nowhere dense, $E \subset C(L)$.

$$C_E(K\|L) = \{f \in C(K) : f|_L \in E\}.$$  

Theorem

$$C_E(K\|L)^* \equiv E^* \oplus_1 C_0(K\|L)^* \quad \& \quad n(C_E(K\|L)) = 1.$$  

Proof.

- Fix $T \in L(C_E(K\|L))$. Take $f_0 \in SC_E(K\|L)$ and $\xi_0 \in K \setminus L$ with $\|Tf_0(\xi_0)\| \sim \|T\|$.

- Consider $V = \{\xi \in K \setminus L : f_0(\xi) \sim f_0(\xi_0)\}$ and take $\varphi : K \rightarrow [0,1]$ continuous with $\text{supp}(\varphi) \subset V$ and $\varphi(\xi_0) = 1$. 


Semigroups of surjective isometries and duality

Spaces \( C_E(K\|L) \)

\[ K \text{ compact, } L \subset K \text{ closed nowhere dense, } E \subset C(L). \]

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Theorem

\[ C_E(K\|L)^* \equiv E^* \oplus_1 C_0(K\|L)^* \quad \& \quad n(C_E(K\|L)) = 1. \]

Proof.

- Fix \( T \in L(C_E(K\|L)) \). Take \( f_0 \in S_{C_E(K\|L)} \) and \( \xi_0 \in K \setminus L \) with \( \|Tf_0(\xi_0)\| \sim \|T\| \).
- Consider \( V = \{ \xi \in K \setminus L : f_0(\xi) \sim f_0(\xi_0) \} \) and take \( \varphi : K \to [0,1] \) continuous with \( \text{supp}(\varphi) \subset V \) and \( \varphi(\xi_0) = 1 \).
- Write \( f_0(\xi_0) = \lambda \omega_1 + (1 - \lambda) \omega_2 \) with \( |\omega_i| = 1 \) and consider the functions \( f_i = (1 - \varphi)f_0 + \varphi \omega_i \equiv f_0 + \varphi(\omega_i - f_0) \in C_E(K\|L) \) for \( i = 1,2 \).
Spaces $C_E(K\|L)$

$K$ compact, $L \subset K$ closed nowhere dense, $E \subset C(L)$.

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- Consider $V = \{\xi \in K \setminus L : f_0(\xi) \sim f_0(\xi_0)\}$ and take $\varphi : K \to [0,1]$ continuous with $\text{supp}(\varphi) \subset V$ and $\varphi(\xi_0) = 1$.
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  $$f_i = (1-\varphi)f_0 + \varphi \omega_i \quad (= f_0 + \varphi(\omega_i - f_0)) \in C_E(K\|L) \text{ for } i = 1,2.$$ 
- Then $\|f_i\| \leq 1$ and $\|f_0 - (\lambda f_1 + (1-\lambda)f_2)\| = \|\varphi f_0 - \varphi f_0(\xi_0)\| \sim 0.$
Semigroups of surjective isometries and duality

Spaces $C_E(K\|L)$

$K$ compact, $L \subset K$ closed nowhere dense, $E \subset C(L)$.

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Theorem

$$C_E(K\|L)^* \equiv E^* \oplus_1 C_0(K\|L)^* \quad \& \quad \text{null}(C_E(K\|L)) = 1.$$

Proof.

- Fix $T \in L(C_E(K\|L))$. Take $f_0 \in S_{C_E(K\|L)}$ and $\xi_0 \in K \setminus L$ with $|(Tf_0)(\xi_0)| \sim \|T\|$.

- Consider $V = \{\xi \in K \setminus L : f_0(\xi) \sim f_0(\xi_0)\}$ and take $\varphi : K \to [0,1]$ continuous with $\text{supp}(\varphi) \subset V$ and $\varphi(\xi_0) = 1$.

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- Then $\|f_i\| \leq 1$ and $\|f_0 - (\lambda f_1 + (1 - \lambda)f_2)\| = \|\varphi f_0 - \varphi f_0(\xi_0)\| \sim 0$.

- Therefore, we may choose $i \in \{1,2\}$ with $|(T(f_i))(\xi_0)| \sim \|T\|$, but now $|f_i(\xi_0)| = 1$. 


Semigroups of surjective isometries and duality

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$$C_E(K\|L)^* \equiv E^* \oplus_1 C_0(K\|L)^* \quad \& \quad n(C_E(K\|L)) = 1.$$ 

Proof.

- Fix $T \in L(C_E(K\|L))$. Take $f_0 \in S_{C_E(K\|L)}$ and $\xi_0 \in K \setminus L$ with $|[T f_0](\xi_0)| \sim \|T\|$.

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- Then $\|f_i\| \leq 1$ and $\|f_0 - (\lambda f_1 + (1 - \lambda)f_2)\| = \|\varphi f_0 - \varphi f_0(\xi_0)\| \sim 0$.

- Therefore, we may choose $i \in \{1,2\}$ with $|[T(f_i)](\xi_0)| \sim \|T\|$, but now $|f_i(\xi_0)| = 1$.

- Equivalently, $|\delta_{\xi_0}(T(f_i))| \sim \|T\|$ and $|\delta_{\xi_0}(f_i)| = 1$, so $v(T) \sim \|T\|$. ✓
**Spaces** $C_E(K\Vert L)$

$K$ compact, $L \subset K$ closed nowhere dense, $E \subset C(L)$.

$$C_E(K\Vert L) = \{f \in C(K) : f|_L \in E\}.$$ 

**Theorem**

$$C_E(K\Vert L)^* \equiv E^* \oplus_1 C_0(K\Vert L)^* \quad \& \quad n(C_E(K\Vert L)) = 1.$$ 

**Consequence: the example**

Take $K = [0, 1]$, $L = \Delta$ (Cantor set), $E = \ell_2 \subset C(\Delta)$.

- $\text{Iso}(C_{\ell_2}([0, 1]\Vert \Delta))$ has no exponential one-parameter semigroups.
- $C_{\ell_2}([0, 1]\Vert \Delta)^* \equiv \ell_2 \oplus_1 C_0([0, 1]\Vert \Delta)^*$, so taken $S \in \text{Iso}(\ell_2)$

$$\implies T = \begin{pmatrix} S & 0 \\ 0 & \text{Id} \end{pmatrix} \in \text{Iso}(C_{\ell_2}([0, 1]\Vert \Delta)^*)$$

Then, $\text{Iso}(C_{\ell_2}([0, 1]\Vert \Delta)^*)$ contains infinitely many exponential one-parameter semigroups.
Some comments
Some comments

In terms of linear dynamical systems

In $C^\ell_2([0, 1] \parallel \Delta)$ there is no $A \in L(X)$ such that the solution to the linear dynamical system $x'(t) = A x(t)$ (where $x(t) : \mathbb{R}^+ \rightarrow C^\ell_2([0, 1] \parallel \Delta)$) is given by a semigroup of isometries. There are infinitely many such $A$'s in $C^\ell_2([0, 1] \parallel \Delta)^*$, in $C^\ell_2([0, 1] \parallel \Delta)^{**}$...

Further results (Koszmider–M.–Merí., 2011) There are unbounded $A$'s on $C^\ell_2([0, 1] \parallel \Delta)$ such that the solution to the linear dynamical system $x'(t) = A x(t)$ is a one-parameter $C_0$ semigroup of isometries. There is $X$ such that $\text{Iso}(X) = \{-\text{Id}, \text{Id}\}$ and $X^* = \ell_2 \oplus^1 L_1(\nu)$.

Therefore, there is no semigroups in $\text{Iso}(X)$, but there are infinitely many exponential one-parameter semigroups in $\text{Iso}(X^*)$. 
Some comments

In terms of linear dynamical systems

- In $C_{\ell^2}([0, 1]\|\Delta)$ there is no $A \in L(X)$ such that the solution to the linear dynamical system

$$x' = Ax \quad (x : \mathbb{R}^+_0 \rightarrow C_{\ell^2}([0, 1]\|\Delta))$$

(which is $x(t) = \exp(t A)(x(0))$) is given by a semigroup of isometries.
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Further results (Koszmider–M.–Merí., 2011)

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Numerical index of Banach spaces

- Basic definitions and examples
- Stability properties
- Duality
- The isomorphic point of view
- Banach spaces with numerical index one
  - Isomorphic properties
  - Isometric properties
  - Asymptotic behavior
- How to deal with numerical index 1 property?
- Some open problems

V. Kadets, M. Martín, and R. Payá.
Recent progress and open questions on the numerical index of Banach spaces.
RACSAM (2006)
Numerical index of Banach spaces: definitions

**Numerical radius**

Let $X$ be a Banach space, $T \in L(X)$. The **numerical radius** of $T$ is

$$v(T) = \sup \{|x^*(Tx)| : x^* \in S_{X^*}, x \in S_X, x^*(x) = 1\}$$
Numerical index of Banach spaces: definitions

Numerical radius

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Remark

The numerical radius is a continuous seminorm in $L(X)$. Actually, $\nu(\cdot) \leq \| \cdot \|$
Numerical index of Banach spaces: definitions

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The numerical radius is a continuous seminorm in \( L(X) \). Actually, \( v(\cdot) \leq \| \cdot \| \)

**Numerical index (Lumer, 1968)**

X Banach space, the numerical index of \( X \) is

\[
n(X) = \inf \{ v(T) : T \in L(X), \ |T| = 1 \}
\]

\[
= \max \ \{ k \geq 0 : k |T| \leq v(T) \ \forall \ T \in L(X) \}
\]

\[
= \inf \ \{ M \geq 0 : \exists T \in L(X), \ |T| = 1, \ |\exp(\rho T)| \leq e^{\rho M} \ \forall \rho \in \mathbb{R} \}
\]
Recalling some basic properties

\[ n(X) = 1 \text{ iff } v \text{ and } \|\cdot\| \text{ coincide.} \]

\[ n(X) = 0 \text{ iff } v \text{ is not an equivalent norm in } L(X). \]

\( X \) complex \( \Rightarrow n(X) \geq \frac{1}{e}. \)

(Bohnenblust–Karlin, 1955; Glickfeld, 1970)

Actually, \( \{ n(X) : X \text{ complex}, \dim(X) = 2 \} = [e^{-1}, 1] \)

\( \{ n(X) : X \text{ real}, \dim(X) = 2 \} = [0, 1] \)

(Duncan–McGregor–Pryce–White, 1970)
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(Duncan–McGregor–Pryce–White, 1970)
### Some examples

1. **H** Hilbert space, \( \dim(H) > 1 \),

   \[
   n(H) = 0 \quad \text{if } H \text{ is real} \\
   n(H) = 1/2 \quad \text{if } H \text{ is complex}
   \]
Numerical index of Banach spaces: examples (I)

Some examples

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   \begin{align*}
   n(H) &= 0 \quad \text{if } H \text{ is real} \\
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   \end{align*}

2. $n(L_1(\mu)) = 1$ \quad $\mu$ positive measure
   $n(C(K)) = 1$ \quad $K$ compact Hausdorff space
   
   (Duncan et al., 1970)
Numerical index of Banach spaces: examples (I)

Some examples

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3. If $A$ is a $C^*$-algebra $\implies \begin{cases} 
   n(A) = 1 & A \text{ commutative} \\
   n(A) = 1/2 & A \text{ not commutative} 
\end{cases}$
   (Huruya, 1977; Kaidi–Morales–Rodríguez, 2000)
Some examples

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   \end{center}

4. If $A$ is a function algebra $\Rightarrow n(A) = 1$
   
   \begin{center}
   (Werner, 1997)
   \end{center}
For $n \geq 2$, the unit ball of $X_n$ is a $2n$ regular polygon:

$$n(X_n) = \begin{cases} 
\tan \left( \frac{\pi}{2n} \right) & \text{if } n \text{ is even,} \\
\sin \left( \frac{\pi}{2n} \right) & \text{if } n \text{ is odd.}
\end{cases}$$

(M.–Mérité, 2007)
For $n \geq 2$, the unit ball of $X_n$ is a $2n$ regular polygon:

$$n(X_n) = \begin{cases} 
\tan \left( \frac{\pi}{2n} \right) & \text{if } n \text{ is even,} \\
\sin \left( \frac{\pi}{2n} \right) & \text{if } n \text{ is odd.}
\end{cases}$$

(M.–Merí, 2007)

Every finite-codimensional subspace of $C[0, 1]$ has numerical index 1

(Boyko–Kadets–M.–Werner, 2007)
Numerical index of Banach spaces: some examples (III)

Even more examples

- Numerical index of $L_p$-spaces, $1 < p < \infty$:
Numerical index of Banach spaces: some examples (III)

Even more examples

- **Numerical index of** $L_p$-spaces, $1 < p < \infty$:
  
  $$n(L_p[0, 1]) = n(\ell_p) = \lim_{m \to \infty} n(\ell_p^{(m)}).$$

  (Ed-Dari, 2005 & Ed-Dari-Khamsi, 2006)
Even more examples

Numerical index of $L^p$-spaces, $1 < p < \infty$:

- $n(L^p[0, 1]) = n(\ell_p) = \lim_{m \to \infty} n(\ell_p^{(m)})$.

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- $n(\ell_p^{(2)})$ ?
### Numerical index of Banach spaces: some examples (III)

#### Even more examples

- **Numerical index of** $L_p$-spaces, $1 < p < \infty$:
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    (Ed-Dari, 2005 & Ed-Dari-Khamsi, 2006)
  - $n(\ell_p^{(2)})$?
  - In the real case,
    
    $\max \left\{ \frac{1}{2^{1/p}}, \frac{1}{2^{1/q}} \right\} M_p \leq n(\ell_p^{(2)}) \leq M_p$

    and $M_p = v \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \max_{t \in [0,1]} \frac{|t^{p-1} - t|}{1 + t^p}$

    (M.–Merí, 2009)
Even more examples

- **Numerical index of $L_p$-spaces, $1 < p < \infty$:**
  - $n(L_p[0, 1]) = n(\ell_p) = \lim_{m \to \infty} n(\ell_p^{(m)})$.
    - (Ed-Dari, 2005 & Ed-Dari-Khamsi, 2006)
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    - (M.–Merí, 2009)
  - In the real case, $n(L_p(\mu)) \geq \frac{M_p}{6 p^p q^q}$.
    - (M.–Merí–Popov, 2011)
Even more examples

Numerical index of $L_p$-spaces, $1 < p < \infty$:

- $n(L_p[0,1]) = n(\ell_p) = \lim_{m \to \infty} n(\ell_p^m)$.

  (Ed-Dari, 2005 & Ed-Dari-Khamsi, 2006)

- $n(\ell_p^{(2)})$?

- In the real case,

  $$\max \left\{ \frac{1}{2^{1/p}}, \frac{1}{2^{1/q}} \right\} M_p \leq n(\ell_p^{(2)}) \leq M_p$$

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  (M.–Merí, 2009)

- In the real case, $n(L_p(\mu)) \geq \frac{M_p}{1 \overline{\text{p}} \overline{\text{q}}}$.

- In particular, $n(L_p(\mu)) > 0$ for $p \neq 2$.

  (M.–Merí–Popov, 2011)
Numerical index: open problems on computing

1. Compute $n(L^p[0, 1])$ for $1 < p < \infty$, $p \neq 2$.

2. Is $n(\ell^p(2^p)) = M_p$ (real case)?

3. Is $n(\ell^p(2^p)) = (p_{1/p}q_{1/q})^{-1}$ (complex case)?

4. Compute the numerical index of real $C^*$-algebras.

5. Compute the numerical index of more classical Banach spaces: $C_m[0, 1], Lip(K)$, Lorentz spaces, Orlicz spaces, etc.
Open problems

1. Compute \( n(L_p[0,1]) \) for \( 1 < p < \infty, \ p \neq 2 \).
Numerical index: open problems on computing

Open problems

1. Compute $n(L_p[0,1])$ for $1 < p < \infty$, $p \neq 2$.

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Open problems

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Open problems

1. Compute $n(L_p[0,1])$ for $1 < p < \infty$, $p \neq 2$.
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4. Compute the numerical index of real $C^*$-algebras.
5. Compute the numerical index of more classical Banach spaces: $C^m[0,1]$, Lip($K$), Lorentz spaces, Orlicz spaces...
Direct sums of Banach spaces (M.–Payá, 2000)

\[ n\left(\bigoplus_{\lambda \in \Lambda} X_\lambda [c_0]\right) = n\left(\bigoplus_{\lambda \in \Lambda} X_\lambda [\ell_1]\right) = n\left(\bigoplus_{\lambda \in \Lambda} X_\lambda [\ell_\infty]\right) = \inf_{\lambda} n(X_\lambda) \]
Stability properties

Direct sums of Banach spaces (M.–Payá, 2000)

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Consequences

- There is a real Banach space \( X \) such that
  \[ \nu(T) > 0 \quad \text{when } T \neq 0, \]
  but \( n(X) = 0 \)
  (i.e. \( \nu(\cdot) \) is a norm on \( L(X) \) which is not equivalent to the operator norm).
Numerical index Stability properties

Direct sums of Banach spaces (M.–Payá, 2000)

\[ n\left(\bigoplus_{\lambda \in \Lambda} X_{\lambda}\big)c_0\right) = n\left(\bigoplus_{\lambda \in \Lambda} X_{\lambda}\big)\ell_1\right) = n\left(\bigoplus_{\lambda \in \Lambda} X_{\lambda}\big)\ell_\infty\right) = \inf_{\lambda} n(X_{\lambda}) \]

Consequences

- There is a real Banach space \( X \) such that \( \nu(T) > 0 \) when \( T \neq 0 \), but \( n(X) = 0 \)
  (i.e. \( \nu(\cdot) \) is a norm on \( L(X) \) which is not equivalent to the operator norm).
- For every \( t \in [0, 1] \), there exist a real \( X_t \) isomorphic to \( c_0 \) (or \( \ell_1 \) or \( \ell_\infty \)) with \( n(X_t) = t \).
- For every \( t \in [e^{-1}, 1] \), there exist a complex \( Y_t \) isomorphic to \( c_0 \) (or \( \ell_1 \) or \( \ell_\infty \)) with \( n(Y_t) = t \).
Vector-valued function spaces (López-M.-Merí-Payá-Villena, 2000's)

$E$ Banach space, $\mu$ positive $\sigma$-finite measure, $K$ compact space. Then

\[ n(C(K,E)) = n(C_w(K,E)) = n(L_1(\mu,E)) = n(L_\infty(\mu,E)) = n(E), \]

and $n(C_{w^*}(K,E^*)) \leq n(E)$
### Stability properties (II)

**Vector-valued function spaces (López-M.-Merí-Payá-Villena, 2000's)**

$E$ Banach space, $\mu$ positive $\sigma$-finite measure, $K$ compact space. Then

$$n(C(K,E)) = n(C_{\text{w}}(K,E)) = n(L_1(\mu,E)) = n(L_\infty(\mu,E)) = n(E),$$

and $n(C_{\text{w}}^*(K,E^*)) \leq n(E)$

---

**Tensor products (Lima, 1980)**

There is no general formula for $n(X\tilde{\otimes}_\varepsilon Y)$ nor for $n(X\tilde{\otimes}_\pi Y)$:

- $n(\ell_1^{(4)} \tilde{\otimes}_\pi \ell_1^{(4)}) = n(\ell_\infty^{(4)} \tilde{\otimes}_\varepsilon \ell_\infty^{(4)}) = 1$.
- $n(\ell_1^{(4)} \tilde{\otimes}_\varepsilon \ell_1^{(4)}) = n(\ell_\infty^{(4)} \tilde{\otimes}_\pi \ell_\infty^{(4)}) < 1$. 
Vector-valued function spaces (López-M.-Merí-Payá-Villena, 2000's)

Let $E$ be a Banach space, $\mu$ a positive $\sigma$-finite measure, and $K$ a compact space. Then

$$n(C(K, E)) = n(C_w(K, E)) = n(L_1(\mu, E)) = n(L_\infty(\mu, E)) = n(E),$$

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Tensor products (Lima, 1980)

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$$n(L_p([0, 1], E)) = n(\ell_p(E)) = \lim_{m \to \infty} n(E \oplus_p \cdots \oplus_p E).$$
Numerical index and duality

Proposition

Let $X$ be a Banach space, and $T \in L(X)$. Then

$$\sup \Re \nu(T) = \lim_{\alpha \to 0^+} \|\Id + \alpha T\| - 1.$$

Then, $\nu(T^*) = \nu(T)$ for every $T \in L(X)$. Therefore,

$$n(X^*) \leq n(X).$$

(Duncan–McGregor–Pryce–White, 1970)

Question (From the 1970's)

Is $n(X) = n(X^*)$?

Negative answer (Boyko–Kadets–M.–Werner, 2007)

Consider the space $X = \{(x, y, z) \in c \oplus \infty c \oplus \infty c : \lim x + \lim y + \lim z = 0\}$.

Then, $n(X) = 1$ but $n(X^*) < 1$. 
Numerical index and duality

**Proposition**

Let $X$ be a Banach space, $T \in L(X)$. Then

$$\sup \text{Re} V(T) = \lim_{\alpha \to 0^+} \frac{\|\text{Id} + \alpha T\| - 1}{\alpha}.$$  

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Numerical index and duality

**Proposition**

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(Duncan–McGregor–Pryce–White, 1970)
Proposition

Let $X$ be a Banach space, $T \in L(X)$. Then

- $\sup \Re V(T) = \lim_{\alpha \to 0^+} \frac{\|\Id + \alpha T\| - 1}{\alpha}$.

- Then, $v(T^*) = v(T)$ for every $T \in L(X)$.

- Therefore, $n(X^*) \leq n(X)$.

(Duncan–McGregor–Pryce–White, 1970)
Proposition

**X** Banach space, **T** ∈ **L**(**X**). Then

- \( \text{sup Re } V(T) = \lim_{\alpha \to 0^+} \frac{\|\text{Id} + \alpha T\| - 1}{\alpha} \).

- Then, \( \nu(T^*) = \nu(T) \) for every \( T \in \text{L}(X) \).

- Therefore, \( n(X^*) \leq n(X) \).

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Proposition

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- Then, \( v(T^*) = v(T) \) for every \( T \in L(X) \).
- Therefore, \( n(X^*) \leq n(X) \).

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Consider the space

\[ X = \{(x, y, z) \in c \oplus_{\infty} c \oplus_{\infty} c : \lim x + \lim y + \lim z = 0\}. \]

Then, \( n(X) = 1 \) but \( n(X^*) < 1 \).
Numerical index and duality. Proof of main example

\[ X = \{ (x, y, z) \in c \oplus_\infty c \oplus_\infty c : \lim x + \lim y + \lim z = 0 \} : \]
\[ n(X) = 1 \quad \text{but} \quad n(X^*) < 1. \]
Numerical index and duality. Proof of main example

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**Proof**

- \( c^* = \ell_1 \oplus_1 \mathbb{K} \lim \implies X^* = \left[ c^* \oplus_1 c^* \oplus_1 c^* \right] / (\lim, \lim, \lim). \)
Numerical index and duality. Proof of main example

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**Proof**

- \( c^* = \ell_1 \oplus_1 K \lim \implies X^* = [c^* \oplus_1 c^* \oplus_1 c^*] / (\lim, \lim, \lim). \)
- Then, writing \( Z = \ell_1^{(3)} / (1, 1, 1) \), we can identify
  \[ X^* \equiv \ell_1 \oplus_1 \ell_1 \oplus_1 \ell_1 \oplus_1 Z, \quad X^{**} \equiv \ell_\infty \oplus_\infty \ell_\infty \oplus_\infty \ell_\infty \oplus_\infty Z^*. \]
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- \( c^* = \ell_1 \oplus_1 K \lim \implies X^* = [c^* \oplus_1 c^* \oplus_1 c^*] / (\lim, \lim, \lim). \)
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- \( A = \{(e_n,0,0,0) : n \in \mathbb{N}\} \cup \{(0,e_n,0,0) : n \in \mathbb{N}\} \cup \{(0,0,e_n,0) : n \in \mathbb{N}\} \subset X^*. \)
Numerical index and duality. Proof of main example

\[ X = \{(x, y, z) \in c \oplus_{\infty} c \oplus_{\infty} c : \lim x + \lim y + \lim z = 0\} : \]

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Proof

- \( c^* = \ell_1 \oplus_1 K \lim \implies X^* = \left[c^* \oplus_1 c^* \oplus_1 c^*\right]/(\lim, \lim, \lim). \)

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- Then \( B_X = \overline{\text{aco}^{w^*}(A)} \) and
  \[ |x^{**}(a)| = 1 \quad \forall x^{**} \in \text{ext}(B_X^{**}) \forall a \in A. \]
Numerical index and duality. Proof of main example

\[ X = \{ (x, y, z) \in c \oplus_\infty c \oplus_\infty c : \lim x + \lim y + \lim z = 0 \} : \]

\[ n(X) = 1 \quad \text{but} \quad n(X^*) < 1. \]

**Proof**

- \( c^* = \ell_1 \oplus_1 \mathbb{K} \lim \implies X^* = [c^* \oplus_1 c^* \oplus_1 c^*] / (\lim, \lim, \lim). \)
- Then, writing \( Z = \ell_1^{(3)} / (1, 1, 1), \) we can identify
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- \( A = \{ (e_n, 0, 0, 0) : n \in \mathbb{N} \} \cup \{ (0, e_n, 0, 0) : n \in \mathbb{N} \} \cup \{ (0, 0, e_n, 0) : n \in \mathbb{N} \} \subset X^*. \)
- Then \( B_{X^*} = \overline{\text{aco}}^{w^*}(A) \) and
  \[ |x^{**}(a)| = 1 \quad \forall x^{**} \in \text{ext}(B_{X^{**}}) \forall a \in A. \]
- Fix \( T \in L(X), \varepsilon > 0. \) Find \( a \in A \) with \( \|T^*(a)\| > \|T^*\| - \varepsilon. \)
Numerical index and duality. Proof of main example

\[ X = \{ (x, y, z) \in c \oplus \infty c \oplus \infty c : \lim x + \lim y + \lim z = 0 \} : \]

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Proof

- \( c^* = \ell_1 \oplus_1 \mathbb{K} \lim \implies X^* = \left[ c^* \oplus_1 c^* \oplus_1 c^* \right] / (\lim, \lim, \lim). \)
- Then, writing \( Z = \ell_1^{(3)} / (1, 1, 1) \), we can identify
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- Then we find \( x^{**} \in \text{ext}(B_{X^{**}}) \) such that
  \[ |x^{**}(T^*(a))| = \|T^*(a)\| > \|T^*\| - \varepsilon. \]
Numerical index and duality. Proof of main example

\[ X = \{ (x, y, z) \in c \oplus \infty c \oplus \infty c : \lim x + \lim y + \lim z = 0 \} : \]
\[ n(X) = 1 \quad \text{but} \quad n(X^*) < 1. \]

Proof

- \( c^* = \ell_1 \oplus_1 \lim \implies X^* = [c^* \oplus_1 c^* \oplus_1 c^*] / (\lim, \lim, \lim). \)
- Then, writing \( Z = \ell_1^{(3)} / (1, 1, 1) \), we can identify
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- \( A = \{(e_n, 0, 0, 0) : n \in \mathbb{N}\} \cup \{(0, e_n, 0, 0) : n \in \mathbb{N}\} \cup \{(0, 0, e_n, 0) : n \in \mathbb{N}\} \subset X^*. \)
- Then \( B_{X^*} = \overline{\text{aco}} \omega^* (A) \) and
  \[ |x^{**}(a)| = 1 \quad \forall x^{**} \in \text{ext}(B_{X^{**}}) \quad \forall a \in A. \]
- Fix \( T \in L(X), \varepsilon > 0. \) Find \( a \in A \) with \( \|T^*(a)\| > \|T^*\| - \varepsilon. \)
- Then we find \( x^{**} \in \text{ext}(B_{X^{**}}) \) such that
  \[ |x^{**}(T^*(a))| = \|T^*(a)\| > \|T^*\| - \varepsilon. \]
- Since \( |x^{**}(a)| = 1 \), this gives that \( v(T^*) > \|T^*\| - \varepsilon \), so \( v(T) = \|T\| \) and \( n(X) = 1. \) √
Numerical index and duality. Proof of main example

\[ X = \{ (x, y, z) \in c \oplus_{\infty} c \oplus_{\infty} c : \lim x + \lim y + \lim z = 0 \} : \]

\[ n(X) = 1 \quad \text{but} \quad n(X^*) < 1. \]

Proof

- \( c^* = \ell_1 \oplus_1 K \lim \implies X^* = [c^* \oplus_1 c^* \oplus_1 c^*] / (\lim, \lim, \lim) \).

- Then, writing \( Z = \ell_1^{(3)} / (1, 1, 1) \), we can identify
  \[ X^* \equiv \ell_1 \oplus_1 \ell_1 \oplus_1 \ell_1 \oplus_1 Z, \quad X^{**} \equiv \ell_\infty \oplus_\infty \ell_\infty \oplus_\infty \ell_\infty \oplus_\infty Z^*. \]

- \( Z \) is an \( L \)-summand of \( X^* \) so
  \[ n(X^*) \leq n(Z). \]
Numerical index and duality. Proof of main example

\[ X = \{ (x, y, z) \in c \oplus_c c \oplus_c c : \lim x + \lim y + \lim z = 0 \} : \]

\[ n(X) = 1 \quad \text{but} \quad n(X^*) < 1. \]

Proof

- \( c^* = \ell_1 \oplus_1 K \lim \implies X^* = [c^* \oplus_1 c^* \oplus_1 c^*] / (\lim, \lim, \lim). \)

- Then, writing \( Z = \ell_1^{(3)}/(1, 1, 1) \), we can identify

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- \( Z \) is an \( L \)-summand of \( X^* \) so

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- But \( n(Z) < 1 ! \) \( \checkmark \)
Numerical index and duality. Proof of main example

\[ X = \{ (x, y, z) \in c \oplus c \oplus c : \lim x + \lim y + \lim z = 0 \} : \]

\[ n(X) = 1 \quad \text{but} \quad n(X^*) < 1. \]

Proof

- \[ c^* = \ell_1 \oplus_1 k \lim \implies X^* = \left[ c^* \oplus_1 c^* \oplus_1 c^* \right] / (\lim, \lim, \lim). \]

- Then, writing \[ Z = \ell_1^{(3)}/(1, 1, 1), \] we can identify

\[ X^* \equiv \ell_1 \oplus_1 \ell_1 \oplus_1 \ell_1 \oplus_1 Z, \quad X^{**} \equiv \ell_\infty \oplus_\infty \ell_\infty \oplus_\infty \ell_\infty \oplus_\infty Z^*. \]

- \( Z \) is an \( L \)-summand of \( X^* \) so

\[ n(X^*) \leq n(Z). \]

- But \( n(Z) < 1 ! \checkmark \]
Numerical index and duality (II)

The above example can be squeezed to get more counterexamples.
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Example 1

- Exists $X$ real with $n(X) = 1$ and $n(X^*) = 0$.
- Exists $X$ complex with $n(X) = 1$ and $n(X^*) = 1/e$. 
The above example can be squeezed to get more counterexamples.

**Example 1**
- Exists $X$ real with $n(X) = 1$ and $n(X^*) = 0$.
- Exists $X$ complex with $n(X) = 1$ and $n(X^*) = 1/e$.

**Example 2**
- Given $t \in ]0, 1]$, exists $X$ real with $n(X) = t$ and $n(X^*) = 0$.
- Given $t \in ]1/e, 1]$, exists $X$ complex with $n(X) = t$ and $n(X^*) = 1/e$. 
Numerical index and duality (III)

Some positive partial answers

One has $n(X) = n(X^*)$ when $X$ is reflexive (evident). $X$ is a $C^*$-algebra or a von Neumann predual (1970's – 2000's). $X$ is $L$-embedded in $X^{**}$ (M., 2009).

If $X$ has RNP and $n(X) = 1$, then $n(X^*) = 1$ (M., 2002).

If $X$ is $M$-embedded in $X^{**}$ and $n(X) = 1$, then $n(Y) = 1$ for $X \subseteq Y \subseteq X^{**}$.

Example $X = C(K(\ell^2([0, 1] \parallel \Delta)))$. Then $n(X) = 1$ and $X^* \equiv K(\ell^2) \oplus 1 C_0(K \parallel \Delta)$ and $X^{**} \equiv L(\ell^2) \oplus \infty C_0(K \parallel \Delta)$.

Therefore, $X^{**}$ is a $C^*$-algebra, but $n(X^*) = 1/2 < n(X) = 1$. 

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Example

\[ X = C_{K(\ell_2)}([0, 1] \| \Delta). \] Then \( n(X) = 1 \) and

\[ X^* \equiv K(\ell_2)^* \oplus_1 C_0(K\|\Delta)^* \quad \text{and} \quad X^{**} \equiv L(\ell_2) \oplus_\infty C_0(K\|\Delta)^{**}. \]

Therefore, \( X^{**} \) is a \( C^* \)-algebra, but \( n(X^*) = 1/2 < n(X) = 1 \).
Numerical index and duality: open problems

Main question
Find isometric or isomorphic properties assuring that $n(X) = n(X^*)$.

Question 1
If $Z$ has a unique predual $X$, does $n(X) = n(X^*)$?

Question 2
If $Z$ is a dual space, does there exist a predual $X$ such that $n(X) = n(X^*)$?

Question 4
If $X$ has the RNP, does $n(X) = n(X^*)$?
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The isomorphic point of view

Renorming and numerical index (Finet–M.–Payá, 2003)

\((X, \| \cdot \|)\) (separable or reflexive) Banach space. Then

\[0, 1 \subseteq \{ n(X, | \cdot |) : | \cdot | \cong \| \cdot \| \} \]

Real case:

Complex case:

\[e^{-1}, 1 \subseteq \{ n(X, | \cdot |) : | \cdot | \cong \| \cdot \| \} \]

Open question

The result is known to be true when \(X\) has a long biorthogonal system. Is it true in general?

Remark

In some sense, any other value of \(n(X)\) but 1 is isomorphically trivial.

\(\star\) What about the value 1?
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Recall that $X$ has numerical index one ($n(X) = 1$) iff

$$\|T\| = \sup\{|x^*(Tx)| : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}$$

(i.e. $\nu(T) = \|T\|$) for every $T \in L(X)$. 

**Observation**

For Hilbert spaces, the above formula is equivalent to

$$\|T\| = \sup\{|\langle Tx, x \rangle| : x \in S_X\}$$

which is known to be valid for every self-adjoint operator $T$. 

**Examples**

$C(K)$, $L^1(\mu)$, $A(D)$, $H^\infty$, finite-codimensional subspaces of $C[0, 1]$. . .
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Isomorphic properties (prohibitive results)

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Does every Banach space admit an equivalent norm with numerical index 1?
**Isomorphic properties (prohibitive results)**

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Does every Banach space admit an equivalent norm with numerical index 1?

**Negative answer (López–M.–Payá, 1999)**
Not every real Banach space can be renormed to have numerical index 1.

More details on this later on.
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Does every Banach space admit an equivalent norm with numerical index 1?

Negative answer (López–M.–Payá, 1999)

Not every real Banach space can be renormed to have numerical index 1. Concretely:

- If $X$ is real, reflexive, and $\dim(X) = \infty$, then $n(X) < 1$. 
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- If $X$ is real, reflexive, and $\dim(X) = \infty$, then $n(X) < 1$.
- Actually, if $X$ is real, $X^{**}/X$ separable and $n(X) = 1$, then $X$ is finite-dimensional.
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- If $X$ is real, reflexive, and $\dim(X) = \infty$, then $n(X) < 1$.
- Actually, if $X$ is real, $X^{**}/X$ separable and $n(X) = 1$, then $X$ is finite-dimensional.
- Moreover, if $X$ is real, RNP, $\dim(X) = \infty$, and $n(X) = 1$, then $X \supset \ell_1$. 
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- Moreover, if $X$ is real, RNP, $\dim(X) = \infty$, and $n(X) = 1$, then $X \supset \ell_1$.

**A very recent result (Avilés–Kadets–M.–Merí–Shepelska)**

If $X$ is real, $\dim(X) = \infty$ and $n(X) = 1$, then $X^* \supset \ell_1$.

More details on this later on.
Lemma

X Banach space, \( n(X) = 1 \Rightarrow \|x^* - 0(x_0)\| = 1 \) for all \( x^* \in \text{ext}(B^X) \) and all denting point \( x_0 \) of \( B^X \).

Proof:

Fix \( \varepsilon > 0 \). As \( x_0 \) denting point, \( \exists y^* \in S^{X^*} \) and \( \alpha > 0 \) such that \( \|z - x_0\| < \varepsilon \) whenever \( z \in B^X \) satisfies \( \Re y^*(z) > 1 - \alpha \).

(Choquet's lemma): \( x^* \in \text{ext}(B^X) \), \( \exists y \in S^X \) and \( \beta > 0 \) such that \( |z^*(y) - x^* - 0(x_0)| < \varepsilon \) whenever \( z^* \in B^X \) satisfies \( \Re z^*(y) > 1 - \beta \).

Let \( T = y^* \otimes y^* \in L(X) \). \( \|T\| = 1 = \Rightarrow v(T) = 1 \).

We may find \( x \in S^X, x^* \in S^{X^*} \), such that \( x^*(x) = 1 \) and \( |x^*(Tx) - x^* - 0(x_0)| = |y^*(x)| > 1 - \min\{\alpha, \beta\} \).

By choosing suitable \( s, t \in T \) we have \( \Re y^*(sx) = |y^*(x)| > 1 - \alpha \) and \( \Re tx^*(y) = |x^*(y)| > 1 - \beta \).

It follows that \( \|sx - x_0\| < \varepsilon \) and \( |tx^*(x_0) - x^* - 0(x_0)| < \varepsilon \), and so \( 1 - |x^* - 0(x_0)| \leq |tx^*(sx) - x^* - 0(x_0)| \leq |tx^*(sx) - tx^*(x_0)| + |tx^*(x_0)| < 2 \varepsilon \).
Lemma

Let $X$ be a Banach space, with $n(X) = 1$. Then $|x^*_0(x_0)| = 1$ for all $x^*_0 \in \text{ext}(B_{X^*})$ and all denting point $x_0$ of $B_X$. 

Proof:

Fix $\varepsilon > 0$. As $x_0$ is a denting point, there exists $y^* \in S_{X^*}$ and $\alpha > 0$ such that $\|z - x_0\| < \varepsilon$ whenever $z \in B_X$ satisfies $\Re y^*(z) > 1 - \alpha$.

(Choquet's lemma):

For $x^*_0 \in \text{ext}(B_{X^*})$, there exist $y^* \in S_{X^*}$ and $\beta > 0$ such that $|z^*(x_0) - x^*_0(x_0)| < \varepsilon$ whenever $z^* \in B_{X^*}$ satisfies $\Re z^*(y) > 1 - \beta$.

Let $T = y^* \otimes y^* \in L(X)$. Then $\|T\| = 1$ implies $v(T) = 1$.

We may find $x \in S_X$, $x^* \in S_{X^*}$ such that $x^*(x) = 1$ and $|x^*_0(x_0)| = |z^*(x_0)|$, whenever $z^* \in B_{X^*}$ satisfies $\Re z^*(y^*) > 1 - \min\{\alpha, \beta\}$.

By choosing suitable $s, t \in T$ we have $\Re y^*(sx) = |y^*(x)| > 1 - \alpha$ and $\Re tx^*(y) = |x^*(y)| > 1 - \beta$.

It follows that $\|sx - x_0\| < \varepsilon$ and $|tx^*(x_0) - x^*_0(x_0)| < \varepsilon$, and so $1 - |x^*_0(x_0)| \leq |tx^*(sx) - x^*_0(x_0)| \leq |tx^*(sx) - tx^*(x_0)| + |tx^*(x_0)| < 2\varepsilon$. 

$\blacksquare$
Lemma

\(X\) Banach space, \(n(X) = 1\)
\[\implies |x_0^*(x_0)| = 1 \text{ for all } x_0^* \in \text{ext } (B_{X^*}) \text{ and all denting point } x_0 \text{ of } B_X.\]

Proof:

\[\text{Fix } \varepsilon > 0. \text{ As } x_0 \text{ denting point, } \exists y^* \in S_{X^*} \text{ and } \alpha > 0 \text{ such that } \|z - x_0\| < \varepsilon \text{ whenever } z \in B_X \text{ satisfies } \Re y^*(z) > 1 - \alpha.\]

(Choquet's lemma):
\[x_0^* \in \text{ext } (B_{X^*}), \exists y \in S_{X^*} \text{ and } \beta > 0 \text{ such that } |z^*(x_0) - x_0^*(x_0)| < \varepsilon \text{ whenever } z^* \in B_{X^*} \text{ satisfies } \Re z^*(y) > 1 - \beta.\]

Let \(T = y^* \otimes y^* \in L(X)\).
\[\|T\| = 1 \implies v(T) = 1.\]

We may find \(x \in S_X, x^* \in S_{X^*}, \) such that \(x^*(x) = 1\) and \(|x^*(Tx) - x^*(x_0)| < \varepsilon\) whenever \(z^* \in B_{X^*} \text{ satisfies } \Re z^*(y) > 1 - \min\{\alpha, \beta\}.\)

By choosing suitable \(s, t \in T\) we have \(\Re y^*(sx) = |y^*(x_0)| > 1 - \alpha \) and \(\Re tx^*(y) = |x^*(y)| > 1 - \beta.\)

It follows that \(\|sx - x_0\| < \varepsilon\) and \(|tx^*(x_0) - x_0^*(x_0)| < \varepsilon\), and so \(1 - |x_0^*(x_0)| \leq |tx^*(sx) - x_0^*(x_0)| \leq 1 - |x_0^*(x_0)| + |tx^*(x_0) - x_0^*(x_0)| < 2 \varepsilon.\) \(\square\)
Lemma

\( X \) Banach space, \( n(X) = 1 \)
\( \implies |x^*_0(x_0)| = 1 \) for all \( x^*_0 \in \text{ext}(B_{X^*}) \) and all denting point \( x_0 \) of \( B_X \).

Proof:
- Fix \( \varepsilon > 0 \). As \( x_0 \) denting point, \( \exists y^* \in S_{X^*} \) and \( \alpha > 0 \) such that
  \[ \|z - x_0\| < \varepsilon \quad \text{whenever } z \in B_X \text{ satisfies } \text{Re} y^*(z) > 1 - \alpha. \]
Lemma

\( X \) Banach space, \( n(X) = 1 \)
\( \implies |x_0^*(x_0)| = 1 \) for all \( x_0^* \in \text{ext}(B_{X^*}) \) and all denting point \( x_0 \) of \( B_X \).

Proof:

• Fix \( \varepsilon > 0 \). As \( x_0 \) denting point, \( \exists y^* \in S_{X^*} \) and \( \alpha > 0 \) such that
  \[ ||z - x_0|| < \varepsilon \quad \text{whenever } z \in B_X \text{ satisfies } \text{Re } y^*(z) > 1 - \alpha. \]

• \textbf{(Choquet’s lemma):} \( x_0^* \in \text{ext}(B_{X^*}), \exists y \in S_X \) and \( \beta > 0 \) such that
  \[ |z^*(x_0) - x_0^*(x_0)| < \varepsilon \quad \text{whenever } z^* \in B_{X^*} \text{ satisfies } \text{Re } z^*(y) > 1 - \beta. \]
Lemma

Let $X$ be a Banach space, $n(X) = 1$ implies $|x^*_0(x_0)| = 1$ for all $x^*_0 \in \text{ext}(B_{X^*})$ and all denting point $x_0$ of $B_X$.

Proof:
- Fix $\varepsilon > 0$. As $x_0$ denting point, there exists $y^* \in S_{X^*}$ and $\alpha > 0$ such that $\|z - x_0\| < \varepsilon$ whenever $z \in B_X$ satisfies $\text{Re} y^*(z) > 1 - \alpha$.
- (Choquet's lemma): $x^*_0 \in \text{ext}(B_{X^*})$, $\exists y \in S_X$ and $\beta > 0$ such that $|z^*(x_0) - x^*_0(x_0)| < \varepsilon$ whenever $z^* \in B_{X^*}$ satisfies $\text{Re} z^*(y) > 1 - \beta$.
- Let $T = y^* \otimes y \in L(X)$. $\|T\| = 1$ implies $v(T) = 1$. 

\[\]
Lemma

Let $X$ be a Banach space, $n(X) = 1$.

Then $|x^*_0(x_0)| = 1$ for all $x^*_0 \in \text{ext}(B_{X^*})$ and all denting point $x_0$ of $B_X$.

**Proof:**

- Fix $\varepsilon > 0$. As $x_0$ denting point, $\exists y^* \in S_{X^*}$ and $\alpha > 0$ such that $\|z - x_0\| < \varepsilon$ whenever $z \in B_X$ satisfies $\text{Re}y^*(z) > 1 - \alpha$.

- *(Choquet’s lemma)*: $x^*_0 \in \text{ext}(B_{X^*})$, $\exists y \in S_X$ and $\beta > 0$ such that $|z^*(x_0) - x^*_0(x_0)| < \varepsilon$ whenever $z^* \in B_{X^*}$ satisfies $\text{Re}z^*(y) > 1 - \beta$.

- Let $T = y^* \otimes y \in L(X)$. $\|T\| = 1 \implies v(T) = 1$.

- We may find $x \in S_X$, $x^* \in S_{X^*}$, such that $x^*(x) = 1$ and $|x^*(Tx)| = |y^*(x)||x^*(y)| > 1 - \min\{\alpha, \beta\}$. 


Lemma

Let $X$ be a Banach space, $n(X) = 1$ if and only if $|x^*_0(x_0)| = 1$ for all $x^*_0 \in \text{ext}(B_{X^*})$ and all denting point $x_0$ of $B_X$.

Proof:

- Fix $\epsilon > 0$. As $x_0$ denting point, $\exists y^* \in S_{X^*}$ and $\alpha > 0$ such that $\|z - x_0\| < \epsilon$ whenever $z \in B_X$ satisfies $\text{Re} y^*(z) > 1 - \alpha$.

- *Choquet’s lemma*: $x^*_0 \in \text{ext}(B_{X^*})$, $\exists y \in S_X$ and $\beta > 0$ such that $|z^*(x_0) - x^*_0(x_0)| < \epsilon$ whenever $z^* \in B_{X^*}$ satisfies $\text{Re} z^*(y) > 1 - \beta$.

- Let $T = y^* \otimes y \in L(X)$. $\|T\| = 1 \implies v(T) = 1$.

- We may find $x \in S_X$, $x^* \in S_{X^*}$, such that $x^*(x) = 1$ and $|x^*(Tx)| = |y^*(x)||x^*(y)| > 1 - \min\{\alpha, \beta\}$.

- By choosing suitable $s, t \in \mathbb{T}$ we have

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\text{Re} y^*(sx) = |y^*(x)| > 1 - \alpha \quad \& \quad \text{Re} tx^*(y) = |x^*(y)| > 1 - \beta.
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Lemma

Let $X$ be a Banach space, $n(X) = 1$ implies $|x_0^*(x_0)| = 1$ for all $x_0^* \in \text{ext}(B_{X^*})$ and all denting point $x_0$ of $B_X$.

Proof:

- Fix $\varepsilon > 0$. As $x_0$ denting point, $\exists y^* \in S_{X^*}$ and $\alpha > 0$ such that $\|z - x_0\| < \varepsilon$ whenever $z \in B_X$ satisfies $\text{Re} \, y^*(z) > 1 - \alpha$.

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- Let $T = y^* \otimes y \in L(X)$. $\|T\| = 1 \implies \nu(T) = 1$.

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- It follows that $\|sx - x_0\| < \varepsilon$ and $|tx^*(x_0) - x_0^*(x_0)| < \varepsilon$, $\square$
Lemma

Let $X$ be a Banach space, $n(X) = 1 \implies |x_0^*(x_0)| = 1$ for all $x_0^* \in \text{ext}(B_{X^*})$ and all denting point $x_0$ of $B_X$.

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- It follows that $\|sx - x_0\| < \varepsilon$ and $|tx^*(x_0) - x_0^*(x_0)| < \varepsilon$, and so

$$1 - |x_0^*(x_0)| \leq |tx^*(sx) - x_0^*(x_0)| \leq |tx^*(sx) - tx^*(x_0)| + |tx^*(x_0) - x_0^*(x_0)| < 2\varepsilon.$$
Proposition

$X$ real, $A \subset S_X$ infinite with $|x^* (a)| = 1$ $\forall x^* \in \text{ext}(B_X^*)$, $\forall a \in A$.

$\Rightarrow X \supseteq c_0$ or $X \supseteq \ell_1$.

Proof:

$X \supseteq \ell_1$ ✓

(Rosenthal $\ell_1$-theorem): Otherwise, $\exists \{a_n\} \subseteq A$ non-trivial weak Cauchy.

Consider $Y$ the closed linear span of $\{a_n: n \in \mathbb{N}\}$.

$\|a_n - a_m\| = 2$ if $n \neq m$ $\Rightarrow \text{dim}(Y) = \infty$.

(Krein-Milman theorem): every $y^* \in \text{ext}(B_Y^*)$ has an extension which belongs to $\text{ext}(B_X^*)$.

So, $|y^*(a_n)| = 1$ $\forall y^* \in \text{ext}(B_Y^*), \forall n \in \mathbb{N}$.

$\{a_n\}$ weak Cauchy $\Rightarrow \{y^*(a_n)\}$ is eventually 1 or $-1$.

Then $\text{ext}(B_Y^*) = \bigcup_{k \in \mathbb{N}} (E_k \cup -E_k)$ where $E_k = \{y^* \in \text{ext}(B_Y^*): y^*(a_n) = 1 \text{ for } n \geq k\}$.

$\{a_n\}$ separates points of $Y$ $\Rightarrow E_k$ finite, so $\text{ext}(B_Y^*)$ countable.

(Fonf): $Y \supseteq c_0$. So, $X \supseteq c_0$.

✓
Proposition

$X$ real, $A \subset S_X$ infinite with $|x^*(a)| = 1 \ \forall x^* \in \text{ext} \ (B_{X^*}), \ \forall a \in A.$

$\implies X \supseteq c_0$ or $X \supseteq \ell_1.$
Proposition

**Proposition**

Let \( X \) be real, \( A \subset S_X \) infinite with \( |x^*(a)| = 1 \) for all \( x^* \in \text{ext}(B_{X^*}) \) and \( a \in A \). Then \( X \supseteq c_0 \) or \( X \supseteq \ell_1 \).

**Proof:**

Consider \( Y \), the closed linear span of \( \{a_n\} \subset A \), and assume it is non-trivial. Let \( \{a_n\} \) be a weak Cauchy sequence. Consider the weak closure of \( \{y^*(a_n)\} \), where \( y^* \in \text{ext}(B_{Y^*}) \). By the Krein-Milman theorem, \( y^* \) has an extension to \( B_{X^*} \). Then \( \{y^*(a_n)\} \) is eventually \( 1 \) or \(-1 \).

Finally, \( \text{ext}(B_{Y^*}) = \bigcup_{k \in \mathbb{N}} (E_k \cup -E_k) \) where \( E_k = \{y^* \in \text{ext}(B_{Y^*}) : y^*(a_n) = 1 \text{ for } n \geq k \} \). Since \( \{a_n\} \) separates points of \( Y^* \), \( E_k \) is finite, and hence \( \text{ext}(B_{Y^*}) \) is countable. Thus, \( Y \supseteq c_0 \). Therefore, \( X \supseteq c_0 \) or \( X \supseteq \ell_1 \).
Proposition

Let $X$ be real, $A \subset S_X$ infinite with $|x^*(a)| = 1$ for all $x^* \in \text{ext}(B_{X^*})$, $\forall a \in A$.

$\implies X \supseteq c_0$ or $X \supseteq \ell_1$.

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Proof:

- $X \supseteq \ell_1$ ✓

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\( X \text{ real, } A \subset S_X \text{ infinite with } |x^*(a)| = 1 \ \forall x^* \in \text{ext} \ (B_{X^*}), \ \forall a \in A. \)

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Proof:

- \( X \supseteq \ell_1 \) ✓
- (Rosenthal \( \ell_1 \)-theorem): Otherwise, \( \exists \ \{a_n\} \subseteq A \) non-trivial weak Cauchy.
- Consider \( Y \) the closed linear span of \( \{a_n : n \in \mathbb{N}\} \).
Proving the 1999 results (II)

**Proposition**

$X$ real, $A \subset S_X$ infinite with $|x^*(a)| = 1 \ \forall x^* \in \text{ext}(B_{X^*}), \ \forall a \in A.$

$\implies X \supseteq c_0$ or $X \supseteq \ell_1.$

**Proof:**

- $X \supseteq \ell_1 \checkmark$
- *(Rosenthal $\ell_1$-theorem):* Otherwise, $\exists \{a_n\} \subseteq A$ non-trivial weak Cauchy.
- Consider $Y$ the closed linear span of $\{a_n : n \in \mathbb{N}\}$.
- $\|a_n - a_m\| = 2$ if $n \neq m \implies \dim(Y) = \infty.$
Proving the 1999 results (II)

**Proposition**

\[
X \text{ real, } A \subset S_X \text{ infinite with } |x^*(a)| = 1 \forall x^* \in \text{ext}(B_{X^*}), \forall a \in A.
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\[\implies X \supseteq c_0 \text{ or } X \supseteq \ell_1.\]

**Proof:**

- \(X \supseteq \ell_1\) \(\checkmark\)
- **(Rosenthal \(\ell_1\)-theorem):** Otherwise, \(\exists \{a_n\} \subseteq A\) non-trivial weak Cauchy.
- Consider \(Y\) the closed linear span of \(\{a_n : n \in \mathbb{N}\}\).
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- **(Krein-Milman theorem):** every \(y^* \in \text{ext}(B_{Y^*})\) has an extension which belongs to \(\text{ext}(B_{X^*})\).
Proposition

$X$ real, $A \subset S_X$ infinite with $|x^*(a)| = 1 \ \forall x^* \in \text{ext} \ (B_{X^*})$, $\forall a \in A$.  

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- So, $|y^*(a_n)| = 1 \ \forall y^* \in \text{ext} \ (B_{Y^*})$, $\forall n \in \mathbb{N}$.
Proposition

\( X \) real, \( A \subset S_X \) infinite with \( |x^*(a)| = 1 \ \forall x^* \in \text{ext} (B_{X^*}), \ \forall a \in A \).

\[ \implies X \supseteq c_0 \text{ or } X \supseteq \ell_1. \]

Proof:

- \( X \supseteq \ell_1 \ check \)
- **(Rosenthal \( \ell_1 \)-theorem)**: Otherwise, \( \exists \{a_n\} \subseteq A \) non-trivial weak Cauchy.
- Consider \( Y \) the closed linear span of \( \{a_n : n \in \mathbb{N}\} \).
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Proposition

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- \( \{a_n\} \) weak Cauchy \( \implies \{y^*(a_n)\} \) is eventually 1 or \(-1.\)
- Then \( \text{ext} (B_{Y^*}) = \bigcup_{k \in \mathbb{N}} (E_k \cup -E_k) \) where
  \[ E_k = \{y^* \in \text{ext} (B_{Y^*}) : y^*(a_n) = 1 \text{ for } n \geq k\}. \]
Proving the 1999 results (II)

**Proposition**

\(X\) real, \(A \subset S_X\) infinite with \(|x^*(a)| = 1\ \forall x^* \in \text{ext } (B_{X^*}), \forall a \in A.\)

\[\implies X \supseteq c_0\ or\ X \supseteq \ell_1.\]

**Proof:**

- **\(X \supseteq \ell_1\)** ✓
- **(Rosenthal \(\ell_1\)-theorem):** Otherwise, \(\exists \{a_n\} \subseteq A\) non-trivial weak Cauchy.
- Consider \(Y\) the closed linear span of \(\{a_n : n \in \mathbb{N}\}\).
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\( \implies X \supseteq c_0 \text{ or } X \supseteq \ell_1. \)

Proof:

- \( X \supseteq \ell_1 \checkmark \)
- (Rosenthal \( \ell_1 \)-theorem): Otherwise, \( \exists \ \{a_n\} \subseteq A \text{ non-trivial weak Cauchy.} \)
- Consider \( Y \) the closed linear span of \( \{a_n : n \in \mathbb{N}\}. \)
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- So, \( |y^*(a_n)| = 1 \ \forall y^* \in \text{ext}(B_{Y^*}), \ \forall n \in \mathbb{N}. \)
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- (Fonf): \( Y \supseteq c_0. \) So, \( X \supseteq c_0. \checkmark \)
Lemma

Let $X$ be a Banach space, $n(X) = 1$.

Then $|x_0^*(x_0)| = 1$ for all $x_0^* \in \text{ext}(B_{X^*})$ and all denting point $x_0$ of $B_X$.

Proposition

Let $X$ be real, $A \subset S_X$ infinite with $|x^*(a)| = 1 \forall x^* \in \text{ext}(B_{X^*})$, $\forall a \in A$.

Then $X \supseteq c_0$ or $X \supseteq \ell_1$. 

Lemma

$X$ Banach space, $n(X) = 1$

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$X$ real, $A \subset S_X$ infinite with $|x^*(a)| = 1 \ \forall x^* \in \text{ext} (B_{X^*}), \forall a \in A.$

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Main consequence

$X$ real, RNP, $\dim(X) = \infty$, and $n(X) = 1 \implies X \supseteq \ell_1$.
Proving the 1999 results (III)

**Lemma**

Let $X$ be a Banach space, $n(X) = 1$.

$$|x_0^*(x_0)| = 1 \quad \text{for all } x_0^* \in \text{ext} \,(B_{X^*}) \text{ and all denting point } x_0 \text{ of } B_X.$$ 

**Proposition**

Let $X$ be real, $A \subset S_X$ infinite with $|x^*(a)| = 1 \quad \forall x^* \in \text{ext} \,(B_{X^*}), \forall a \in A$.

$$X \supseteq c_0 \text{ or } X \supseteq \ell_1.$$ 

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Let $X$ be real, RNP, $\dim(X) = \infty$, and $n(X) = 1$.

$$X \supseteq \ell_1.$$ 

**Proof.**
Lemma

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Proposition

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Proof.

- $X$ RNP, $\dim(X) = \infty$ $\implies \exists$ infinitely many denting points of $B_X$. 


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- Therefore, $X \supseteq c_0$ or $X \supseteq \ell_1$. 
Lemma

$X$ Banach space, $n(X) = 1$ \[ \Rightarrow |x_0^*(x_0)| = 1 \text{ for all } x_0^* \in \text{ext}(B_{X^*}) \text{ and all denting point } x_0 \text{ of } B_X. \]

Proposition

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$X$ real, RNP, $\dim(X) = \infty$, and $n(X) = 1 \Rightarrow X \supseteq \ell_1.$

Proof.

- $X$ RNP, $\dim(X) = \infty \Rightarrow \exists$ infinitely many denting points of $B_X.$
- Therefore, $X \supseteq c_0 \text{ or } X \supseteq \ell_1.$
- If $X$ RNP, then $X \not\supseteq c_0.$ ✓
Proving the 1999 results (III)

**Lemma**

$X$ Banach space, $n(X) = 1$

$\implies |x_0^*(x_0)| = 1$ for all $x_0^* \in \text{ext} (B_{X^*})$ and all denting point $x_0$ of $B_X$.

**Proposition**

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**Main consequence**

$X$ real, RNP, $\dim(X) = \infty$, and $n(X) = 1$ $\implies X \supseteq \ell_1$.

**Corollary**

$X$ real, $\dim(X) = \infty$, $n(X) = 1$.

- $X$ is not reflexive.
- $X^{**}/X$ is non-separable.
Isomorphic properties (positive results)

A renorming result (Boyko–Kadets–M.–Merí, 2009)

If $X$ is separable, $X \supset c_0$, then $X$ can be renormed to have numerical index $1$.

Consequence: $X$ separable containing $c_0$ $\Rightarrow$ there is $Z \cong X$ such that $n(Z) = 1$ and

- $\{n(Z^*) = 0\}$ real case
- $\{n(Z^*) = e^{-1}\}$ complex case

Open questions:

Find isomorphic properties which assures renorming with numerical index $1$.

In particular, if $X \supset \ell_1$, can $X$ be renormed to have numerical index $1$?

Negative result (Bourgain–Delbaen, 1980)

There is $X$ such that $X^* \cong \ell_1$ and $X$ has the RNP. Then, $X$ can not be renormed with numerical index $1$ (in such a case, $X \supset \ell_1$).
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Isomorphic properties (positive results)
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If $X$ is separable, $X \supset c_0$, then $X$ can be renormed to have numerical index 1.

### Consequence

$X$ separable containing $c_0$ $\implies$ there is $Z \simeq X$ such that

\[
n(Z) = 1 \quad \text{and} \quad \begin{cases} 
n(Z^*) = 0 & \text{real case} \\
n(Z^*) = e^{-1} & \text{complex case} \end{cases}
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Isomorphic properties (positive results)

A renorming result (Boyko–Kadets–M.–Merí, 2009)

If $X$ is separable, $X \supset c_0$, then $X$ can be renormed to have numerical index $1$.

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$X$ separable containing $c_0 \implies$ there is $Z \simeq X$ such that

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Open questions

Find isomorphic properties that assure renorming with numerical index $1$.

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There is $X$ such that $X^* \simeq \ell_1$ and $X$ has the RNP. Then, $X$ cannot be renormed with numerical index $1$ (in such a case, $X \supset \ell_1$!)

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Isomorphic properties (positive results)

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Isomorphic properties (positive results)

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If \( X \) is separable, \( X \supseteq c_0 \), then \( X \) can be renormed to have numerical index 1.

Consequence

\( X \) separable containing \( c_0 \) \( \implies \) there is \( Z \cong X \) such that

\[
\begin{align*}
n(Z) &= 1 \\
\begin{cases}
n(Z^*) = 0 & \text{real case} \\
n(Z^*) = e^{-1} & \text{complex case}
\end{cases}
\end{align*}
\]

Open questions

- Find isomorphic properties which assures renorming with numerical index 1
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### Isomorphic properties (positive results)

<table>
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Isometric properties: finite-dimensional spaces

A real or complex finite-dimensional space $X$ is said to have numerical index one if for every $x^* \in \text{ext}(B_{X^*})$, $x \in \text{ext}(B_X)$, we have

$$|x^*(x)| = 1.$$ 

This shows a rough behavior of the norm of a finite-dimensional space with numerical index 1: the space is not smooth and not strictly convex.

**Question**

What is the situation in the infinite-dimensional case?
Isometric properties: finite-dimensional spaces

Finite-dimensional spaces (McGregor, 1971; Lima, 1978)

$X$ real or complex finite-dimensional space. TFAE:

- $n(X) = 1$. 

Remark: This shows a rough behavior of the norm of a finite-dimensional space with numerical index $1$:

- The space is not smooth.
- The space is not strictly convex.

Question: What is the situation in the infinite-dimensional case?
Finite-dimensional spaces (McGregor, 1971; Lima, 1978)

Let $X$ be a real or complex finite-dimensional space. TFAE:

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- $|x^*(x)| = 1$ for every $x^* \in \text{ext } (B_{X^*})$, $x \in \text{ext } (B_X)$.

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Isometric properties: finite-dimensional spaces

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Let $X$ be a real or complex finite-dimensional space. TFAE:

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- $|x^*(x)| = 1$ for every $x^* \in \text{ext}(B_{X^*})$, $x \in \text{ext}(B_X)$.
- $B_X = \text{aconv}(F)$ for every maximal convex subset $F$ of $S_X$ ($X$ is a CL-space).

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- $|x^*(x)| = 1$ for every $x^* \in \text{ext} \,(B_{X^*})$, $x \in \text{ext} \,(B_X)$.
- $B_X = \text{aconv}(F)$ for every maximal convex subset $F$ of $S_X$ $(X$ is a CL-space).

Remark

This shows a rough behavior of the norm of a finite-dimensional space with numerical index 1:

- The space is not smooth.
- The space is not strictly convex.
**Finite-dimensional spaces (McGregor, 1971; Lima, 1978)**

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This shows a rough behavior of the norm of a finite-dimensional space with numerical index 1:

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**Question**

What is the situation in the infinite-dimensional case?
Isometric properties: infinite-dimensional spaces

Theorem (Kadets–M.–Merí–Payá, 2009)

Let $X$ be an infinite-dimensional Banach space, and assume $\text{num}(X) = 1$. Then $X^*$ is neither smooth nor strictly convex. The norm of $X$ cannot be Fréchet-smooth. There is no WLUR points in $S_X$. 
Isometric properties: infinite-dimensional spaces

**Theorem (Kadets–M.–Merí–Payá, 2009)**

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Theorem (Kadets–M.–Merí–Payá, 2009)

Let $X$ be an infinite-dimensional Banach space with numerical index $n(X) = 1$. Then

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Proving that $X^*$ is not smooth:
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Proving that $X^*$ is not smooth:

- $\dim(X) > 1$, exists $x_0 \in S_X$ and $x_0^* \in S_{X^*}$ such that $x_0^*(x_0) = 0$. Then, consider $T = x_0^* \otimes x_0$ which satisfies $T^2 = 0$, $\|T\| = 1$. 

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- We may find $\lambda_n \in \mathbb{T}$ and $(x_n^*, x_n^{**}) \in S_{X^*} \times S_{X^{**}}$ such that

$$\lambda_n x_n^{**}(x_n^*) = 1 \quad \text{and} \quad [T_n^{**}(x_n^{**})](x_n^*) = x_n^{**}(T_n^*(x_n^*)) = 1.$$
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- \( \dim(X) > 1 \), exists \( x_0 \in S_X \) and \( x_0^* \in S_{X^*} \) such that \( x_0^*(x_0) = 0 \). Then, consider \( T = x_0^* \otimes x_0 \) which satisfies \( T^2 = 0, \|T\| = 1 \).
- \( \text{(AcostaPayá1993)}: \) exists \( \{T_n\} \rightarrow T \) such that \( \|T_n\| = 1, T_n^* \) attains its numerical radius \( v(T_n^*) = v(T_n) = \|T_n\| = 1 \).
- We may find \( \lambda_n \in \mathbb{T} \) and \( (x_n^*, x_n^{**}) \in S_{X^*} \times S_{X^{**}} \) such that \( \lambda_n x_n^{**}(x_n^*) = 1 \) and \( [T_n^{**}(x_n^{**})](x_n^*) = x_n^{**}(T_n^*(x_n^*)) = 1 \).
- If \( X^* \) is smooth: \( T_n^{**}(x_n^{**}) = \lambda_n x_n^{**} \). Thus, \( \|T_n^{**}(x_n^{**})\|^2 = \|\lambda_n x_n^{**}\|^2 = 1 \).
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\[ \| [T_n^{**}]^2 (x_n^{**}) \| = \| \lambda_n^2 x_n^{**} \| = 1. \]

- But, since $T_n \rightarrow T$ and $T^2 = 0$, then $[T_n^{**}]^2 \rightarrow 0$ !!
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Corollary

$X = C(\mathbb{T})/A(\mathbb{D})$. $X^* = H^1$ is smooth $\implies n(X) < 1 \& n(H^1) < 1$. 
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Example without completeness

- There is \( X \) (non-complete) strictly convex with \( X^* \equiv L_1(\mu) \), so \( n(X) = 1 \).
- \( \tilde{X} \) completion of \( X \). For \( F \subseteq S_{\tilde{X}} \) maximal face, \( B_{\tilde{X}} = \text{aconv}(F) \).
Isometric properties: infinite-dimensional spaces

**Theorem (Kadets–M.–Merí–Payá, 2009)**

Let $X$ be an infinite-dimensional Banach space with $n(X) = 1$. Then

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**Corollary**

If $X = C(T)/A(D)$. Then $X^* = H^1$ is smooth implies $n(X) < 1$ and $n(H^1) < 1$.

**Example without completeness**

- There is $X$ (non-complete) strictly convex with $X^* \equiv L_1(\mu)$, so $n(X) = 1$.
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**Open question**

Is there $X$ with $n(X) = 1$ which is smooth or strictly convex?
Numerical index  
Banach spaces with numerical index one

Asymptotic behavior of the set of spaces with numerical index one
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Theorem (Oikhberg, 2005)

There is a universal constant $c$ such that

$$\text{dist}(X, \ell_2^m) \geq c \ m^{\frac{1}{4}}$$

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- What is the diameter of the set of all $m$-dimensional $X$’s with $n(X) = 1$ ?
How to deal with numerical index 1 property?

One the one hand: weaker properties

In a general Banach space, we only can construct compact (actually, finite-rank) operators. Actually, we only may easily calculate the norm of rank-one operators. All the results given before for Banach spaces in which we use numerical index 1 only need \( v(T) = \|T\| \) for every rank-one operator \( T \). This is called the alternative Daugavet property (ADP) and we will present it in the next section.

One the other hand: stronger properties

We do not know any operator-free characterization of Banach spaces with numerical index 1. When we know that a Banach space has numerical index 1 (or that it can be renormed with numerical index 1), we actually prove more. Later we will study sufficient geometrical conditions. The weakest property is called lushness.
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How to deal with numerical index 1 property?

One of the key ideas to get interesting results for Banach spaces with numerical index 1 is to study when the three properties below are equivalent.

- lushness
- Numerical index 1
- ADP

with SCD property (RNP, Asplund...)

A very interesting property appears: the slicely countably determination. We will study this property later on.
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**Relationship between the properties**

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Numerical index
How to deal with numerical index 1 property?
Some interesting open problems

Open problems

Characterize (without operators) Banach spaces with numerical index 1.

\[ X \ni n(X) = 1, \quad \text{dim}(X) = \infty \]

\[ X \supset c_0 \text{ or } \supset \ell_1 ? \]

Characterize those \( X \) admitting a renorming with numerical index 1.

If \( X \supset c_0 \text{ or } \supset \ell_1 \) can \( X \) be renormed with numerical index 1?

Find isomorphic or isometric conditions assuring that \( n(X) = n(X^*) \).

The oldest open problem

Calculate the numerical index of "classical" spaces.

- In particular, calculate \( n(L^p(\mu)) \).
Some interesting open problems

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The alternative Daugavet property

- The Daugavet property
- The alternative Daugavet property
  - Geometric characterizations
  - $C^*$-algebras and preduals
  - Some results

M. Martín and T. Oikberg

*An alternative Daugavet property*

M. Martín

*The alternative Daugavet property of $C^*$-algebras and $JB^*$-triples*
In a Banach space $X$ with the Radon-Nikodým property the unit ball has many denting points.
The Daugavet property: motivation

- In a Banach space $X$ with the Radon-Nikodým property the unit ball has many denting points.
- $x \in S_X$ is a denting point of $B_X$ if for every $\varepsilon > 0$ one has
  $$x \notin \overline{co}(B_X \setminus (x + \varepsilon B_X)).$$
In a Banach space $X$ with the Radon-Nikodým property the unit ball has many denting points.

$x \in S_X$ is a denting point of $B_X$ if for every $\varepsilon > 0$ one has

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$C[0,1]$ and $L_1[0,1]$ have an extremely opposite property: for every $x \in S_X$ and every $\varepsilon > 0$

$$\overline{co} \left( B_X \setminus (x + (2 - \varepsilon)B_X) \right) = B_X.$$
In a Banach space $X$ with the Radon-Nikodým property the unit ball has many denting points.

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This geometric property is equivalent to a property of operators on the space.
The alternative Daugavet property

The Daugavet property: definition

**The Daugavet equation**

$X$ Banach space, $T \in L(X)$

$$\|\text{Id} + T\| = 1 + \|T\| \quad \text{(DE)}$$
The Daugavet property: definition

The Daugavet equation

$X$ Banach space, $T \in L(X)$

$$\|\text{Id} + T\| = 1 + \|T\| \quad \text{(DE)}$$

Classical examples

1. **Daugavet, 1963:**
   Every compact operator on $C[0,1]$ satisfies (DE).

2. **Lozanoskii, 1966:**
   Every compact operator on $L_1[0,1]$ satisfies (DE).

3. **Abramovich, Holub, and more, 80’s:**
   $X = C(K)$, $K$ perfect compact space
   or $X = L_1(\mu)$, $\mu$ atomless measure
   $\implies$ every weakly compact $T \in L(X)$ satisfies (DE).
The Daugavet property: definition

The Daugavet equation

\[ X \text{ Banach space, } T \in L(X) \]
\[ \|\text{Id} + T\| = 1 + \|T\| \quad (DE) \]

The Daugavet property

A Banach space \( X \) is said to have the \textbf{Daugavet property} iff every rank-one operator on \( X \) satisfies (DE).

\( \star \) Then, every weakly compact operator on \( X \) satisfies (DE).

\( (Kadets–Shvidkoy–Sirotkin–Werner, 1997 & 2000) \)
The Daugavet property: geometric characterizations

Theorem [KSSW]

$X$ Banach space. TFAE:

- $X$ has the Daugavet property.

Every rank-one operator $T \in L(X)$ satisfies

$$\|\text{Id} + T\| = 1 + \|T\|.$$
The Daugavet property: geometric characterizations

**Theorem [KSSW]**

Let $X$ be a Banach space. TFAE:

- $X$ has the Daugavet property.

- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\epsilon > 0$, there exists $y \in S_X$ such that
  \[ \Re x^*(y) > 1 - \epsilon \quad \text{and} \quad \|x - y\| \geq 2 - \epsilon. \]

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The Daugavet property: geometric characterizations

**Theorem [KSSW]**

Let $X$ be a Banach space. The following are equivalent:

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- For every $x \in S_X$ and every $\varepsilon > 0$, we have
  \[ \overline{co} \left( B_X \setminus (x + (2 - \varepsilon)B_X) \right) = B_X. \]
Some propaganda

\( X \) with the Daugavet property. Then:

- \( X \) does not have the Radon-Nikodým property.

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- $X$ does not embed into a unconditional sum of Banach spaces without a copy of $\ell_1$.
  
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Observation (Duncan-McGregor-Price-White, 1970) \[ X \] Banach space, \( T \in \mathcal{L}(X) \):

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\sup \text{Re} \, V(T) = \|T\| \iff \|\text{Id} + T\| = 1 + \|T\|.
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\[ X \] Banach space:

Daugavet property (DPr): every rank-one \( T \) satisfies \( \|\text{Id} + T\| = 1 + \|T\| \) (DE)

numerical index \( 1 \): every \( T \) satisfies \( \max_{\theta \in T} \|\text{Id} + \theta T\| = 1 + \|T\| \) (aDE)

The alternative Daugavet property (M.–Oikhberg, 2004)

alternative Daugavet property (ADP): every rank-one \( T \in \mathcal{L}(X) \) satisfies (aDE).

\[ \star \] Then, every weakly compact operator satisfies (aDE).
Observation (Duncan-McGregor-Price-White, 1970)

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The DPr, the ADP and numerical index 1

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Relations between the properties

\[ \text{Daugavet property} = \text{ADP} \neq \text{Numerical index} = 1 \]

Examples

- \( C \left( [0, 1], K(\ell_2) \right) \) has DPr, but has not numerical index 1
- \( c_0 \) has numerical index 1, but has not DPr
- \( c_0 \oplus \infty C \left( [0, 1], K(\ell_2) \right) \) has ADP, neither DPr nor numerical index 1

Remarks

For RNP or Asplund spaces, \( \text{ADP} \Rightarrow \text{numerical index} = 1 \).

Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.
Relations between the properties

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Geometric characterizations of the ADP

**Theorem**

Let $X$ be a Banach space. TFAE:

- $X$ has the ADP.

Every rank-one operator $T \in L(X)$ (equivalently, every weakly compact operator) satisfies

$$\max_{|\omega|=1} \|\text{Id} + \omega T\| = 1 + \|T\|.$$
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  $$B_X = \overline{\operatorname{co}} \left( T \{ y \in B_X : \|x - y\| \geq 2 - \varepsilon \} \right).$$
Let $V_*$ be the predual of the von Neumann algebra $V$. 
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The Daugavet property of $V_\ast$ is equivalent to:

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- $|v(v_*)| = 1$ for $v \in \text{ext}(B_V)$ and $v_* \in \text{ext}(B_{V_*})$, or
- $V = C \oplus \infty N$, where $C$ is commutative and $N$ has no atomic projections.
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C*-algebras and preduals (II)

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- $\exists$ a commutative ideal $Y$ such that $X/Y$ has the Daugavet property.
Some results on the ADP: isomorphic properties

Remark
Since when we use the numerical index \( n = 1 \), only rank-one operators may be used, most of the known results are valid for the ADP.

Theorem (López–M.—Payá, 1999)
Not every real Banach space can be renormed with the ADP.

\[ X \text{ real reflexive with ADP} \implies \dim(X) = \infty. \]

Moreover, if \( X \) is real, \( \text{RNP} \), \( \dim(X) = \infty \), and ADP, then \( X \supset \ell_1 \).

A very recent result (Avilés–Kadets–M.—Merí–Shepelska)
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Also some isometric properties of Banach spaces with numerical index 1 are actually true for ADP.
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X infinite-dimensional with the ADP. Then
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$X = C(\mathbb{T})/A(\mathbb{D})$. Since $X^* = H^1$ is smooth $\implies$ nor $X$ nor $H^1$ have the ADP.
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**Open question**

Is there $X$ with the ADP which is smooth or strictly convex?
Lush spaces

6 Lush spaces
- Definition and examples
- Lush renorming
- Reformulations of lushness and applications
- Lushness is not equivalent to numerical index one

K. Boyko, V. Kadets, M. Martín, and J. Merí.
Properties of lush spaces and applications to Banach spaces with numerical index 1.

K. Boyko, V. Kadets, M. Martín, and D. Werner.
Numerical index of Banach spaces and duality.

V. Kadets, M. Martín, J. Merí, and R. Payá.
Convexity and smoothness of Banach spaces with numerical index one.

V. Kadets, M. Martín, J. Merí, and V. Shepelska.
Lushness, numerical index one and duality.
Lush spaces

Motivation

Remark

Usually, when we show that a Banach space has numerical index 1, we actually prove more. We do not have an operator-free characterization of the spaces with numerical index 1. Hence, it makes sense to study geometrical sufficient conditions.

Some sufficient conditions

Let $X$ be a Banach space. Consider:

(a) Lindenstrauss, 1964: $X$ has the 3.2.I.P. if the intersection of every family of three mutually intersecting balls is not empty.

(b) Fullerton, 1961: $X$ is a CL-space if $B_X$ is the absolutely convex hull of every maximal face of $S_X$.

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\[ n(X) = 1 \]
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Observation

Showing that $(c) \implies n(X) = 1$, one realizes that $(c)$ is too much.
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Let \( X \) be a Banach space. Consider:

(a) **Lindenstrauss, 1964:** \( X \) has the **3.2.I.P.** if the intersection of every family of three mutually intersecting balls is not empty.

(b) **Fullerton, 1961:** \( X \) is a **CL-space** if \( B_X \) is the absolutely convex hull of every maximal face of \( S_X \).

(c) **Lima, 1978:** \( X \) is an **almost-CL-space** if \( B_X \) is the closed absolutely convex hull of every maximal face of \( S_X \).

\[
(a) \iff (b) \iff (c) \iff n(X) = 1
\]

Observation

Showing that (c) \( \implies n(X) = 1 \), one realizes that (c) is too much.

Lushness (Boyko–Kadets–M.–Werner, 2007)

\( X \) is **lush** if given \( x, y \in S_X, \varepsilon > 0 \), there is \( x^* \in S_{X^*} \) such that

\[
x \in S(B_X, x^*, \varepsilon) \quad \text{and} \quad \text{dist}(y, aconv(S(B_X, x^*, \varepsilon))) < \varepsilon.
\]
Lush spaces

Definition and examples

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Definition and first property

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**Proof.**

- \( T \in L(X) \) with \( \|T\| = 1, \varepsilon > 0 \). Find \( y_0 \in S_X \) which \( \|Ty_0\| > 1 - \varepsilon. \)
Lush spaces

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Proof.

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- Use lushness for \( x_0 = Ty_0/\|Ty_0\| \) and \( y_0 \) to get \( x^* \in S_{X^*} \) and

\[
v = \sum_{i=1}^{n} \lambda_i \theta_i x_i \quad \text{where} \quad x_i \in S(B_X, x^*, \varepsilon), \lambda_i \in [0,1], \sum \lambda_i = 1, \theta_i \in T,
\]

with \( \text{Re} x^*(x_0) > 1 - \varepsilon \) and \( \|v - y_0\| < \varepsilon \).
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  with \( \text{Re} \, x^*(x_0) > 1 - \varepsilon \) and \( \|v - y_0\| < \varepsilon \).
- Then \( |x^*(Tv)| = |x^*(x_0) - x^*(T(\frac{y_0}{\|Ty_0\| - v}))| \sim \|T\| \).
**Definition and first property**

**Lushness (Boyko–Kadets–M.–Werner, 2007)**

A space $X$ is **lush** if given $x, y \in S_X$, $\varepsilon > 0$, there is $x^* \in S_{X^*}$ such that

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**Theorem**

$X$ lush $\implies n(X) = 1$.

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- $T \in L(X)$ with $\|T\| = 1$, $\varepsilon > 0$. Find $y_0 \in S_X$ which $\|Ty_0\| > 1 - \varepsilon$.
- Use lushness for $x_0 = Ty_0 / \|Ty_0\|$ and $y_0$ to get $x^* \in S_{X^*}$ and

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- Then $|x^*(Tv)| = |x^*(x_0) - x^* \left( T \left( \frac{y_0}{\|Ty_0\|} - v \right) \right) | \sim \|T\|$.

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**Theorem**

If $X$ is lush, then $n(X) = 1$.

**Proof.**

- Let $T \in L(X)$ with $\|T\| = 1$, $\varepsilon > 0$. Find $y_0 \in S_X$ such that $\|Ty_0\| > 1 - \varepsilon$.
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with $\text{Re} x^*(x_0) > 1 - \varepsilon$ and $\|v - y_0\| < \varepsilon$.
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- By a convexity argument, $\exists i$ such that $|x^*(Tx_i)| \sim \|T\|$ and $\text{Re} x^*(x_i) \sim 1$.
- Then $\max_{\omega \in \mathbb{T}} \|\text{Id} + \omega T\| \sim 1 + \|T\| \implies v(T) \sim \|T\|$. ✔️
Examples of lush spaces

2. In particular, $C(K), L_1(\mu), C_0(L)$.
3. Preduals of $L_1(\mu)$-spaces.
4. $C$-rich subspaces $K$ compact, $X$ subspace of $C(K)$ is $C$-rich iff $\forall U$ open nonempty and $\forall \varepsilon > 0$ exists $h: K \rightarrow [0, 1]$ continuous, $\text{supp}(h) \subseteq U$ such that $\text{dist}(h, X) < \varepsilon$.
5. More examples of lush spaces
6. $C$-rich subspaces of $C(K)$.
7. In particular, finite-codimensional subspaces of $C[0, 1]$.
8. $C_E(K)$, where $L$ nowhere dense in $K$ and $E \subseteq C(L)$.
9. $Y$ if $c_0 \subseteq Y \subseteq \ell_\infty$ (canonical copies).
Examples of lush spaces

- Almost-CL-spaces.

Preduals of $L_1(\mu)$-spaces.

Examples of lush spaces

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Examples of lush spaces

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Examples of lush spaces

- Almost-CL-spaces.
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Examples of lush spaces

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$K$ compact, $X$ subspace of $C(K)$ is **C-rich** iff $\forall U$ open nonempty and $\forall \varepsilon > 0$ exists $h : K \rightarrow [0, 1]$ continuous, $\text{supp}(h) \subseteq U$ such that $\text{dist}(h, X) < \varepsilon$. 
Examples of lush spaces

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C-rich subspaces

A subspace $X$ of $C(K)$ is $C$-rich if for all open nonempty $U$ and all $\epsilon > 0$, there exists a continuous $h: K \to [0,1]$ such that $\text{supp}(h) \subseteq U$, $\text{dist}(h, X) < \epsilon$.

More examples of lush spaces

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Examples of lush spaces

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## Examples of lush spaces

**Examples of lush spaces**

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A compact, $X$ subspace of $C(K)$ is **C-rich** iff $\forall U$ open nonempty and $\forall \varepsilon > 0$ exists $h : K \to [0, 1]$ continuous, $\text{supp}(h) \subseteq U$ such that $\text{dist}(h, X) < \varepsilon$.

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Lush renorming

The goal

When we may get a lush equivalent norm?
Lush renorming

The goal
When we may get a lush equivalent norm?

Proposition

$X$ separable, $X \supseteq c_0 \implies$ exists $\| \cdot \| \simeq \| \cdot \|$ and $T : (X, \| \cdot \|) \to \ell_\infty$ with $T$ isometric embedding & $c_0 \subseteq T(X)$ (canonical copy).
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When we may get a lush equivalent norm?

Proposition

\(X\) separable, \(X \supseteq c_0\) \(\implies\) exists \(\| \cdot \| \simeq \| \cdot \|\) and \(T : (X, \| \cdot \|) \to \ell_\infty\) with \(T\) isometric embedding \& \(c_0 \subseteq T(X)\) (canonical copy).

Recall this family of examples of lush spaces

\(Y\) if \(c_0 \subseteq Y \subseteq \ell_\infty\) (canonical copies).
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\(\ell_\infty\) if \(c_0 \subseteq Y \subseteq \ell_\infty\) (canonical copies).

Theorem
\(X\) separable, \(X \supseteq c_0 \implies X\) admits an equivalent lush norm.
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When we may get a lush equivalent norm?

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\[ X \text{ separable, } X \supseteq c_0 \implies \text{exists } \| \cdot \| \simeq \| \cdot \| \text{ and } T : (X, \| \cdot \|) \to \ell_\infty \text{ with } T \text{ isometric embedding } \& \ c_0 \subseteq T(X) \text{ (canonical copy).} \]

Recall this family of examples of lush spaces

- \( Y \) if \( c_0 \subseteq Y \subseteq \ell_\infty \) (canonical copies).

Theorem

\[ X \text{ separable, } X \supseteq c_0 \implies X \text{ admits an equivalent lush norm.} \]

Corollary

Every closed subspace of \( c_0 \) admits an equivalent lush norm.
Lush spaces

Lush renorming

The goal

When we may get a lush equivalent norm?

Proposition

\( X \) separable, \( X \supseteq c_0 \Rightarrow \exists \| \cdot \| \simeq \| \cdot \| \)

and \( T : (X, \| \cdot \|) \to \ell_\infty \) with

\( T \) isometric embedding & \( c_0 \subseteq T(X) \) (canonical copy).

Recall this family of examples of lush spaces

\( Y \) if \( c_0 \subseteq Y \subseteq \ell_\infty \) (canonical copies).

Open problems

Theorem

\( X \) separable, \( X \supseteq c_0 \Rightarrow X \) admits an equivalent lush norm.

Corollary

Every closed subspace of \( c_0 \) admits an equivalent lush norm.
Lush renorming

The goal
When we may get a lush equivalent norm?

Proposition
$X$ separable, $X \supseteq c_0 \Rightarrow \exists |||\cdot||| \simeq \|\cdot\|$ and $T : (X, |||\cdot|||) \rightarrow \ell_\infty$ isometric embedding & $c_0 \subseteq T(X)$ (canonical copy).

Open problems
- Find more sufficient conditions to get equivalent lush norms.

Recall
$Y$ if $c_0 \subseteq Y \subseteq \ell_\infty$ (canonical copies).

Theorem
$X$ separable, $X \supseteq c_0 \implies X$ admits an equivalent lush norm.

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Every closed subspace of $c_0$ admits an equivalent lush norm.
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When we may get a lush equivalent norm?

Proposition
\[ X \text{ separable, } X \supseteq c_0 \implies \exists |||\cdot||| \simeq \|\cdot\| \text{ and } T : (X, |||\cdot|||) \to \ell_\infty \text{ with } T \text{ isometric embedding} \]

Recall this family of examples of lush spaces.
\[ Y \text{ if } c_0 \subseteq Y \subseteq \ell_\infty \text{ (canonical copies).} \]

Open problems
- Find more sufficient conditions to get equivalent lush norms.
- When \( X \supseteq \ell_1 \)?

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\[ X \text{ separable, } X \supseteq c_0 \implies X \text{ admits an equivalent lush norm.} \]

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When we may get a lush equivalent norm?

Proposition
X separable, \( X \supseteq c_0 = \begin{array}{l}
\Rightarrow \exists |||\cdot||| \simeq \|\cdot\| \\
T: (X, |||\cdot|||) \to \ell_\infty \text{ with } T \text{ isometric embedding} & c_0 \subseteq T(X) \text{ (canonical copy)}.
\end{array}

Open problems
- Find more sufficient conditions to get equivalent lush norms.
- When \( X \supseteq \ell_1 \)?
- When \( X \supseteq \ell_\infty \)?

Recall this family of examples of lush spaces
\( Y \) if \( c_0 \subseteq Y \subseteq \ell_\infty \) (canonical copies).

Theorem
\( X \) separable, \( X \supseteq c_0 \implies X \) admits an equivalent lush norm.

Corollary
Every closed subspace of \( c_0 \) admits an equivalent lush norm.
Even more examples of lush spaces

Observation

X

Banach space. Consider the following assertions.

(a) Exists $A \subset B_X^*$ norming, $|x^{**}(a^*)| = 1$ for all $a^* \in A$ and $x^{**} \in \text{ext}(B_X^{**})$.

(b) For $x \in S_{B_X}$ and $\varepsilon > 0$, exists $x^* \in S_{B_X^*}$ such that $x \in S(B_X, x^*, \varepsilon)$ and $B_X = a_{\text{conv}}(S(B_X, x^*, \varepsilon))$.

Definition (Werner, 1997)

$X$ is nicely embedded in $C_b(\Omega)$ if exists $J: X \rightarrow C_b(\Omega)$ linear isometry with

(N1) $\|J^*\delta_s\| = 1$ for all $s \in \Omega$,

(N2) span$(J^*\delta_s)$ is a $L^1$-summand in $X$ for all $s \in \Omega$.

Nicely embedded Banach spaces (they fulfil (a)).

In particular, function algebras (as $A(D)$ and $H_\infty$).
Observation

Let $X$ be a Banach space. Consider the following assertions.

(a) Exists $A \subset B_{X^*}$ norming, $|x^{**}(a^*)| = 1$ for all $a^* \in A$ and $\forall x^{**} \in \text{ext}(B_{X^{**}})$.

(b) For $x \in S_X$ and $\varepsilon > 0$, exists $x^* \in S_{X^*}$ such that

$$x \in S(B_X, x^*, \varepsilon) \quad \text{and} \quad B_X = \overline{\text{aconv}}(S(B_X, x^*, \varepsilon)).$$

(a) $\iff$ (b) $\implies$ lushness
Even more examples of lush spaces

Observation

$X$ Banach space. Consider the following assertions.

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(b) For \( x \in S_X \) and \( \varepsilon > 0 \), exists \( x^* \in S_{X^*} \) such that

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\( \text{(N1)} \quad \|J^*\delta_s\| = 1 \ \forall s \in \Omega, \)

\( \text{(N2)} \quad \text{span}(J^*\delta_s) \ L\text{-summand in } X^* \ \forall s \in \Omega. \)

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Even more examples of lush spaces

Nicely embedded Banach spaces (they fulfil (a)).

In particular, function algebras (as \( A(D) \) and \( H_\infty \)).
Observation

A Banach space. Consider the following assertions.

(a) Exists \( A \subset B_{X^*} \) norming, \(|x^{**}(a^*)| = 1 \) \( \forall a^* \in A \) and \( \forall x^{**} \in \text{ext}(B_{X^{**}}) \).

(b) For \( x \in S_X \) and \( \epsilon > 0 \), exists \( x^* \in S_{X^*} \) such that

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x \in S(B_X, x^*, \epsilon) \quad \text{and} \quad B_X = \text{aconv}(S(B_X, x^*, \epsilon)).
\]

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\]

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Definition (Werner, 1997)

X is nicely embedded in $C_b(\Omega)$ if exists $J : X \to C_b(\Omega)$ linear isometry with

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Even more examples of lush spaces

8 Nicely embedded Banach spaces (they fulfil (a)).

9 In particular, function algebras (as $A(\mathbb{D})$ and $H^\infty$).
Some reformulations of lushness
Some reformulations of lushness

**Proposition**

Let $X$ be a Banach space. TFAE:

- $X$ is lush,
- Every separable $E \subset X$ is contained in a separable lush $Y$ with $E \subset Y \subset X$. 

In other words, $B_X$ is the aconvex hull of the set of $x \in B_X$ such that $x^*(x) = 1$ for any $x^* \in G$.
Some reformulations of lushness

**Proposition**

X Banach space. TFAE:
- X is lush,
- Every separable $E \subset X$ is contained in a separate lush $Y$ with $E \subset Y \subset X$.

**Separable lush spaces**

X separable. TFAE:
- X is lush.
- There is $G \subseteq S_{X^*}$ norming such that
  \[ B_X = \overline{\text{aconv}}(S(B_X, x^*, \varepsilon)) \quad (\varepsilon > 0, \ x^* \in G). \]
- Therefore, $|x^{**}(x^*)| = 1$ for every $x^{**} \in \text{ext} (B_{X^{**}})$ and every $x^* \in G$.
- This implies that $B_X = \overline{\text{aconv}} \left( \{ x \in B_X : x^*(x) = 1 \} \right)$ $\forall x^* \in G$. 
Some reformulations of lushness

**Proposition**

Let $X$ be a Banach space. TFAE:

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**Separable lush spaces**

Let $X$ be separable. TFAE:

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  $$B_X = \overline{\text{aconv}}(S(B_X, x^*, \varepsilon)) \quad (\varepsilon > 0, \ x^* \in G).$$

- Therefore, $|x^{**}(x^*)| = 1$ for every $x^{**} \in \text{ext}(B_{X^{**}})$ and every $x^* \in G$.
- This implies that $B_X = \overline{\text{aconv}} \left( \{ x \in B_X : x^*(x) = 1 \} \right) \ \forall x^* \in G$.

We almost returned to the almost-CL-space definition!!
Some reformulations of lushness

**Proposition**

$X$ Banach space. TFAE:

- $X$ is lush,
- Every separable $E \subset X$ is contained in a separable lush $Y$ with $E \subset Y \subset X$.

**Separable lush spaces**

$X$ separable. TFAE:

- $X$ is lush.
- There is $G \subseteq S_{X^*}$ norming such that
  \[ B_X = \overline{\text{aconv}}(S(B_X, x^*, \varepsilon)) \quad (\varepsilon > 0, \ x^* \in G). \]

- Therefore, $|x^{**}(x^*)| = 1$ for every $x^{**} \in \text{ext}(B_{X^{**}})$ and every $x^* \in G$.
- This implies that $B_X = \overline{\text{aconv}} \left( \{ x \in B_X : x^*(x) = 1 \} \right)$ $\forall x^* \in G$.

**Consequence**

$X \subseteq C[0,1]$ strictly convex or smooth $\implies C[0,1]/X$ contains $C[0,1]$. 
An important consequence

Remark

\[ X \text{ lush separable, } \dim(X) = \infty = \Rightarrow \text{ there is } G \in S_{X^*} \text{ infinite such that } \left| x^{**}(x^*) \right| = 1 \left( x^{**} \in \text{ext}(B_{X^{**}}), x^* \in G \right). \]

Proposition (López–M.–Payá, 1999)

\[ X \text{ real, } A \subset S_X \text{ infinite such that } \left| x^*(a) \right| = 1 \left( x^* \in \text{ext}(B_X), a \in A \right). \]

Then,

\[ X \supseteq c_0 \text{ or } X \supseteq \ell_1. \]

Main consequence

\[ X \text{ real lush, } \dim(X) = \infty = \Rightarrow X^{**} \supseteq \ell_1. \]
An important consequence

Remark

$X$ lush separable, $\dim(X) = \infty \implies$ there is $G \in S_{X^*}$ infinite such that

$$|x^{**}(x^*)| = 1 \quad (x^{**} \in \text{ext} (B_{X^{**}}), \ x^* \in G).$$
An important consequence

**Remark**

X lush separable, \( \dim(X) = \infty \implies \) there is \( G \in S_{X^*} \) infinite such that

\[ |x^{**}(x^*)| = 1 \quad (x^{**} \in \text{ext}(B_{X^{**}}), \ x^* \in G). \]

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X real, \( A \subset S_X \) infinite such that

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Then, \( X \supseteq c_0 \) or \( X \supseteq \ell_1 \).
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Remark

\( X \) lush separable, \( \dim(X) = \infty \implies \) there is \( G \in S_{X^*} \) infinite such that

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Then, \( X \supseteq c_0 \) or \( X \supseteq \ell_1 \).

Main consequence

\( X \) real lush, \( \dim(X) = \infty \implies X^* \supseteq \ell_1. \)
### An important consequence

#### Remark

$LUSH$ spaces

If $X$ is lush separable, $\dim(X) = \infty \implies \exists G \in S_{X*}$ infinite such that

$$|x^{**}(x^*)| = 1 \quad (x^{**} \in \text{ext}(B_{X^{**}}), \ x^* \in G).$$

#### Proposition (López–M.–Payá, 1999)

If $X$ is real, $A \subset S_X$ infinite such that

$$|x^*(a)| = 1 \quad (x^* \in \text{ext}(B_{X^*}), \ a \in A).$$

Then, $X \supseteq c_0$ or $X \supseteq \ell_1$.

#### Main consequence

If $X$ is real lush, $\dim(X) = \infty \implies X^* \supseteq \ell_1$.

**Proof.**
An important consequence

**Remark**

\( X \) lush separable, \( \dim(X) = \infty \implies \) there is \( G \in S_{X^*} \) infinite such that

\[
|x^{**}(x^*)| = 1 \quad (x^{**} \in \text{ext}(B_{X^{**}}), \ x^* \in G).
\]

**Proposition (López–M.–Payá, 1999)**

\( X \) real, \( A \subset S_X \) infinite such that

\[
|x^*(a)| = 1 \quad (x^* \in \text{ext}(B_{X^*}), \ a \in A).
\]

Then, \( X \supseteq c_0 \) or \( X \supseteq \ell_1 \).

**Main consequence**

\( X \) real lush, \( \dim(X) = \infty \implies X^* \supseteq \ell_1 \).

**Proof.**

- There is \( E \subset X \) separable and lush.
An important consequence

Remark

$X$ lush separable, $\dim(X) = \infty \implies$ there is $G \in S_{X^*}$ infinite such that

$$|x^{**}(x^*)| = 1 \quad (x^{**} \in \text{ext}(B_{X^{**}}), \ x^* \in G).$$

Proposition (López–M.–Payá, 1999)

$X$ real, $A \subset S_X$ infinite such that

$$|x^*(a)| = 1 \quad (x^* \in \text{ext}(B_{X^*}), \ a \in A).$$

Then, $X \supseteq c_0$ or $X \supseteq \ell_1$.

Main consequence

$X$ real lush, $\dim(X) = \infty \implies X^* \supseteq \ell_1$.

Proof.

- There is $E \subseteq X$ separable and lush.
- Then $E^* \supseteq c_0$ or $E^* \supseteq \ell_1 \implies E^* \supseteq \ell_1$. 
An important consequence

Remark

\(X \text{ lush separable, } \dim(X) = \infty \implies \text{ there is } G \in S_{X^*} \text{ infinite such that}
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\(X \text{ real, } A \subset S_X \text{ infinite such that}
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Then, \(X \supseteq c_0 \text{ or } X \supseteq \ell_1.\)

Main consequence

\(X \text{ real lush, } \dim(X) = \infty \implies X^* \supseteq \ell_1.\)

Proof.

• There is \(E \subseteq X \text{ separable and lush.}\)
• Then \(E^* \supseteq c_0 \text{ or } E^* \supseteq \ell_1 \implies E^* \supseteq \ell_1.\)
• By “lifting” property of \(\ell_1 \implies X^* \supseteq \ell_1. \checkmark\)
An important consequence

Remark

Let $X$ be lush separable, $\dim(X) = \infty \implies$ there is $G \in S_{X^*}$ infinite such that

$$|x^{**}(x^*)| = 1 \quad (x^{**} \in \text{ext}(B_{X^{**}}), \ x^* \in G).$$

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Then, $X \supseteq c_0$ or $X \supseteq \ell_1$.

Main consequence

Let $X$ be real lush, $\dim(X) = \infty \implies X^* \supseteq \ell_1$.

Question

What happens if just $n(X) = 1$?
An important consequence

Remark

X lush separable, \( \dim(X) = \infty \implies \) there is \( G \in S_{X^*} \) infinite such that

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Main consequence

X real lush, \( \dim(X) = \infty \implies X^* \supseteq \ell_1 \).

Question

What happens if just \( n(X) = 1 \)? The same, we will prove later.
Lush spaces

Lushness is not equivalent to numerical index one

Example

There is a separable Banach space $X$ such that $X^*$ is lush but $X$ is not lush. Since $n(X^*) = 1$, also $n(X) = 1$.

The set $\{x^* \in S_{X^*} : |x^{**}(x^*)| = 1 \text{ for every } x^{**} \in \text{ext}(B_{X^{**}})\}$ is empty.

Consequence

$X$ lush $\neq \neq X^*$ lush

Proposition

$X^{**}$ lush $\neq \neq X$ lush
Lushness is not equivalent to numerical index one

Example

There is a separable Banach space $\mathcal{X}$ such that

- $\mathcal{X}^*$ is lush but $\mathcal{X}$ is not lush.
Lush spaces  Lushness is not equivalent to numerical index one

Example

There is a separable Banach space $\mathcal{X}$ such that

- $\mathcal{X}^*$ is lush but $\mathcal{X}$ is not lush.
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- The set

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Lushness is not equivalent to numerical index one

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is empty.

Consequence

$$X \text{ lush } \iff X^* \text{ lush}$$
Lushness is not equivalent to numerical index one

**Example**

There is a separable Banach space $\mathcal{X}$ such that

- $\mathcal{X}^*$ is lush but $\mathcal{X}$ is not lush.
- Since $n(\mathcal{X}^*) = 1$, also $n(\mathcal{X}) = 1$.
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**Consequence**

\[ X \text{ lush } \iff X^* \text{ lush} \]

**Proposition**

\[ X^{**} \text{ lush } \iff X \text{ lush} \]
Slicely countably determined spaces

7 Slicely countably determined spaces
- Slicely Countably Determined sets and spaces
- Applications to numerical index 1 spaces
- SCD operators
- Open questions

A. Avilés, V. Kadets, M. Martín, J. Merí, and V. Shepelska
Slicely Countably Determined Banach spaces
SCD sets: Definitions and preliminary remarks

A Banach space $X$ is Slicely Countably Determined (SCD) if there is a sequence $\{S_n : n \in \mathbb{N}\}$ of slices of $A$ satisfying one of the following equivalent conditions:

- Every slice of $A$ contains one of the $S_n$'s.
- $A \subseteq \text{conv}(B)$ if $B \subseteq A$ satisfies $B \cap S_n \neq \emptyset$ for all $n$.
- Given $\{x_n : n \in \mathbb{N}\}$ with $x_n \in S_n$ for all $n$, $A \subseteq \text{conv}(\{x_n : n \in \mathbb{N}\})$.

Remarks:

A is SCD iff A is separable.

If A is SCD, then it is separable.
X Banach space, $A \subset X$ bounded and convex.

**SCD sets**

$A$ is **Slicely Countably Determined (SCD)** if there is a sequence $\{S_n : n \in \mathbb{N}\}$ of slices of $A$ satisfying one of the following equivalent conditions:
SCD sets: Definitions and preliminary remarks

$X$ Banach space, $A \subset X$ bounded and convex.

SCD sets

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- every slice of $A$ contains one of the $S_n$’s,
- $A \subseteq \overline{\text{conv}}(B)$ if $B \subseteq A$ satisfies $B \cap S_n \neq \emptyset \ \forall n$,
- given $\{x_n\}_{n \in \mathbb{N}}$ with $x_n \in S_n \ \forall n \in \mathbb{N}$, $A \subseteq \overline{\text{conv}}(\{x_n : n \in \mathbb{N}\})$. 

Remarks

$A$ is SCD iff $A$ is SCD.

If $A$ is SCD, then it is separable.
SCD sets: Definitions and preliminary remarks

$X$ Banach space, $A \subset X$ bounded and convex.

**SCD sets**

$A$ is **Slicely Countably Determined** (SCD) if there is a sequence $\{S_n : n \in \mathbb{N}\}$ of slices of $A$ satisfying one of the following equivalent conditions:

- every slice of $A$ contains one of the $S_n$’s,
- $A \subseteq \text{conv}(B)$ if $B \subseteq A$ satisfies $B \cap S_n \neq \emptyset \ \forall n$,
- given $\{x_n\}_{n \in \mathbb{N}}$ with $x_n \in S_n \ \forall n \in \mathbb{N}$, $A \subseteq \overline{\text{conv}}(\{x_n : n \in \mathbb{N}\})$.

**Remarks**

- $A$ is SCD iff $\overline{A}$ is SCD.
- If $A$ is SCD, then it is separable.
SCD sets: Elementary examples I

Example A separable and $A = \text{conv}(\text{dent}(A)) \Rightarrow A$ is SCD.

Proof. Take $\{a_n: n \in \mathbb{N}\}$ denting points with $A = \text{conv}(\{a_n: n \in \mathbb{N}\})$.

For every $n, m \in \mathbb{N}$, take a slice $S_{n, m}$ containing $a_n$ and of diameter $1/m$.

If $B \cap S_{n, m} \neq \emptyset \forall n, m \in \mathbb{N} \Rightarrow a_n \in B \forall n \in \mathbb{N}$.

Therefore, $A = \text{conv}(\{a_n: n \in \mathbb{N}\}) \subseteq \text{conv}(B) = \text{conv}(B)$. ✓

Example In particular, $A_{RNP}$ separable $\Rightarrow A$ SCD.

Corollary If $X$ is separable LUR $\Rightarrow B(X)$ is SCD.

So, every separable space can be renormed such that $B(X, |\cdot|)$ is SCD.
SCD sets: Elementary examples I

Example

$A$ separable and $A = \overline{\text{conv}}(\text{dent}(A)) \implies A$ is SCD.
Example

A separable and \( A = \overline{\text{conv}}(\text{dent}(A)) \implies A \text{ is SCD.} \)

Proof.
Example

A separable and \( A = \overline{\text{conv}}(\text{dent}(A)) \implies A \text{ is SCD.} \)

Proof.

- Take \( \{ a_n : n \in \mathbb{N} \} \) denting points with \( A = \overline{\text{conv}}(\{ a_n : n \in \mathbb{N} \}) \).
Example

A separable and $A = \overline{\text{conv}}(\text{dent}(A)) \implies A$ is SCD.

Proof.

- Take $\{a_n : n \in \mathbb{N}\}$ denting points with $A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\})$.
- For every $n, m \in \mathbb{N}$, take a slice $S_{n,m}$ containing $a_n$ and of diameter $1/m$. 
SCD sets: Elementary examples I

Example

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- If $B \cap S_{n,m} \neq \emptyset \ \forall n, m \in \mathbb{N} \implies a_n \in \overline{B} \ \forall n \in \mathbb{N}$. 
SCD sets: Elementary examples I

Example

A separable and \( A = \overline{\text{conv}}(\text{dent}(A)) \implies A \) is SCD.

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- Take \( \{a_n : n \in \mathbb{N}\} \) denting points with \( A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\}) \).
- For every \( n, m \in \mathbb{N} \), take a slice \( S_{n,m} \) containing \( a_n \) and of diameter \( 1/m \).
- If \( B \cap S_{n,m} \neq \emptyset \ \forall n, m \in \mathbb{N} \implies a_n \in \overline{B} \ \forall n \in \mathbb{N} \).
- Therefore, \( A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\}) \subseteq \overline{\text{conv}(B)} = \overline{\text{conv}(B)} \).  ✓
SCD sets: Elementary examples I

Example

A separable and \( A = \overline{\text{conv}}(\text{dent}(A)) \implies A \) is SCD.

Proof.

- Take \( \{a_n : n \in \mathbb{N}\} \) denting points with \( A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\}) \).
- For every \( n, m \in \mathbb{N} \), take a slice \( S_{n,m} \) containing \( a_n \) and of diameter \( 1/m \).
- If \( B \cap S_{n,m} \neq \emptyset \ \forall n, m \in \mathbb{N} \implies a_n \in \overline{B} \ \forall n \in \mathbb{N} \).
- Therefore, \( A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\}) \subseteq \overline{\text{conv}(B)} = \overline{\text{conv}(B)}. \) \( \checkmark \)

Example

In particular, \( A \) RNP separable \( \implies A \) SCD.
SCD sets: Elementary examples I

Example

A separable and \( A = \overline{\text{conv}}(\text{dent}(A)) \) \( \implies \) \( A \) is SCD.

Proof.

- Take \( \{a_n : n \in \mathbb{N}\} \) denting points with \( A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\}) \).
- For every \( n, m \in \mathbb{N} \), take a slice \( S_{n,m} \) containing \( a_n \) and of diameter \( 1/m \).
- If \( B \cap S_{n,m} \neq \emptyset \ \forall n, m \in \mathbb{N} \) \( \implies \) \( a_n \in \overline{B} \ \forall n \in \mathbb{N} \).
- Therefore, \( A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\}) \subseteq \overline{\text{conv}}(\overline{B}) = \overline{\text{conv}}(B) \).  ✓

Example

In particular, \( A \) RNP separable \( \implies \) \( A \) SCD.

Corollary

- If \( X \) is separable LUR \( \implies \) \( B_X \) is SCD.
- So, every separable space can be renormed such that \( B_{(X, \| \cdot \|)} \) is SCD.
Example

If $X^*$ is separable $\Rightarrow A$ is SCD.

Proof.
Take $\{x^*_n: n \in \mathbb{N}\}$ dense in $S X^*$. For every $n, m \in \mathbb{N}$, consider $S_n^*, m = S(A, x^*_n, 1/m)$. It is easy to show that any slice of $A$ contains one of the $S_n^*, m$.

Negative example

If $X$ has the Daugavet property $\Rightarrow B^*_X$ is not SCD.

Therefore, $B^*_{C[0,1]}$, $B^*_1[0,1]$ are not SCD.

Proof.
Fix $x^*_0 \in B^*_X$ and $\{S_n^*\}$ sequence of slices of $B^*_X$. By [KSSW] there is a sequence $(x_n) \subset B^*_X$ such that $x_n \in S_n^*$ for every $n \in \mathbb{N}$, $(x_n)_{n \geq 0}$ is equivalent to the basis of $\ell_1$, so $x^*_0 / \in lin\{x_n: n \in \mathbb{N}\}$.
Example

If $X^*$ is separable $\implies A$ is SCD.
Example
If $X^*$ is separable $\iff A$ is SCD.

Proof.
SCD sets: Elementary examples II

Example

If $X^*$ is separable $\implies A$ is SCD.

Proof.

- Take $\{x^*_n : n \in \mathbb{N}\}$ dense in $S_{X^*}$.
Example

If \( X^* \) is separable \( \implies \) \( A \) is SCD.

Proof.

- Take \( \{x_n^* : n \in \mathbb{N}\} \) dense in \( S_{X^*} \).
- For every \( n, m \in \mathbb{N} \), consider \( S_{n,m} = S(A, x_n^*, 1/m) \).
SCD sets: Elementary examples II

Example

If $X^*$ is separable $\iff A$ is SCD.

Proof.

- Take $\{x_n^*: n \in \mathbb{N}\}$ dense in $S_{X^*}$.
- For every $n, m \in \mathbb{N}$, consider $S_{n,m} = S(A, x_n^*, 1/m)$.
- It is easy to show that any slice of $A$ contains one of the $S_{n,m}$. ✓
Example
If $X^*$ is separable $\implies A$ is SCD.

Proof.
- Take $\{x_n^* : n \in \mathbb{N}\}$ dense in $S_{X^*}$.
- For every $n, m \in \mathbb{N}$, consider $S_{n,m} = S(A, x_n^*, 1/m)$.
- It is easy to show that any slice of $A$ contains one of the $S_{n,m}$. ✓

Negative example
If $X$ has the Daugavet property $\implies B_X$ is not SCD.
Therefore, $B_{C[0,1]}, B_{L_1[0,1]}$ are not SCD.
SCD sets: Elementary examples II

**Example**

If $X^*$ is separable $\implies A$ is SCD.

Proof.

- Take $\{x_n^* : n \in \mathbb{N}\}$ dense in $S_{X^*}$.
- For every $n, m \in \mathbb{N}$, consider $S_{n,m} = S(A, x_n^*, 1/m)$.
- It is easy to show that any slice of $A$ contains one of the $S_{n,m}$. ✓

**Negative example**

If $X$ has the Daugavet property $\implies B_X$ is not SCD.

Therefore, $B_{C[0,1]}$, $B_{L_1[0,1]}$ are not SCD.

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- It is easy to show that any slice of $A$ contains one of the $S_{n,m}$. ✓

Negative example
If $X$ has the Daugavet property $\implies B_X$ is not SCD.
Therefore, $B_{C[0,1]}, B_{L_1[0,1]}$ are not SCD.

Proof.

- Fix $x_0 \in B_X$ and $\{S_n\}$ sequence of slices of $B_X$. 

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Slicely countably determined spaces SCD sets & spaces

SCD sets: Elementary examples II

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SCD sets: Elementary examples II

**Example**

If $X^*$ is separable $\implies A$ is SCD.

Proof.

- Take $\{x^*_n : n \in \mathbb{N}\}$ dense in $S_{X^*}$.
- For every $n, m \in \mathbb{N}$, consider $S_{n,m} = S(A, x^*_n, 1/m)$.
- It is easy to show that any slice of $A$ contains one of the $S_{n,m}$. ✓

**Negative example**

If $X$ has the Daugavet property $\implies B_X$ is not SCD. Therefore, $B_{C[0,1]}, B_{L_1[0,1]}$ are not SCD.

Proof.

- Fix $x_0 \in B_X$ and $\{S_n\}$ sequence of slices of $B_X$.
- By [KSSW] there is a sequence $(x_n) \subset B_X$ such that
  - $x_n \in S_n$ for every $n \in \mathbb{N}$,
  - $(x_n)_{n \geq 0}$ is equivalent to the basis of $\ell_1$,
  - so $x_0 \notin \overline{\text{lin}}\{x_n : n \in \mathbb{N}\}$. ✓
SCD sets: Further examples I

Convex combination of slices

\[ W = \sum_{k=1}^{m} \lambda_k S_k \subset A \]

where \( \lambda_k \geq 0 \), \( \sum \lambda_k = 1 \), \( S_k \) slices.

**Proposition**

In the definition of SCD we can use a sequence \( \{S_n : n \in \mathbb{N}\} \) of convex combinations of slices.

Small combinations of slices of \( A \) has small combinations of slices iff every slice of \( A \) contains convex combinations of slices of \( A \) with arbitrary small diameter.

**Example**

If \( A \) has small combinations of slices + separable \( \Rightarrow A \) is SCD.

**Particular case**

\( A \) strongly regular + separable \( \Rightarrow A \) is SCD.
Convex combination of slices

\[ W = \sum_{k=1}^{m} \lambda_k S_k \subset A \text{ where } \lambda_k \geq 0, \sum \lambda_k = 1, S_k \text{ slices.} \]
Convex combination of slices

\[ W = \sum_{k=1}^{m} \lambda_k S_k \subset A \text{ where } \lambda_k \geq 0, \sum \lambda_k = 1, S_k \text{ slices.} \]

Proposition

In the definition of SCD we can use a sequence \( \{S_n : n \in \mathbb{N}\} \) of convex combination of slices.
SCD sets: Further examples I

**Convex combination of slices**

\[ W = \sum_{k=1}^{m} \lambda_k S_k \subset A \text{ where } \lambda_k \geq 0, \sum \lambda_k = 1, S_k \text{ slices.} \]

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\( A \) has **small combinations of slices** iff every slice of \( A \) contains convex combinations of slices of \( A \) with arbitrary small diameter.
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If \( A \) has small combinations of slices + separable \( \implies A \) is SCD.
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Particular case

A strongly regular + separable \( \implies \) A is SCD.
SCD sets: Further examples II
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**Bourgain’s lemma**

Every relative weak open subset of $A$ contains a convex combination of slices.
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In the definition of SCD we can use a sequence $\{S_n : n \in \mathbb{N}\}$ of relative weak open subsets.
SCD sets: Further examples II

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In the definition of SCD we can use a sequence $\{S_n : n \in \mathbb{N}\}$ of relative weak open subsets.

**$\pi$-bases**

A $\pi$-base of the weak topology of $A$ is a family $\{V_i : i \in I\}$ of weak open sets of $A$ such that every weak open subset of $A$ contains one of the $V_i$'s.
SCD sets: Further examples II

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Proposition
If $(A, \sigma(X, X^*))$ has a countable $\pi$-base $\implies A$ is SCD.
Theorem

A separable without \( \ell_1 \)-sequences \( \Rightarrow \) \((A, \sigma(X, X^*))\) has a countable \( \pi \)-base.

Proof.

We see \((A, \sigma(X, X^*)) \subset C(T)\) where \(T = (B_{X^*}, \sigma(X^*, X))\).

By Rosenthal \( \ell_1 \) theorem, \((A, \sigma(X, X^*))\) is a relatively compact subset of the space of first Baire class functions on \(T\).

By a result of Todor ˇcevi´c, \((A, \sigma(X, X^*))\) has a \(\sigma\)-disjoint \(\pi\)-base.

\(\{V_i : i \in I\}\) is \(\sigma\)-disjoint if \(I = \bigcup_{n \in \mathbb{N}} I_n\) and each \(\{V_i : i \in I_n\}\) is pairwise disjoint.

A \(\sigma\)-disjoint family of open subsets in a separable space is countable. ✓
SCD sets: Further examples III

Theorem

A separable without $\ell_1$-sequences $\implies (A, \sigma(X, X^*))$ has a countable $\pi$-base.
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SCD sets: Further examples III

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**Example**

A separable without $\ell_1$-sequences $\implies A$ is SCD.
SCD spaces: definition and examples

SCD space $X$ is Slicely Countably Determined (SCD) if so are its convex bounded subsets.

Examples of SCD spaces:
1. $X$ separable strongly regular. In particular, RNP, CPCP spaces.
2. $X$ separable $X^* \not\subseteq \ell_1$. In particular, if $X^*$ is separable.

Examples of NOT SCD spaces:
1. $X$ having the Daugavet property.
2. In particular, $C[0,1]$, $L_1[0,1]$
3. There is $X$ with the Schur property which is not SCD.

Remark:
Every subspace of a SCD space is SCD. This is false for quotients.
SCD spaces: definition and examples

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SCD spaces: stability properties

Theorem

\[ Z \subset X \text{. If } Z \text{ and } X/Z \text{ are SCD } \Rightarrow X \text{ is SCD.} \]

Corollary

\[ \text{If } \ell_1 \cong Y \subset X = \Rightarrow X/Y \text{ contains a copy of } \ell_1. \]

\[ \text{If } \ell_1 \cong Y_1 \subset X = \Rightarrow \text{there is } \ell_1 \cong Y_2 \subset X \text{ with } Y_1 \cap Y_2 = 0. \]

Corollary

\[ X_1, \ldots, X_m \text{ SCD } \Rightarrow X_1 \oplus \cdots \oplus X_m \text{ SCD.} \]
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\( X_1, \ldots, X_m \text{ SCD } \implies X_1 \oplus \cdots \oplus X_m \text{ SCD.} \)
SCD spaces: stability properties II

Theorem

\[ X_1, X_2, \ldots, SCD, E \text{ with unconditional basis.} \]

\[ E \nsubseteq c_0 \Rightarrow [\bigoplus_{n \in \mathbb{N}} X_n] E SCD. \]

\[ E \nsubseteq \ell_1 \Rightarrow [\bigoplus_{n \in \mathbb{N}} X_n] E SCD. \]

Examples

1. \( c_0 (\ell_1) \) and \( \ell_1 (c_0) \) are SCD.

2. \( c_0 \otimes \varepsilon c_0, c_0 \otimes \pi c_0, c_0 \otimes \varepsilon \ell_1, c_0 \otimes \pi \ell_1, \ell_1 \otimes \varepsilon \ell_1, \) and \( \ell_1 \otimes \pi \ell_1 \) are SCD.

3. \( K (c_0) \) and \( K (c_0, \ell_1) \) are SCD.

4. \( \ell_2 \otimes \varepsilon \ell_2 \equiv K (\ell_2) \) and \( \ell_2 \oplus \pi \ell_2 \equiv L_1 (\ell_2) \) are SCD.
SCD spaces: stability properties II

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$X_1, X_2, \ldots$ SCD, $E$ with unconditional basis.

- $E \not\ni c_0 \implies [\bigoplus_{n \in \mathbb{N}} X_n]_E$ SCD.
- $E \not\ni \ell_1 \implies [\bigoplus_{n \in \mathbb{N}} X_n]_E$ SCD.
SCD spaces: stability properties II

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**Examples**

1. $c_0(\ell_1)$ and $\ell_1(c_0)$ are SCD.
2. $c_0 \otimes_\varepsilon c_0$, $c_0 \otimes_\pi c_0$, $c_0 \otimes_\varepsilon \ell_1$, $c_0 \otimes_\pi \ell_1$, $\ell_1 \otimes_\varepsilon \ell_1$, and $\ell_1 \otimes_\pi \ell_1$ are SCD.
3. $K(c_0)$ and $K(c_0, \ell_1)$ are SCD.
4. $\ell_2 \otimes_\varepsilon \ell_2 \equiv K(\ell_2)$ and $\ell_2 \oplus_\pi \ell_2 \equiv L_1(\ell_2)$ are SCD.
The DPr, the ADP and numerical index 1

Recalling the properties

Kadets-Shvidkoy-Sirotkin-Werner, 1997:

The space $X$ has the Daugavet property (DPr) if

$$\|Id + T\| = 1 + \|T\|$$

for every rank-one $T \in L(X)$.

⋆ Then every weakly compact $T$ also satisfies (DE).

Lumer, 1968:

The space $X$ has numerical index 1 if

$$\max_{\theta \in T} \|Id + \theta T\| = 1 + \|T\|$$

⋆ Equivalently, $v(T) = \|T\|$ for every $T \in L(X)$.

M.-Oikhberg, 2004:

The space $X$ has the alternative Daugavet property (ADP) if

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Lumer, 1968:

$X$ has numerical index 1 if every operator on $X$ satisfies

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Then every weakly compact $T$ also satisfies (DE).
Recalling the properties

1. **Kadets-Shvidkoy-Sirotkin-Werner, 1997:**
   \( X \) has the **Daugavet property (DPr)** if
   \[
   \|\text{Id} + T\| = 1 + \|T\| \quad \text{(DE)}
   \]
   for every rank-one \( T \in L(X) \).
   \[\star\] Then every weakly compact \( T \) also satisfies (DE).

2. **Lumer, 1968:** \( X \) has **numerical index 1** if **EVERY** operator on \( X \) satisfies
   \[
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The DPr, the ADP and numerical index 1

Recalling the properties

1. **Kadets-Shvidkoy-Sirotkin-Werner, 1997:**
   \( X \) has the **Daugavet property (DPr)** if
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   ★ Then every weakly compact \( T \) also satisfies (DE).

2. **Lumer, 1968:** \( X \) has **numerical index 1** if EVERY operator on \( X \) satisfies
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   ★ Equivalently, \( v(T) = \| T \| \) for EVERY \( T \in L(X) \).

3. **M.-Oikhberg, 2004:** \( X \) has the **alternative Daugavet property (ADP)** if
every rank-one \( T \in L(X) \) satisfies (aDE).
   ★ Then every weakly compact \( T \) also satisfies (aDE).
Relations between these properties
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Examples

- $C([0,1], K(\ell_2))$ has DPr, but has not numerical index 1
- $c_0$ has numerical index 1, but has not DPr
- $c_0 \oplus \infty C([0,1], K(\ell_2))$ has ADP, neither DPr nor numerical index 1
Relations between these properties

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### Remarks
- For RNP or Asplund spaces, $\text{ADP} \implies \text{numerical index 1}$.
- Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.
ADP + SCD $\implies$ numerical index 1
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Characterizations of the ADP

$X$ Banach space. TFAE:

- $X$ has ADP (i.e. $\max_{\theta \in T} \| \text{Id} + \theta T \| = 1 + \| T \|$ for all $T$ rank-one).
Characterizations of the ADP

Let $X$ be a Banach space. The following are equivalent (TFAE):

1. $X$ has ADP (i.e. $\max_{\theta \in \mathbb{T}} \|\text{Id} + \theta T\| = 1 + \|T\|$ for all $T$ rank-one).
2. Given $x \in S_X$, a slice $S$ of $B_X$ and $\varepsilon > 0$, there is $y \in S$ with
   $$\max_{\theta \in \mathbb{T}} \|x + \theta y\| > 2 - \varepsilon.$$
ADP + SCD $\implies$ numerical index 1

**Characterizations of the ADP**

X Banach space. TFAE:

- X has ADP (i.e. $\max_{\theta \in T} \| \text{Id} + \theta T \| = 1 + \| T \|$ for all $T$ rank-one).
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- Given $x \in S_X$, a sequence $\{S_n\}$ of slices of $B_X$, and $\varepsilon > 0$, there is $y^* \in S_{X^*}$ such that $x \in S(B_X, y^*, \varepsilon)$ and

$$\text{conv}(T S(B_X, y^*, \varepsilon)) \cap S_n \neq \emptyset \quad (n \in \mathbb{N}).$$
Characterizations of the ADP

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  $$\overline{\text{conv}}(\bigcap_{\theta \in T} S(B_X, y^*, \varepsilon)) \cap S_n \neq \emptyset \quad (n \in \mathbb{N}).$$

Theorem

$X$ ADP $+ B_X$ SCD $\implies$ given $x \in S_X$ and $\varepsilon > 0$, there is $y^* \in S_{X^*}$ such that

$$x \in S(B_X, y^*, \varepsilon) \quad \text{and} \quad B_X = \overline{\text{conv}}(\bigcap_{\theta \in T} S(B_X, y^*, \varepsilon)).$$

★ This implies lushness and so, numerical index 1.
Some consequences
Some consequences

**Corollary**

- $\text{ADP} + \text{strongly regular} \implies \text{numerical index} 1$ (actually, lushness).
- $\text{ADP} + X \nsubseteq \ell_1 \implies \text{numerical index} 1$ (actually, lushness).
Some consequences

**Corollary**
- ADP + strongly regular $\implies$ numerical index 1 (actually, lushness).
- ADP + $X \not\in \ell_1$ $\implies$ numerical index 1 (actually, lushness).

**Corollary**

$$X \text{ real } + \dim(X) = \infty + \text{ADP} \implies X^* \supseteq \ell_1.$$
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- ADP + $X \not\subseteq \ell_1$ $\implies$ numerical index 1 (actually, lushness).

Corollary

$X$ real + $\dim(X) = \infty$ + ADP $\implies$ $X^* \supseteq \ell_1$.

Proof.

- If $X \supseteq \ell_1$ $\implies$ $X^*$ contains $\ell_\infty$ as a quotient, so $X^*$ contains $\ell_1$ as a quotient, and the lifting property gives $X^* \supseteq \ell_1$. ✓
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Proof.

- If \( X \supseteq \ell_1 \) \( \implies \) \( X^* \) contains \( \ell_\infty \) as a quotient, so \( X^* \) contains \( \ell_1 \) as a quotient, and the lifting property gives \( X^* \supseteq \ell_1 \).
- If \( X \nsubseteq \ell_1 \) \( \implies \) \( X \) contains \( E \) separable, \( \dim(E) = \infty \) with ADP. \( E \) is SCD + ADP, so \( E \) is lush.
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$X$ real + $\dim(X) = \infty$ + ADP $\implies$ $X^* \supseteq \ell_1$.

Proof.

- If $X \supseteq \ell_1 \implies X^*$ contains $\ell_\infty$ as a quotient, so $X^*$ contains $\ell_1$ as a quotient, and the lifting property gives $X^* \supseteq \ell_1$. ✓
- If $X \not\supset \ell_1 \implies X$ contains $E$ separable, $\dim(E) = \infty$ with ADP. $E$ is SCD + ADP, so $E$ is lush.
- Lush + $\dim(E) = \infty \implies E^* \supseteq \ell_1 \implies X^* \supseteq \ell_1$. ✓
Some consequences

**Corollary**

- ADP + strongly regular $\implies$ numerical index 1 (actually, lushness).
- ADP + $X \not\cong \ell_1$ $\implies$ numerical index 1 (actually, lushness).

**Corollary**

$$X \text{ real } + \dim(X) = \infty + \text{ADP} \implies X^* \supseteq \ell_1.$$  

In particular,
Some consequences

Corollary

- ADP + strongly regular $\implies$ numerical index 1 (actually, lushness).
- $\text{ADP} + X \not\subseteq \ell_1 \implies$ numerical index 1 (actually, lushness).

Corollary

$X$ real + dim$(X) = \infty$ + ADP $\implies$ $X^* \supseteq \ell_1$.

In particular,

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$X$ real + dim$(X) = \infty$ + numerical index 1 $\implies$ $X^* \supseteq \ell_1$. 
Some consequences

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- ADP + strongly regular $\implies$ numerical index 1 (actually, lushness).
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Corollary
$X$ real + $\dim(X) = \infty +$ ADP $\implies X^* \supseteq \ell_1$.

In particular,

Corollary
$X$ real + $\dim(X) = \infty +$ numerical index 1 $\implies X^* \supseteq \ell_1$.

Open question
$X$ real, $\dim(X) = \infty$, $n(X) = 1$ $\implies X \supset c_0$ or $X \supset \ell_1$?
SCD operators

A SCD operator $T \in \mathcal{L}(X)$ is an SCD-operator if $T(B_X)$ is an SCD-set.

Examples:
1. $T(B_X)$ is separable and $T(B_X)$ is RPN,
2. $T(B_X)$ has no $\ell_1$ sequences,
3. $T$ does not fix copies of $\ell_1$.

Theorem

$X$ ADP $+$ SCD operator $\Rightarrow \max_{\theta \in T} \|\text{Id} + \theta T\| = 1 + \|T\|$.  

$X$ DPr $+$ SCD operator $\Rightarrow \|\text{Id} + T\| = 1 + \|T\|$.

Main corollary

$X$ ADP $+$ SCD operator $\Rightarrow \max_{\theta \in T} \|\text{Id} + \theta T\| = 1 + \|T\|$.  

$X$ DPr $+$ SCD operator $\Rightarrow \|\text{Id} + T\| = 1 + \|T\|$.  


SCD operators

**SCD operator**

$T \in L(X)$ is an **SCD-operator** if $T(B_X)$ is an SCD-set.
SCD operators

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**Examples**

$T$ is an SCD-operator when $T(B_X)$ is separable and

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### SCD operators

**SCD operator**

\[ T \in L(X) \text{ is an SCD-operator if } T(B_X) \text{ is an SCD-set.} \]

**Examples**

\( T \) is an SCD-operator when \( T(B_X) \) is separable and

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**Theorem**

- \( X \text{ ADP + } T \text{ SCD-operator} \implies \max_{\theta \in \mathbb{T}} \| \text{Id} + \theta T \| = 1 + \| T \|. \)
- \( X \text{ DPr + } T \text{ SCD-operator} \implies \| \text{Id} + T \| = 1 + \| T \|. \)
SCD operators

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**Theorem**

- \( X \text{ ADP} + T \text{ SCD-operator} \implies \max_{\theta \in T} \|\text{Id} + \theta T\| = 1 + \|T\| \).
- \( X \text{ DPr} + T \text{ SCD-operator} \implies \|\text{Id} + T\| = 1 + \|T\| \).

**Main corollary**

\( X \text{ ADP} + T \) does not fix copies of \( \ell_1 \) \implies \( \max_{\theta \in T} \|\text{Id} + \theta T\| = 1 + \|T\| \).
SCD operators

SCD operator

\( T \in L(X) \) is an **SCD-operator** if \( T(B_X) \) is an SCD-set.

Examples

\( T \) is an SCD-operator when \( T(B_X) \) is separable and

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Theorem

- \( X \) ADP + \( T \) SCD-operator \( \iff \) \( \max_{\theta \in T} \| \text{Id} + \theta T \| = 1 + \| T \| \).
- \( X \) DPr + \( T \) SCD-operator \( \iff \) \( \max_{\theta \in T} \| \text{Id} + \theta T \| = 1 + \| T \| \).

Remark

Separability is not needed!

Main corollary

\( X \) ADP + \( T \) does not fix copies of \( \ell_1 \) \( \iff \) \( \max_{\theta \in T} \| \text{Id} + \theta T \| = 1 + \| T \| \).
Open questions

On SCD-sets

- Find more sufficient conditions for a set to be SCD.
- For instance, if $X$ has 1-symmetric basis, is $B_X$ an SCD-set?
- Is SCD equivalent to the existence of a countable $\pi$-base for the weak topology?
Open questions

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- Find more sufficient conditions for a set to be SCD.
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On SCD-spaces
- $E$ with unconditional basis. Is $E$ SCD?
- $X, Y$ SCD. Are $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$ SCD?
On the containment of $c_0$ or $\ell_1$

Remarks on the containment of $c_0$ and $\ell_1$

A. Avilés, V. Kadets, M. Martín, J. Merí, and V. Shepelska.
Slicely countably determined Banach spaces.

V. Kadets, M. Martín, J. Merí, and R. Payá.
Smoothness and convexity for Banach spaces with numerical index 1.
Open question (Godefroy, private communication)

\[ X \text{ real}, \dim(X) = \infty, n(X) = 1 \implies X \supset c_0 \text{ or } X \supset \ell_1 \]
Open question (Godefroy, private communication)

\[ X \text{ real, } \dim(X) = \infty, \ n(X) = 1 \Rightarrow X \supset c_0 \text{ or } X \supset \ell_1 ? \]

★ Old approaches to this problem:
Containment of $c_0$ or $\ell_1$

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Proof of the last statement:
Open question (Godefroy, private communication)

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Proof of the last statement:

- If $X \supset \ell_1$ we use the “lifting” property of $\ell_1$
Containment of $c_0$ or $\ell_1$

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Proof of the last statement:
- If $X \supset \ell_1$ we use the “lifting” property of $\ell_1$ ✓
- **(AKMMS 2010):** If $X \not\supset \ell_1 \implies X$ is lush.
Containment of $c_0$ or $\ell_1$

Open question (Godefroy, private communication)

$X$ real, $\dim(X) = \infty$, $n(X) = 1 \implies X \supset c_0$ or $X \supset \ell_1$?

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- (AKMMS 2010): If $X \not\supset \ell_1 \implies X$ is lush.

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- (KMMP 2009): In the separable case, lushness implies $|x^{**}(x^*)| = 1$ for every $x^{**} \in \text{ext}(B_{X^{**}})$ and every $x^* \in G$, $G$ norming for $X$. 
Containment of $c_0$ or $\ell_1$

Open question (Godefroy, private communication)

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- (LMP 1999): This gives $X^* \supset c_0 \text{ or } X^* \supset \ell_1 \implies X^* \supset \ell_1$ ✓
Containment of $c_0$ or $\ell_1$

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Equivalent reformulation of the problem:

Equivalent open problem

$X$ real separable, $X \not\supset \ell_1$, exists $G \subseteq S_{X^*}$ norming with $B_{X^*} = aconv\{x \in B_X : x^*(x) = 1\}$ ($x^* \in G$).

Does $X \supseteq c_0$?
Open question (Godefroy, private communication)

\[ X \text{ real, } \dim(X) = \infty, n(X) = 1 \implies X \supset c_0 \text{ or } X \supset \ell_1? \]

**Old approaches to this problem:**

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**Equivalent reformulation of the problem:**
Containment of $c_0$ or $\ell_1$

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$X$ real separable, $X \not
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Does $X \supset c_0$?
Numerical index of $L_p$-spaces

- The 2000’s results on the numerical index on $L_p$-spaces
- The new results on the numerical index of $L_p$-spaces

M. Martín, and J. Merí.
A note on the numerical index of the $L_p$-space of dimension two.
Linear Mult. Algebra (2009)

M. Martín, J. Merí, and M. Popov.
On the numerical index of real $L_p(\mu)$-spaces.
Israel J. Math. (2011)

M. Martín, J. Merí, and M. Popov.
On the numerical radius of operators on Lebesgue spaces.

M. Martín, J. Merí, M. Popov, and B. Randrianantoanina.
Numerical index of absolute sums of Banach spaces.
Known results on the numerical index of $L_p$-spaces

\[ n(\ell_p) \leq n(\ell_{m+1}) \leq n(\ell_m) \]

for $m \in \mathbb{N}$.

(M. Payá, 2000)

\[ n(L_p[0,1]) = n(\ell_p) = \lim_{m \to \infty} n(\ell_m) = \inf_{m \in \mathbb{N}} n(\ell_m). \]


In the real case,

\[ \max\{1^{1/p}, 1^{1/q}\} \leq n(\ell_{2}) \leq \max_{t \in [0,1]} |t^p - 1 - t|^{1/p} + t^p. \]

(M. Merí, 2009)
Numerical index of $L_p$-spaces

Known results on the numerical index of $L_p$-spaces

$\ell_p \leq \ell_{p(m+1)} \leq \ell_{p(m)}$ for $m \in \mathbb{N}$.

(M.–Payá, 2000)
Known results on the numerical index of $L_p$-spaces

1. $n(\ell_p) \leq n(\ell_p^{(m+1)}) \leq n(\ell_p^{(m)})$ for $m \in \mathbb{N}$.
   (M.–Payá, 2000)

2. $n(L_p[0,1]) = n(\ell_p) = \lim_{m \to \infty} n(\ell_p^{(m)}) = \inf_{m \in \mathbb{N}} n(\ell_p^{(m)})$.
Numerical index of $L_p$-spaces

Known results on the numerical index of $L_p$-spaces

1. $n(\ell_p) \leq n(\ell_p^{(m+1)}) \leq n(\ell_p^{(m)})$ for $m \in \mathbb{N}$.
   (M.–Payá, 2000)

2. $n(L_p[0,1]) = n(\ell_p) = \lim_{m \to \infty} n(\ell_p^{(m)}) = \inf_{m \in \mathbb{N}} n(\ell_p^{(m)})$.

3. In the real case,
   \[
   \max \left\{ \frac{1}{2^{1/p}}, \frac{1}{2^{1/q}} \right\} \nu \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leq n(\ell_p^{(2)}) \leq \nu \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
   \]
   \[
   \text{and } \nu \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \max_{t \in [0,1]} \frac{|t^{p-1} - t|}{1 + t^p}
   \]
   (M.–Merí, 2009)
Ideas behind the proofs I

Ideas behind the proofs I

Numerical index of $L_p$ The 2000’s results

The numerical index decreases with the dimension $n$:

$$\ell_p \leq n \left( \ell_{m+1} \right)^p \leq n \left( \ell_m \right)^p$$

for $m \in \mathbb{N}$.

Proposition (M.–Paya, 2000)

$Z = U \oplus V$ with absolute sum (i.e. $\|u+v\| = f(\|u\|, \|v\|)$ for $u \in U$, $v \in V$).

Proof of the decreasing $\ell_m$ is an absolute summand in both $\ell_{m+1}$ and in $\ell_p$.
Ideas behind the proofs I

The numerical index decreases with the dimension

\[ n(\ell_p) \leq n(\ell_p^{(m+1)}) \leq n(\ell_p^{(m)}) \text{ for } m \in \mathbb{N}. \]
The numerical index decreases with the dimension

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**Proposition (M.–Payá, 2000)**

\[ Z = U \oplus V \text{ with absolute sum (i.e. } \|u + v\| = f(\|u\|, \|v\|) \text{ for } u \in U, v \in V). \]

\[ \Rightarrow n(Z) \leq \min\{n(U), n(V)\}. \]
Ideas behind the proofs I

The numerical index decreases with the dimension

\[ n(\ell_p) \leq n(\ell^{(m+1)}_p) \leq n(\ell^{(m)}_p) \text{ for } m \in \mathbb{N}. \]

Proposition (M.–Payá, 2000)

\[ Z = U \oplus V \text{ with absolute sum (i.e. } \|u + v\| = f(\|u\|, \|v\|) \text{ for } u \in U, \ v \in V). \]

\[ \implies n(Z) \leq \min\{n(U), n(V)\}. \]

Proof of the decreasing
Ideas behind the proofs I

The numerical index decreases with the dimension

\[ n(\ell_p) \leq n(\ell_p^{(m+1)}) \leq n(\ell_p^m) \text{ for } m \in \mathbb{N}. \]

Proposition (M.–Payá, 2000)

\[ Z = U \oplus V \text{ with absolute sum (i.e. } \|u + v\| = f(\|u\|, \|v\|) \text{ for } u \in U, \ v \in V). \]

\[ \implies n(Z) \leq \min\{n(U), n(V)\}. \]

Proof of the decreasing

- \( \ell_p^m \) is an absolute summand in both \( \ell_p^{(m+1)} \) and in \( \ell_p \).
Ideas behind the proofs II
Ideas behind the proofs II

One inequality

\[ n(L_p[0,1]) \leq \lim_{m \to \infty} n(\ell_p^m). \]
One inequality

\[ n(L_p[0,1]) \leq \lim_{m \to \infty} n(\ell_p(m)). \]

Proposition (M.–Merí–Popov–Randrianantoanina, 2011)

\( E \) order continuous Köthe space, \( X \) Banach space

\[ \implies n(E(X)) \leq n(X). \]
Ideas behind the proofs II

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Proof of the inequality
Ideas behind the proofs II

One inequality

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\[ \implies n(E(X)) \leq n(X). \]

Proof of the inequality

- \( E = L_p[0,1], \ X = \ell_p^m. \)
One inequality

\[ n(L_p[0,1]) \leq \lim_{m \to \infty} n(\ell_p^m). \]

Proposition (M.–Merí–Popov–Randrianantoanina, 2011)

\(E\) order continuous Köthe space, \(X\) Banach space

\[ \implies n(E(X)) \leq n(X). \]

Proof of the inequality

- \(E = L_p[0,1], \ X = \ell_p^m\).
- \(E \equiv E(X)\) so \(n(E) \leq n(\ell_p^m)\).
Numerical index of $L_p$

Ideas behind the proofs III

Proposition (M.–Merí–Popov–Randrianantoanina, 2011)

Z

Banach space, \( \{Z_i\}_{i \in I} \) increasing family of one-complemented subspaces whose union is dense. Then,

\[ n(Z) \geq \limsup_{i \in I} n(Z_i). \]

Corollary

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Banach space with monotone basis \( (e_m) \),

\[ Z_m = \text{span} \left\{ e_k : 1 \leq k \leq m \right\}. \]

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Proof of the inequality

Z

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E

= \ell_0, \ell_1, \( (e_m) \) Haar system \( \Rightarrow Z_m \equiv \ell_p(m) \) for \( m = 2^k \) \((k \in \mathbb{N})\).
The reversed inequality

\[ n(L_p[0,1]) \geq \lim_{m \to \infty} n(\ell_p^m) \quad \text{and} \quad n(\ell_p) \geq \lim_{m \to \infty} n(\ell_p^m). \]
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Ideas behind the proofs III

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Ideas behind the proofs III

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The two-dimensional case

In the real case,

$$\max \left\{ \frac{1}{2}, \frac{1}{2} \right\} M_p \leq n \left( \ell \left( 2 \right)^p \right) \leq M_p$$

where

$$M_p = \max_{t \in [0,1]} |t^p - 1 - t| + t^p.$$
Ideas behind the proofs IV

The two-dimensional case

In the real case,

$$\max \left\{ \frac{1}{2^{1/p}}, \frac{1}{2^{1/q}} \right\} M_p \leq n(\ell_p^{(2)}) \leq M_p$$

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$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$ operator in $\ell_p^{(2)}$. Then

$$v(T) = \max \left\{ \max_{t \in [0,1]} \frac{|a + d t^p| + |b t + c t^{p-1}|}{1 + t^p}, \max_{t \in [0,1]} \frac{|d + a t^p| + |c t + b t^{p-1}|}{1 + t^p} \right\}.$$
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Proof of the result
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Proof of the result

- $$n(\ell_p^{(2)}) \leq M_p$$ since

$$\left\| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\| = 1$$

and

$$v \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = M_p.$$
The two-dimensional case

In the real case,

\[
\max \left\{ \frac{1}{2^{1/p}}, \frac{1}{2^{1/q}} \right\} M_p \leq n(\ell_p^2) \leq M_p \quad \text{where} \quad M_p = \max_{t \in [0,1]} \frac{|t^{p-1} - t|}{1 + t^p}
\]

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T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ operator in } \ell_p^2. \text{ Then}
\]

\[
v(T) = \max \left\{ \max_{t \in [0,1]} \frac{|a + d t^p| + |b t + c t^{p-1}|}{1 + t^p}, \max_{t \in [0,1]} \frac{|d + a t^p| + |c t + b t^{p-1}|}{1 + t^p} \right\}.
\]

Proof of the result

- \( n(\ell_p^2) \leq M_p \) since \( \| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \| = 1 \) and \( v \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = M_p. \)

- We compare \( v(T) \) with \( M_p \), but we use \( \| T \|_1 \) and \( \| T \|_\infty \) instead of \( \| T \|_p \).
Questions
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1. Is \( n(\ell_p^{(m+1)}) = n(\ell_p^{(m)}) \) for \( m \geq 2 \) ?
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The 2010’s results

- We left the finite-dimensional approach and introduce the absolute numerical radius.
Questions

1. Is \( n(\ell_p^{(m+1)}) = n(\ell_p^{(m)}) \) for \( m \geq 2 \) ?

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3. We do not have results for the complex case, even for dimension two.

The 2010’s results

- We left the finite-dimensional approach and introduce the absolute numerical radius.
- This allows to show that \( n(L_p[0,1]) > 0 \) in the real case.
The absolute numerical radius in $L_p$
The absolute numerical radius in $L_p$

The numerical radius in $L_p$

- For $x \in L_p(\mu)$, write $x^# = |x|^{p^{-1}} \text{sign}(\bar{x})$.
- It is the unique element in $L_q(\mu)$ such that

$$
\|x\|_p^p = \|x^#\|_q^q \quad \text{and} \quad \int x x^# \, d\mu = \|x\|_p \|x^#\|_q = \|x\|_p^p.
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The absolute numerical radius in $L_p$

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  \[ \|x\|_p^p = \|x^\#\|_q^q \quad \text{and} \quad \int x x^\# \, d\mu = \|x\|_p \|x^\#\|_q = \|x\|_p^p. \]
- Therefore, for $T \in L(L_p(\mu))$ one has
  \[ v(T) = \sup \left\{ \left| \int x^\# T x \, d\mu \right| : x \in L_p(\mu), \|x\|_p = 1 \right\} \]
  \[ = \sup \left\{ \left| \int |x|^{p-1} \text{sign}(\overline{x}) \, T x \, d\mu \right| : x \in L_p(\mu), \|x\|_p = 1 \right\} \]
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- Therefore, for $T \in L(L_p(\mu))$ one has
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Absolute numerical radius

For $T \in L(L_p(\mu))$,

\[ |v|(T) := \sup \left\{ \int |x^#T x| \, d\mu : x \in L_p(\mu), \|x\|_p = 1 \right\} \]
\[ = \sup \left\{ \int |x|^{p-1} |T x| \, d\mu : x \in L_p(\mu), \|x\|_p = 1 \right\}. \]
The absolute numerical index of $L_p$
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Obvious remark

$v(T) \leq |v|(T) \leq \|T\|$ for every $T \in L(L_p(\mu))$. 
The absolute numerical index of $L_p$

**Obvious remark**

$$v(T) \leq |v|(T) \leq \|T\| \text{ for every } T \in L(L_p(\mu)).$$

**Absolute numerical index**

$$|n|(L_p(\mu)) = \inf \{ |v|(T) : T \in L(L_p(\mu)), \|T\| = 1 \}$$

$$= \max \{ k \geq 0 : k\|T\| \leq |v|(T) \ \forall \ T \in L(L_p(\mu)) \}.$$
The absolute numerical index of $L_p$

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- $n(L_p(\mu))$ is the greatest constant $M \geq 0$ such that

$$\sup \left\{ \left| \int |x|^{p-1} \text{sign}(\bar{x}) Tx \, d\mu \right| : x \in L_p(\mu), \|x\|_p = 1 \right\} \geq M \|T\|$$

for every $T \in L(L_p(\mu)).$
The absolute numerical index of \( L_p \)

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\[ v(T) \leq |v|(T) \leq \|T\| \text{ for every } T \in L(L_p(\mu)). \]

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\]

for every \( T \in L(L_p(\mu)) \).

- \( |n|(L_p(\mu)) \) is the greatest constant \( K \geq 0 \) such that

\[
\sup \left\{ \left| \int |x|^{p-1}|T \, x| \, d\mu \right| : x \in L_p(\mu), \|x\|_p = 1 \right\} \geq K \|T\|
\]

for every \( T \in L(L_p(\mu)) \).
Giving an estimation of $n(L_p(\mu))$
Giving an estimation of $n(L_p(\mu))$

**Roadmap**

We would like to give an estimation of $n(L_p(\mu))$ in two steps:

- First, we study the relationship between $v(T)$ and $|v|(T)$ for all operators $T$.
- Second, we study the relationship between $|v|(T)$ and $\|T\|$ for all operators $T$. Here, we actually calculate $|n|(L_p(\mu))$. 
Relating the numerical radius and the absolute numerical radius

Numerical index of $L_p$

The new results

Write

$$M_p = \max_{t \in [0,1]} |t^p - 1 - t| + t^p = v(0,1 - 1,0)$$

the numerical radius taken in the real $\ell_2$.

Remark

It is not difficult to see that in every $L_p(\mu)$ space there is an operator $T$ with $\|T\| = 1$ and $v(T) = M_p$.

⋆ We may use $M_p$ to relate $v$ and $|v|$: Theorem (M.–Merí–Popov, 2011)

In the real case, $v(T) \geq M_p v(T)$ for every $T \in L(L_p(\mu))$. 

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Relating the numerical radius and the absolute numerical radius

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Relating the numerical radius and the absolute numerical radius

**The constant**

Write

\[ M_p = \max_{t \in [0,1]} \frac{|t^{p-1} - t|}{1 + t^p} = v \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

the numerical radius taken in the real \( \ell^2_p \).

**Remark**

It is not difficult to see that in every \( L_p(\mu) \) space there is an operator \( T \) with \( \|T\| = 1 \) and \( v(T) = M_p \).
Relating the numerical radius and the absolute numerical radius

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★ We may use $M_p$ to relate $v$ and $|v|$:
Relating the numerical radius and the absolute numerical radius

The constant

Write

\[ M_p = \max_{t \in [0,1]} \frac{|t^{p-1} - t|}{1 + tp} = \nu \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

the numerical radius taken in the real \( \ell^2_p \).

Remark

It is not difficult to see that in every \( L_p(\mu) \) space there is an operator \( T \) with \( \|T\| = 1 \) and \( \nu(T) = M_p \).

⭐ We may use \( M_p \) to relate \( \nu \) and \( |\nu| \):

Theorem (M.–Merí–Popov, 2011)

In the real case,

\[ \nu(T) \geq \frac{M_p}{6} |\nu|(T) \]

for every \( T \in L(L_p(\mu)) \).
Calculating $|n|(L_p(\mu))$
Calculating $|n|(L_p(\mu))$ 1

The constant

Set $\kappa_p := \max_{\tau > 0} \frac{\tau^{p-1}}{1 + \tau^p} = \max_{\lambda \in [0,1]} \lambda^{\frac{1}{q}} (1 - \lambda)^{\frac{1}{p}} = \frac{1}{p^{1/p} q^{1/q}}$. 
Calculating $|n|(L_p(\mu))$ 1

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The best possibility for $|n|(L_p(\mu))$

If $\dim(L_p(\mu)) \geq 2$, then there is a (positive) operator $T \in L(L_p(\mu))$ with

$$\|T\| = 1, \quad |\nu|(T) = \kappa_p.$$
Calculating $|n|(L_p(\mu))$ I

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The best possibility for $|n|(L_p(\mu))$

If $\dim(L_p(\mu)) \geq 2$, then there is a (positive) operator $T \in L(L_p(\mu))$ with

$$\|T\| = 1, \quad |v|(T) = \kappa_p.$$ 

The examples for $\ell_p$ and $L_p[0,1]$: 

For $\ell_p$: consider the extension by zero of the matrix

\[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\]

For $L_p[0,1]$: $T f = 2 \left[ \int_{1/2}^1 f(s) \, ds \right] \chi_{[1/2,1]} (f \in L_p[0,1])$. 

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Calculating $|n|(L_p(\mu))$ 1

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If dim($L_p(\mu)$) $\geq 2$, then there is a (positive) operator $T \in L(L_p(\mu))$ with

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The examples for $\ell_p$ and $L_p[0,1]$:

- For $\ell_p$: consider the extension by zero of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
Calculating $|n|(L_p(\mu))$

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Set $\kappa_p := \max_{\tau > 0} \frac{\tau^{p-1}}{1 + \tau^p} = \max_{\lambda \in [0,1]} \lambda^\frac{1}{q} (1 - \lambda)^\frac{1}{p} = \frac{1}{p^{1/p} q^{1/q}}$.

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If $\dim(L_p(\mu)) \geq 2$, then there is a (positive) operator $T \in L(L_p(\mu))$ with $\|T\| = 1$, $|v|(T) = \kappa_p$.

The examples for $\ell_p$ and $L_p[0,1]$:  

- For $\ell_p$: consider the extension by zero of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
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Numerical index of $L_p$ The new results

Calculating $|n|(L_p(\mu))$ II
Theorem (M.–Merí–Popov, 2011)

\[ |n|(L_p(\mu)) \geq \kappa_p \]
Calculating $|n|(L_p(\mu))$ II

**Theorem (M.–Merí–Popov, 2011)**

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**Proof for positive operators:**

...
Calculating $|n|(L_p(\mu))$ II

**Theorem (M.–Merí–Popov, 2011)**

$|n|(L_p(\mu)) \geq \kappa_p$

**Proof for positive operators:**

- Fix $T \in L(L_p(\mu))$ positive with $\|T\| = 1$, $\tau > 0$ and $\varepsilon > 0$. 

Calculating $|n|(L_p(\mu))$ II

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$$|n|(L_p(\mu)) \geq \kappa_p$$

**Proof for positive operators:**

- Fix $T \in L(L_p(\mu))$ positive with $\|T\| = 1$, $\tau > 0$ and $\epsilon > 0$.
- Find $x \geq 0$ with $\|x\| = 1$ and $\|Tx\|^p > 1 - \epsilon$, set
  $$y = x \lor \tau Tx \quad \text{and} \quad A = \{\omega \in \Omega : x(\omega) \geq \tau(Tx)(\omega)\},$$
  and observe that
  $$\|y\|^p = \int_A x^p \, d\mu + \int_{\Omega \setminus A} (\tau Tx)^p \, d\mu \leq 1 + \tau^p \quad \text{and} \quad y^# = x^{p-1} \lor (\tau Tx)^{p-1}. $$
Calculating $|n|(L_p(\mu))$ II

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$$y = x \lor \tau Tx \quad \text{and} \quad A = \{\omega \in \Omega : x(\omega) \geq \tau(Tx)(\omega)\},$$

and observe that

$$\|y\|^p = \int_A x^p \, d\mu + \int_{\Omega \setminus A} (\tau Tx)^p \, d\mu \leq 1 + \tau^p \quad \text{and} \quad y^\# = x^{p-1} \lor (\tau Tx)^{p-1}.$$  

- Now,

$$|v|(T) \geq \frac{1}{\|y\|^p} \int_{\Omega} y^\# Ty \, d\mu \geq \frac{1}{1 + \tau^p} \int_{\Omega} y^\# Ty \, d\mu.$$
Calculating $|n|(L_p(\mu))$ II

**Theorem (M.-Merí–Popov, 2011)**

$|n|(L_p(\mu)) \geq \kappa_p$

**Proof for positive operators:**

- Fix $T \in L(L_p(\mu))$ **positive** with $\|T\| = 1$, $\tau > 0$ and $\varepsilon > 0$.
- Find $x \geq 0$ with $\|x\| = 1$ and $\|Tx\|^p > 1 - \varepsilon$, set
  
  $y = x \vee \tau Tx \quad \text{and} \quad A = \{\omega \in \Omega : x(\omega) \geq \tau(Tx)(\omega)\},$

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  $\|y\|^p = \int_A x^p \, d\mu + \int_{\Omega \setminus A} (\tau Tx)^p \, d\mu \leq 1 + \tau^p \quad \text{and} \quad y^# = x^{p-1} \vee (\tau Tx)^{p-1}.$

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  $\geq \frac{1}{1 + \tau^p} \int_{\Omega} (\tau Tx)^{p-1} Tx \, d\mu = \frac{\tau^{p-1}}{1 + \tau^p} \int_{\Omega} (Tx)^p \, d\mu \geq \frac{\tau^{p-1}}{1 + \tau^p} (1 - \varepsilon).$
Calculating $|n|(L_p(\mu))$ II

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- Taking supremum on $\tau > 0$ and $\varepsilon > 0$, we get $|v|(T) \geq \kappa_p$.  

The main consequence:
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\[ n(L_p(\mu)) \geq \frac{M_p \kappa_p}{6} \text{ in the real case.} \]
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Corollary

In the real case, \( n(L_p(\mu)) > 0 \) for every \( p \neq 2 \).
Further results
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- If $T \in L(L_p[0,1])$ is rank-one $\implies v(T) \geq \kappa_p^2 \|T\|.$
- If $T \in L(L_p[0,1])$ is compact, then

$$v(T) \geq \kappa_p^2 \|T\| \text{ (complex case), } \quad v(T) \geq \max_{\tau > 0} \frac{\kappa_p \tau^{p-1} - \tau}{1 + \tau^p} \|T\| \text{ (real case).}$$
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Open problems with conjectures

- Is \( n(L_p(\mu)) = M_p(\dim \geq 2) \) in the real case?
  It is enough to prove that \( n(L_p[0,1]) \geq M_p \) or \( n(\ell_p) \geq M_p \).

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Numerical index of $L_p$  The new results
Extremely non-complex Banach spaces

1. Motivation
2. Extremely non-complex Banach spaces
3. Surjective isometries

V. Kadets, M. Martín, and J. Merí.
Norm equalities for operators on Banach spaces.

P. Koszmider, M. Martín, and J. Merí.
Extremely non-complex $C(K)$ spaces.

P. Koszmider, M. Martín, and J. Merí.
Isometries on extremely non-complex Banach spaces.
Isometries and duality. Reminder

Example (produced with numerical ranges)

There is a Banach space $X$ such that $\text{Iso}(X)$ has no exponential one-parameter semigroups. $\text{Iso}(X^*)$ contains infinitely many exponential one-parameter semigroups.

⋆ In terms of linear dynamical systems:

There is no $A \in L(X)$ such that the solution of $x' = Ax$ ($x : \mathbb{R}_+ \to X$) is given by a semigroup of isometries. There are infinitely many such $A$'s on $X$.

But there are unbounded $A$'s on $X$ such that the solution of the linear dynamical system is a one-parameter $C_0$ semigroup of isometries.

We would like to find $X$ such that $\text{Iso}(X)$ has no $C_0$ semigroup of isometries. $\text{Iso}(X^*)$ has exponential semigroup of isometries.
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Numerical range of unbounded operators (1960’s)

Let $X$ be a Banach space, $T : D(T) \to X$ a linear operator, then

$$V(T) = \{ x^*(Tx) : x^* \in X^*, x \in D(T), \ x^*(x) = \|x^*\| = \|x\| = 1 \}.$$
Numerical range of unbounded operators (1960’s)

$X$ Banach space, $T : D(T) \rightarrow X$ linear,

$$V(T) = \{ x^*(Tx) : x^* \in X^*, x \in D(T), x^*(x) = \|x^*\| = \|x\| = 1 \}.$$  

Teorema (Stone, 1932)

$H$ Hilbert space, $A$ densely defined operator. TFAE:

- $A$ generates an strongly continuous one-parameter semigroup of unitary operators (onto isometries).
- $A^* = -A$.
- $\text{Re}(Ax | x) = 0$ for every $x \in D(A)$. 

Numerical range of unbounded operators. II

Motivation

Which Banach spaces have unbounded operators with numerical range zero?

Examples

In $C^0(\mathbb{R})$, $\Phi(t)(f)(s) = f(t+s)$ is a strongly continuous one-parameter semigroup of isometries (generated by the derivative).

In $C^\infty([0,1] \rightarrow \mathbb{R})$ there are also strongly continuous one-parameter semigroups of isometries.

Consequence

We have to completely change our approach to the problem.
Numerical range of unbounded operators. II

**Difficulty**

Which Banach spaces have unbounded operators with numerical range zero?
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**Examples**

- In $C_0(\mathbb{R})$, $\Phi(t)(f)(s) = f(t + s)$ is an strongly continuous one-parameter semigroup of isometries (generated by the derivative).
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### Numerical range of unbounded operators. II

#### Difficulty
Which Banach spaces have unbounded operators with numerical range zero?

#### Examples
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We have to completely change our approach to the problem.
Complex structures

**Definition**

$X$ has **complex structure** if there is $T \in L(X)$ such that $T^2 = -\text{Id}$.
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**Some remarks**

- This gives a structure of vector space over $\mathbb{C}$:

  $$(\alpha + i \beta) x = \alpha x + \beta T(x) \quad (\alpha + i \beta \in \mathbb{C}, \ x \in X)$$
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- Defining

  \[\|x\| = \max\{\|e^{i\theta} x\| : \theta \in [0, 2\pi]\}\]  

  \[(x \in X)\]

  one gets that \((X, \| \cdot \|)\) is a complex Banach space.
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- If $T$ is an isometry, then actually the given norm of $X$ is complex.
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- If *T* is an isometry, then actually the given norm of *X* is complex.

- Conversely, if *X* is a complex Banach space, then

  \[T(x) = i x \quad (x \in X)\]

  satisfies *T*² = −Id and *T* is an isometry.
If $\dim(X) < \infty$, $X$ has complex structure iff $\dim(X)$ is even.

If $X \cong \mathbb{Z} \oplus \mathbb{Z}$ (in particular, $X \cong X^2$), then $X$ has complex structure.


$X$ is even if admits a complex structure but its hyperplanes does not. $X$ is odd if its hyperplanes are even (and so $X$ does not admit a complex structure).

Definition: $X$ is extremely non-complex if $\text{dist}(T^2, -\text{Id})$ is the maximum possible, i.e. $\|\text{Id} + T^2\| = 1 + \|T^2\|$ ($T \in \mathcal{L}(X)$).
Some examples

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3. There are infinite-dimensional Banach spaces without complex structure:
   - Dieudonné, 1952: the James' space $J$ (since $J^{\ast\ast} \equiv J \oplus \mathbb{R}$).
   - Szarek, 1986: uniformly convex examples.
   - Ferenczi-Medina Galego, 2007: there are odd and even infinite-dimensional spaces $X$.

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## The Daugavet equation

### What Daugavet did in 1963

The norm equality

\[ \| \text{Id} + T \| = 1 + \| T \| \]

holds for every compact \( T \in L(C[0,1]) \).
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\( X \) Banach space, \( T \in L(X) \), \( \|\text{Id} + T\| = 1 + \|T\| \) \hspace{1cm} \text{(DE)}.

Classical examples

1. **Daugavet, 1963:**
   Every compact operator on \( C[0,1] \) satisfies (DE).

2. **Lozanoskii, 1966:**
   Every compact operator on \( L_1[0,1] \) satisfies (DE).

3. **Abramovich, Holub, and more, 80’s:**
   \( X = C(K) \), \( K \) perfect compact space
   or \( X = L_1(\mu) \), \( \mu \) atomless measure
   \( \implies \) every weakly compact \( T \in L(X) \) satisfies (DE).
The Daugavet property

The Daugavet property (Kadets–Shvidkoy–Sirotkin–Werner, 1997)

A Banach space $X$ is said to have the Daugavet property iff every rank-one operator on $X$ satisfies (DE).
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- Every weakly compact operator on $X$ satisfies (DE).
- $X$ contains $\ell_1$.
- $X$ does not embed into a Banach space with unconditional basis.
- **Geometric characterization**: $X$ has the Daugavet property iff for each $x \in S_X$

$$
\overline{co} \left( B_X \setminus (x + (2 - \varepsilon)B_X) \right) = B_X.
$$

The Daugavet property II
More examples

The following spaces have the Daugavet property.

- **Wojtaszczyk, 1992:** The disk algebra and $H^\infty$.
- **Werner, 1997:** “Nonatomic” function algebras.
- **Oikhberg, 2005:** Non-atomic $C^*$-algebras and preduals of non-atomic von Neumann algebras.
- **Becerra–M., 2005:** Non-atomic $JB^*$-triples and their preduals.
- **Becerra–M., 2006:** Preduals of $L_1(\mu)$ without Fréchet-smooth points.
- **Ivankhno, Kadets, Werner, 2007:** $\text{Lip}(K)$ when $K \subseteq \mathbb{R}^n$ is compact and convex.
Daugavet–type inequalities

Benyamini–Lin, 1985:
For every $1 < p < \infty$, $p \neq 2$, there exists $\psi_p$: $(0, \infty) \rightarrow (0, \infty)$ such that
$$\| \operatorname{Id} + T \| \geq 1 + \psi_p(\|T\|)$$
for every compact operator $T$ on $L^p[0, 1]$.

If $p = 2$, then there is a non-null compact $T$ on $L^2[0, 1]$ such that
$$\| \operatorname{Id} + T \| = 1.$$

Boyko–Kadets, 2004:
If $\psi_p$ is the best possible function above, then
$$\lim_{p \to 1^+} \psi_p(t) = t$$
for $t > 0$.

Oikhberg, 2005:
If $K(\ell_2) \subseteq X \subseteq L(\ell_2)$, then
$$\| \operatorname{Id} + T \| \geq 1 + \frac{1}{8\sqrt{2}}\|T\|$$
for every compact $T$ on $X$. 
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Daugavet–type inequalities

Some examples

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Extremely non-complex

Motivation
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Motivation

Norm equalities for operators

Motivating question

Are there other norm equalities which could define interesting properties of Banach spaces?

Concretely

We looked for non-trivial norm equalities of the forms

\[
\|Id + T\| = f(\|T\|)
\]

or

\[
\|g(T)\| = f(\|T\|)
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or

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\|Id + g(T)\| = f(\|g(T)\|)
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(\(g\) analytic, \(f\) arbitrary) satisfied by all rank-one operators on a Banach space.

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We proved that there are few possibilities.
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Equalities of the form $\|\text{Id} + T\| = f(\|T\|)$
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Proposition

$X$ real or complex, $f : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ arbitrary, $a, b \in K$. If the norm equality

$$\|a \text{Id} + b T\| = f(\|T\|)$$

holds for every rank-one operator $T \in L(X)$, then

$$f(t) = |a| + |b| t \quad (t \in \mathbb{R}_0^+).$$

If $a \neq 0$, $b \neq 0$, then $X$ has the Daugavet property.
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Then, we have to look for Daugavet-type equalities in which $\text{Id} + T$ is replaced by something different.
Proof

We have...

\[ \| a \text{Id} + bT \| = f(\|T\|) \quad \forall T \in L(X) \quad \text{rank-one} \]
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- It follows that

\[ |a| + |b| t \geq f(t) = \|a \text{Id} + b T_t\| \geq \|[a \text{Id} + b T_t](x_0)\| \]

\[ = \|a x_0 + b \omega_0 t x_0\| = |a + b \omega_0 t| \|x_0\| = \left| a + b \frac{b}{|b|} \frac{a}{|a|} t \right| = |a| + |b t| \geq |a| + |b| t \]

Finally, for rank-one \( T \in L(X) \), write \( S = a b^T \) and observe

\[ |a| (1 + \|T\|) \geq |a| + |b| \|S\| = \|a \text{Id} + b S\| = |a| + |b| t \]

\[ \blacksquare \]
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We have...

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Let $X$ be real or complex with $\dim(X) \geq 2$. Suppose that the norm equality

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holds for every rank-one operator $T \in L(X)$, where

- $g : \mathbb{K} \to \mathbb{K}$ is analytic,
- $f : \mathbb{R}_0^+ \to \mathbb{R}$ is arbitrary.

Then, there are $a, b \in \mathbb{K}$ such that

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**Corollary**

Only three norm equalities of the form

$$\|g(T)\| = f(\|T\|)$$

are possible:

- $b = 0$: $\|a \text{Id}\| = |a|$,
- $a = 0$: $\|b \, T\| = |b| \|T\|$,  \hspace{1cm} (trivial cases)
- $a \neq 0, b \neq 0$: $\|a \text{Id} + b \, T\| = |a| + |b| \|T\|$,  \hspace{1cm} (Daugavet property)
Proof (complex case)

\[ \|g(T)\| = f(\|T\|) \quad \forall T \in L(X) \quad \text{rank-one} \]

We want...

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Proof (complex case)

We have . . .

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\[ ? \]

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\[ \tilde{g} = \tilde{g}(\lambda) T_1 \quad (\lambda \in \mathbb{C}) \]

\[ \|a_0 \text{Id} + \tilde{g}(\lambda) T_1\| = \|g(\lambda T_1)\| = f(|\lambda|) = \|g(\lambda T_0)\| = \|a_0 \text{Id} + a_1 \lambda T_0\| \]

We use the triangle inequality to get

\[ |\tilde{g}(\lambda)| \leq 2 |a_0| + |a_1| |\lambda| \quad (\lambda \in \mathbb{C}) \]

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<thead>
<tr>
<th>We have...</th>
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**Remark**

If $X$ has the Daugavet property and $g$ is analytic, then

$$\|\text{Id} + g(T)\| = |1 + g(0)| - |g(0)| + \|g(T)\|$$

for every rank-one $T \in L(X)$. 
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- From now on, we have to separate the complex and the real case.
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- **Complex case:**
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**Proposition**

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We obtain two different cases:

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**Theorem**

If $\text{Re} g(0) \neq -1/2$ and

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If $\Re g(0) = -1/2$ and $\|\text{Id} + g(T)\| = f(\|g(T)\|)$ for every rank-one $T$, then there exists $\theta_0 \in \mathbb{R}$ such that $\|\text{Id} + e^{i\theta_0} T\| = \|\text{Id} + T\|$ for every rank-one $T \in L(X)$. 

Example

If $X = C[0,1] \oplus_2 C[0,1]$, then $\|\text{Id} + e^{i\theta} T\| = \|\text{Id} + T\|$ for every $\theta \in \mathbb{R}$, rank-one $T \in L(X)$. $X$ does not have the Daugavet property.
Extremely non-complex Motivation

Equalities of the form $\|\text{Id} + g(T)\| = f(\|g(T)\|)$. Complex case

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- They work when $g$ is onto.
Equalities of the form $\|\text{Id} + g(T)\| = f(\|g(T)\|)$. Real case

- **Real case:**

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  $g(0) = -1/2$:

  **Example**
  If $X = C[0,1] \oplus_2 C[0,1]$, then
  - $\|\text{Id} - T\| = \|\text{Id} + T\|$ for every rank-one $T \in L(X)$.
  - $X$ does not have the Daugavet property.
The question

Godefroy, private communication

Is there any real Banach space $X$ (with $\dim(X) > 1$) such that

$$\|\text{Id} + T^2\| = 1 + \|T^2\|$$

for every operator $T \in L(X)$?

In other words, are there extremely non-complex spaces other than $\mathbb{R}$?
The first attempts

The first attempts

We may try to check whether the known spaces without complex structure are actually extremely non-complex.

Some examples

1. If $\dim(X) < \infty$, $X$ has complex structure iff $\dim(X)$ is even.

2. Dieudonné, 1952: the James' space $J$ ($J^{\ast\ast} \equiv J \oplus \mathbb{R}$).


5. Ferenczi-Medina Galego, 2007: there are odd and even infinite-dimensional spaces $X$.

$X$ is even if admits a complex structure but its hyperplanes do not. $X$ is odd if its hyperplanes are even (and so $X$ does not admit a complex structure).

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This did not work and we moved to $C(K)$ spaces.
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3. **Szarek, 1986**: uniformly convex examples.
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Koszmider, 2004; Plebanek, 2004

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There are compact spaces $K$ such that $C(K)$ has “few operators”: every operator is a weak multiplication.

Weak multiplication

Let $K$ be a compact space. $T \in L(C(K))$ is a **weak multiplication** if

$$T = g \text{Id} + S$$

where $g \in C(K)$ and $S$ is weakly compact.
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Let $K$ be a compact space. $T \in L(C(K))$ is a weak multiplication if

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where $g \in C(K)$ and $S$ is weakly compact.

Theorem

If $K$ is perfect, $T = g \text{Id} + S \in L(C(K))$ weak multiplication

$$\implies \|\text{Id} + T^2\| = 1 + \|T^2\|$$
Proof of the theorem

We have $X = C(K)$, $K$ perfect, $T = gI + S$, $\max \|I + T\| = 1$ (true for every $K$ and every $T$) $\|I + S\| = 1 + \|S\|$ (if $S \in W(X)$, $K$ perfect).

We need $\|I + T^2\| = 1 + \|T^2\|$. If $T = gI + S$, then $T^2 = g^2I + S'$ with $S'$ weakly compact.

We will prove that $\|I + g^2I + S\| = 1 + \|g^2I + S\|$ for $g \in C(K)$ and $S$ weakly compact.

Step 1: We assume $\|g^2\| \leq 1$ and $\min g^2(K) > 0$.

Step 2: We can avoid the assumption that $\min g^2(K) > 0$.

Step 3: Finally, for every $g$ the above gives $\|I + g^2I + S\| = 1 + \|g^2I + S\|$ which gives us the result. ✓
Proof of the theorem

We have $X = C(K)$, $K$ perfect, $T = g\text{Id} + S$

- $\max \|\text{Id} \pm T\| = 1 + \|T\|$ (true for every $K$ and every $T$)
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- If $T = g \text{Id} + S$, then $T^2 = g^2 \text{Id} + S'$ with $S'$ weakly compact.

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We have \( X = C(K), K \) perfect, \( T = g\text{Id} + S \)

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  for \( g \in C(K) \) and \( S \) weakly compact.
- **Step 1:** We assume \( \|g^2\| \leq 1 \) and \( \min g^2(K) > 0 \).

Proof

- It is enough to show that
  \[ \|\text{Id} - (g^2\text{Id} + S)\| < 1 + \|g^2\text{Id} + S\|. \]
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We have \( X = C(K) \), \( K \) perfect, \( T = g\text{Id} + S \)

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- $\|g^2 \text{Id} + S\| = \|\text{Id} + S + (g^2 \text{Id} - \text{Id})\| \geq \|\text{Id} + S\| - \|g^2 \text{Id} - \text{Id}\|
  = 1 + \|S\| - (1 - \min g^2(K)) = \|S\| + \min g^2(K).$
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**Proof**

Just think that the set of operators satisfying (DE) is closed.
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Proof

If $\|u + v\| = \|u\| + \|v\|$ $\Rightarrow$ $\|\alpha u + \beta v\| = \alpha \|u\| + \beta \|v\|$ for $\alpha, \beta \in \mathbb{R}_0^+$. 

We need $\|\text{Id} + T^2\| = 1 + \|T^2\|$
The first example: weak multiplications. II

Weak multiplication

Let $K$ be a compact space. $T \in L(C(K))$ is a weak multiplication if

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Theorem

$K$ perfect, $T = g \text{Id} + S \in L(C(K))$ weak multiplication

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**Example (Koszmider, 2004; Plebanek, 2004)**

There are perfect compact spaces $K$ such that all operators on $C(K)$ are weak multiplications.
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**Consequence**

Therefore, there are extremely non-complex $C(K)$ spaces.
More examples: weak multipliers

**Weak multiplier**

Let $K$ be a compact space. $T \in L(C(K))$ is a weak multiplier if

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There are infinitely many different perfect compact spaces $K$ such that all operators on $C(K)$ are weak multipliers.
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**Corollary**

There are infinitely many non-isomorphic extremely non-complex Banach spaces.
Further examples

Proposition
There is a compact infinite totally disconnected and perfect space $K$ such that all operators on $C(K)$ are weak multipliers.

Consequence
There is a family $(K_i)_{i \in I}$ of pairwise disjoint perfect and totally disconnected compact spaces such that every operator on $C(K_i)$ is a weak multiplier, for $i \neq j$, every $T \in L(C(K_i), C(K_j))$ is weakly compact.

Theorem
There are some compactifications $\tilde{K}$ of the above family $(K_i)_{i \in I}$ such that the corresponding $C(\tilde{K})$'s are extremely non-complex.
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Main consequence

There are perfect compact spaces $K_1$, $K_2$ such that:

- $C(K_1)$ and $C(K_2)$ are extremely non-complex,
- $C(K_1)$ contains a complemented copy of $C(\Delta)$,
- $C(K_2)$ contains a $1$-complemented isometric copy of $\ell_\infty$.

Observation

$C(K_1)$ and $C(K_2)$ have operators which are not weak multipliers.

They are not indecomposable spaces.
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Related open questions

Question 1
Find topological characterization of the compact Hausdorff spaces $K$ such that the spaces $C(K)$ are extremely non-complex.

Question 2
Find topological consequences on $K$ when $C(K)$ is extremely non-complex. For instance:
If $C(K)$ is extremely non-complex and $\psi : K \to K$ is continuous, are there an open subset $U$ of $K$ such that $\psi|_U = \text{id}$ and $\psi(K \setminus U)$ is finite?

We will show latter that $\phi : K \to K$ homeomorphism $\Rightarrow \phi = \text{id}$. 

Extremely non-complex Banach spaces
Related open questions

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Extremely non-complex Banach spaces

Definition

$X$ is extremely non-complex if $\text{dist}(T^2, -\text{Id})$ is the maximum possible, i.e.

$$\|\text{Id} + T^2\| = 1 + \|T^2\| \quad (T \in L(X))$$

Examples

There are several extremely non-complex $C(K)$ spaces:

- If $T = g\text{Id} + S$ for every $T \in L(C(K))$ (Koszmider).

- If $T^* = g\text{Id} + S$ for every $T \in L(C(K))$ (K weak Koszmider).

- One $C(K)$ containing a complemented copy of $C(\Delta)$.

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- One $C(K)$ containing an isometric (1-complemented) copy of $\ell_\infty$. 
Theorem

$X$ extremely non-complex.

- $T \in \text{Iso}(X) \implies T^2 = \text{Id}$.
- $T_1, T_2 \in \text{Iso}(X) \implies T_1 T_2 = T_2 T_1$.
- $T_1, T_2 \in \text{Iso}(X) \implies \|T_1 - T_2\| \in \{0, 2\}$.
- $\Phi : \mathbb{R}_0^+ \rightarrow \text{Iso}(X)$ one-parameter semigroup $\implies \Phi(\mathbb{R}_0^+) = \{\text{Id}\}$.
Isometries on extremely non-complex spaces. 1

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- Then $\text{Id} = \frac{1}{2} T^2 + \frac{1}{2} T^{-2}.$
- Since $\text{Id}$ is an extreme point of $B_{L(X)} \implies T^2 = T^{-2} = \text{Id}.$  ✓
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\[
\text{Id} = (T_1 T_2)(T_1 T_2) \\
\implies T_1 T_2 = T_1(T_1 T_2 T_1 T_2)T_2 = (T_1 T_1) T_2 T_1 (T_2 T_2) = T_2 T_1. \quad \checkmark
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**Proof.**

$$\Phi(t) = \Phi(t/2 + t/2) = \Phi(t/2)^2 = \text{Id}.$$  ✓
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- $\text{Iso}(X)$ is a Boolean group for the composition operation.
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- $\text{Iso}(X)$ identifies with the set $\text{Unc}(X)$ of unconditional projections on $X$:

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- $\text{Iso}(X) \equiv \text{Unc}(X)$ is a Boolean algebra
  $\iff P_1 P_2 \in \text{Unc}(X)$ when $P_1, P_2 \in \text{Unc}(X)$
  $\iff \left\| \frac{1}{2} (\text{Id} + T_1 + T_2 - T_1 T_2) \right\| = 1$ for every $T_1, T_2 \in \text{Iso}(X).$
Extremely non-complex $C_E(K\|L)$ spaces.

Theorem: $K$ perfect weak Koszmider, $L$ closed nowhere dense, $E \subset C(L)$.

Proposition: $K$ perfect $\Rightarrow \exists L \subset K$ closed nowhere dense with $C[0,1] \subset C(L)$.

Observation: exists a non-$C(K)$ extremely non-complex space $C\ell_2(K\|L)$ is not isomorphic to a $C(K')$ space since $\ell_2 \subset \text{comp}-C\ell_2(K\|L)^*$.

Important consequence: Example

Take $K$ perfect weak Koszmider, $L \subset K$ closed nowhere dense with $E = \ell_2 \subset C[0,1] \subset C(L)$:

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But we are able to give a better result...
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$K$ perfect weak Koszmider, $L$ closed nowhere dense, $E \subset C(L)$
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$C_{\ell^2}(K\|L)$ is not isomorphic to a $C(K')$ space since $\ell^2 \overset{\text{comp}}{\longrightarrow} C_{\ell^2}(K\|L)^*$. 
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**Theorem (Banach-Stone like)**

$C_E(K\|L)$ extremely non-complex, $T \in \text{ Iso}(C_E(K\|L))$

$\implies$ exists $\theta : K \setminus L \longrightarrow \{-1,1\}$ continuous such that

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- Consider \( \phi : D_0 \to D_0 \) and \( \theta : D_0 \to \{-1, 1\} \text{ with} \)

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Isometries on extremely non-complex $C_E(K∥L)$ spaces

Theorem (Banach-Stone like)

$C_E(K∥L)$ extremely non-complex, $T ∈ Iso(C_E(K∥L))$ \implies \text{exists } θ : K \setminus L \rightarrow \{-1, 1\} \text{ continuous such that}

$[T(f)](x) = θ(x)f(x)$ \quad (x ∈ K \setminus L, f ∈ C_E(K∥L))

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- $D_0 = \{x ∈ K \setminus L : \exists y ∈ K \setminus L, θ_0 ∈ \{-1, 1\} \text{ with } T^*(δ_x) = θ_0δ_y\}$ dense in $K$.
- Consider $φ : D_0 \rightarrow D_0$ and $θ : D_0 \rightarrow \{-1, 1\}$ with

  $T^*(δ_x) = θ(x)δ_{φ(x)}$

- $φ^2 = \text{id}$, $θ(x)θ(φ(x)) = 1$, $φ$ homeomorphism.
- $φ(x) = x$ for all $x ∈ D_0$.
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- $θ$ is continuous. ✓
Isometries on extremely non-complex $C_E(K\|L)$ spaces

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**Consequences: cases $E = C(L)$ and $E = 0$**

- $C(K)$ extremely non-complex, $\varphi : K \rightarrow K$ homeomorphism $\implies \varphi = \text{id}$
- $C_0(K \setminus L) \equiv C_0(K\|L)$ extremely non-complex, $\varphi : K \setminus L \rightarrow K \setminus L$ homeomorphism $\implies \varphi = \text{id}$
- In both cases, the group of surjective isometries identifies with a Boolean algebra of clopen sets.
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**Consequences: general case**

- If for every $x \in L$, there is $f \in E$ with $f(x) \neq 0$
  $\implies$ $\theta$ extends to the whole $K$ and

  $$[T(f)](x) = \theta(x)f(x) \quad (x \in K, f \in C_E(K\|L))$$

  for every surjective isometry $T$. 

If this happens, then $0 \not\in \text{ext}(B_E^*)$ w$^*$ by (V. Kadets).

But for $E = \ell^2$, $0 \in \text{ext}(B_E^*)$ w$^*$. 

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Isometries on extremely non-complex $C_E(K\|L)$ spaces

**Theorem (Banach-Stone like)**

$C_E(K\|L)$ extremely non-complex, $T \in \text{Iso}(C_E(K\|L))$

$\implies$ exists $\theta : K \setminus L \to \{-1, 1\}$ continuous such that

$$[T(f)](x) = \theta(x)f(x) \quad (x \in K \setminus L, f \in C_E(K\|L))$$

**Consequences: general case**

- If for every $x \in L$, there is $f \in E$ with $f(x) \neq 0$
  $\implies$ $\theta$ extends to the whole $K$ and

  $$[T(f)](x) = \theta(x)f(x) \quad (x \in K, f \in C_E(K\|L))$$

  for every surjective isometry $T$.

- If this happens, then $0 \not\in \text{ext} \left(B_E^* \right)^{w^*}$ (V. Kadets).
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- But for $E = \ell_2$, $0 \in \text{ext} (B_E^*)^{w*}$.
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**Consequence: connected case**

If $K$ and $K \setminus L$ are connected, then

$$\text{Iso}(C_E(K\|L)) = \{-\text{Id}, +\text{Id}\}$$
The main example

Koszmider, 2004

∃K weak Koszmider space such that K \(\setminus\) F is connected if \(|F| < \infty\).

Observation on the above construction

There is \(L \subset K\) closed nowhere dense with \(K \setminus L\) connected

\([0, 1] \subseteq C(L)\)

The best example

Consider \(X = C_\ell_2(K\|L)\). Then:

\[\text{Iso}(X) = \{-\text{Id}, +\text{Id}\}\]

and \(\text{Iso}(X^*) \supset \text{Iso}(\ell_2)\).

Proof.

\(K\) weak Koszmider, \(L\) nowhere dense, \(\ell_2 \subset C(L)\) = \(\Rightarrow X\) well-defined and extremely non-complex.

\(K \setminus L\) connected = \(\Rightarrow \text{Iso}(X) = \{-\text{Id}, +\text{Id}\}\).

\(X^* = \ell_2 \oplus_1 C_0(K\|L)^*,\) so \(\text{Iso}(\ell_2) \subset \text{Iso}(X^*)\).

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\[ \exists \mathcal{K} \text{ weak Koszmider space such that } \mathcal{K} \setminus F \text{ is connected if } |F| < \infty. \]

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There is \( \mathcal{L} \subset \mathcal{K} \) closed nowhere dense with

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Proof.
- \( \mathcal{K} \) weak Koszmider, \( \mathcal{L} \) nowhere dense, \( \ell_2 \subset C(\mathcal{L}) \)
  \( \implies \) \( X \) well-defined and extremely non-complex.
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- \(\mathcal{K} \setminus \mathcal{L}\) connected \(\implies\) \(\text{Iso}(X) = \{-\text{Id}, +\text{Id}\}\).
- \(X^* = \ell_2 \oplus_1 C_{0}(\mathcal{K}||\mathcal{L})^*\), so \(\text{Iso}(\ell_2) \subset \text{Iso}(X^*)\). \(\checkmark\)
Open questions on extremely non-complex Banach spaces

- Does $X$ have the Daugavet property?
- Stronger: Does $Y$ have the Daugavet property if $\| \text{Id} + T \|_2 = 1 + \| T \|_2$ for every rank-one $T \in L(Y)$?
- Is it true that $\mathcal{N}(X) = 1$?
- We actually know that $\mathcal{N}(X) \geq C > 0$.
- Is $\text{Iso}(X) \equiv \text{Unc}(X)$ a Boolean algebra?
- If $Y \leq X$ is 1-codimensional, is $Y$ extremely non complex?
- Is it possible that $X \simeq Z \oplus Z \oplus Z$?
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X extremely non complex
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That's all Folks!
Schedule of the talk

1. Basic notation
2. Numerical range of operators
3. Two results on surjective isometries
4. Numerical index of Banach spaces
5. The alternative Daugavet property
6. Lush spaces
7. Slicely countably determined spaces
8. Remarks on the containment of $c_0$ and $\ell_1$
9. Numerical index of $L_p$-spaces
10. Extremely non-complex Banach spaces