On the numerical radius of operators
on Lebesgue spaces

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**Notation**

**Basic notation**

- $X$ Banach space.
- $\mathbb{K}$ base field (it may be $\mathbb{R}$ or $\mathbb{C}$),
- $S_X$ unit sphere, $B_X$ unit ball,
- $X^*$ dual space,
- $L(X)$ bounded linear operators,
- $T^* \in L(X^*)$ adjoint operator of $T \in L(X)$.
- For $z \in \mathbb{K}$,
  - $\bar{z}$ is the conjugate ($= z$ in the real case),
  - $\text{Re} \ z$ is the real part ($= z$ in the real case).
- $L_p(\mu)$ (real or complex) Banach space of $\mu$-measurable scalar functions with
  \[
  \|x\|_p := \left( \int_{\Omega} |x|^p \, d\mu \right)^{\frac{1}{p}} < \infty
  \]
- $\mathbb{\ell}_p^{(m)}$ $m$-dimensional $L_p$-space
The results

The result for the absolute numerical radius

For $1 < p < \infty$, let $\kappa_p = p\left(\frac{1}{p} - \frac{1}{q}\right)$. Then

$$\sup \left\{ \int |x|^{p-1} |Tx| \, d\mu : x \in L_p(\mu), \|x\|_p = 1 \right\} \geq \kappa_p \|T\|$$

for every $T \in L(L_p(\mu))$ (real and complex cases).

- This inequality is best possible when $\dim(L_p(\mu)) \geq 2$.

Positivity of the numerical index

For $1 < p < \infty$, $p \neq 2$, there is a positive constant $n(L_p)$ such that

$$\sup \left\{ \left| \int |x|^{p-1} \text{sign}(x) Tx \, d\mu \right| : x \in L_p(\mu), \|x\| = 1 \right\} \geq n(L_p) \|T\|$$

for every $T \in L(L_p(\mu))$ (real case).

- We do not know the best possibility for $n(L_p)$.
Schedule of the talk

1. Introduction
2. Numerical index of Banach spaces
3. The 2000's results on the numerical index on $L_p$-spaces
4. The new results on the numerical index of $L_p$-spaces
Numerical index of Banach spaces

- Numerical range
- Numerical index: definition and basic properties
- Examples
- Stability properties
- Duality
- Renorming
- Open problems

F. F. Bonsall and J. Duncan
*Numerical Ranges. Vol I and II.*

V. Kadets, M. Martín, and R. Payá.
Recent progress and open questions on the numerical index of Banach spaces.
*RACSAM* (2006)
**Numerical range: Hilbert spaces**

### Hilbert space numerical range (Toeplitz, 1918)

- A $n \times n$ real or complex matrix
  \[
  W(A) = \{(Ax \mid x) : x \in \mathbb{K}^n, (x \mid x) = 1\}.
  \]
- $H$ real or complex Hilbert space, $T \in L(H)$,
  \[
  W(T) = \{(Tx \mid x) : x \in H, \|x\| = 1\}.
  \]

### Some properties

- $H$ Hilbert space, $T \in L(H)$:
  - $W(T)$ is convex.
  - In the complex case, $\overline{W(T)}$ contains the spectrum of $T$.
  - If $T$ is normal, then $\overline{W(T)} = \overline{\text{co}	ext{Sp}(T)}$. 
Numerical range: Banach spaces

Banach spaces numerical range (Bauer 1962; Lumer, 1961)

Let $X$ be a Banach space, $T \in L(X)$, then

$$V(T) = \left\{ x^*(Tx) : x^* \in S_{X^*}, \ x \in S_X, \ x^*(x) = 1 \right\}$$

Some properties

Let $X$ be a Banach space, $T \in L(X)$:

- $V(T)$ is connected (not necessarily convex).
- In the complex case, $\overline{V(T)}$ contains the spectrum of $T$.
- In fact,

$$\overline{\text{coSp}(T)} = \bigcap V(T),$$

intersection taken over all numerical ranges $V(T)$ corresponding to equivalent norms on $X$. 
Some motivations for the numerical range

**For Hilbert spaces**
- It is a comfortable way to study the spectrum.
- It is useful to work with some concepts like hermitian operator, skew-hermitian operator, dissipative operator...
- It is useful to estimate spectral radii of small perturbations of matrices.

**For Banach spaces**
- It allows to carry to the general case the concepts of hermitian operator, skew-hermitian operator, dissipative operators...
- It gives a description of the Lie algebra corresponding to the Lie group of all onto isometries on the space.
- It gives an easy and quantitative proof of the fact that Id is an strongly extreme point of $B_{L(X)}$ (MLUR point).
Numerical index of Banach spaces: definitions

**Numerical radius**

$X$ Banach space, $T \in L(X)$. The numerical radius of $T$ is

$$v(T) = \sup \{ |x^*(Tx)| : x^* \in S_{X^*}, x \in S_X, x^*(x) = 1 \}$$

**Remark**

The numerical radius is a continuous seminorm in $L(X)$. Actually, $v(\cdot) \leq \|\cdot\|$.

**Exponential formula**

$$\|\exp(\rho T)\| \leq e^{\rho v(T)} \quad (\rho \in \mathbb{K})$$ and $v(T)$ is best possible.

**Numerical index (Lumer, 1968)**

$X$ Banach space, the numerical index of $X$ is

$$n(X) = \inf \{ v(T) : T \in L(X), \|T\| = 1 \}$$

$$= \max \{ k \geq 0 : k\|T\| \leq v(T) \quad \forall \ T \in L(X) \}$$
Some basic properties

- \( n(X) = 1 \) iff \( v \) and \( \| \cdot \| \) coincide.
- \( n(X) = 0 \) iff \( v \) is not an equivalent norm in \( L(X) \).
- If \( X \) is complex, then \( n(X) \geq 1/e \).
  (Bohnenblust–Karlin, 1955; Glickfeld, 1970)

- Actually,

\[
\{ n(X) : X \text{ complex, } \dim(X) = 2 \} = [e^{-1}, 1] \\
\{ n(X) : X \text{ real, } \dim(X) = 2 \} = [0, 1]
\]
  (Duncan–McGregor–Pryce–White, 1970)
Numerical index of Banach spaces: some examples

Examples

1. $H$ Hilbert space, $\dim(H) > 1$,
   
   \[ n(H) = 0 \quad \text{if } H \text{ is real} \]
   \[ n(H) = 1/2 \quad \text{if } H \text{ is complex} \]

2. \[ n\left( L_1(\mu) \right) = 1 \quad \mu \text{ positive measure} \]
   \[ n\left( C(K) \right) = 1 \quad K \text{ compact Hausdorff space} \]
   (Duncan et al., 1970)

3. If $A$ is a $C^*$-algebra \[ \Rightarrow \begin{cases} n(A) = 1 & A \text{ commutative} \\ n(A) = 1/2 & A \text{ not commutative} \end{cases} \]
   (Huruya, 1977; Kaidi–Morales–Rodríguez, 2000)

4. If $A$ is a function algebra \[ \Rightarrow n(A) = 1 \]
   (Werner, 1997)
More examples

5. For \( n \geq 2 \), the unit ball of \( X_n \) is a \( 2n \) regular polygon:

\[
n(X_n) = \begin{cases} 
\tan \left( \frac{\pi}{2n} \right) & \text{if } n \text{ is even}, \\
\sin \left( \frac{\pi}{2n} \right) & \text{if } n \text{ is odd}.
\end{cases}
\]

(M.–Merí, 2007)

6. Every finite-codimensional subspace of \( C[0,1] \) has numerical index 1

(Boyko–Kadets–M.–Werner, 2007)
Stability properties

Direct sums of Banach spaces (M.–Payá, 2000)

\[ n\left(\bigoplus_{\lambda \in \Lambda} X_{\lambda}\right)_{c_0} = n\left(\bigoplus_{\lambda \in \Lambda} X_{\lambda}\right)_{\ell_1} = n\left(\bigoplus_{\lambda \in \Lambda} X_{\lambda}\right)_{\ell_\infty} = \inf_{\lambda} n(X_{\lambda}) \]

Vector-valued function spaces (López–M.–Merí–Payá–Villena, 200’s)

Let \( E \) be a Banach space, \( \mu \) a positive measure, and \( K \) a compact space. Then

\[ n(C(K,E)) = n(C_w(K,E)) = n(L_1(\mu,E)) = n(L_\infty(\mu,E)) = n(E), \]

and \( n(C_w^*(K,E^*)) \leq n(E) \)

Tensor products (Lima, 1980)

There is no general formula neither for \( n(X \tilde{\otimes}_\varepsilon Y) \) nor for \( n(X \tilde{\otimes}_\pi Y) \):

- \( n\left(\ell_1^{(4)} \tilde{\otimes}_\pi \ell_1^{(4)}\right) = n\left(\ell_{\infty}^{(4)} \tilde{\otimes}_\varepsilon \ell_{\infty}^{(4)}\right) = 1. \)
- \( n\left(\ell_1^{(4)} \tilde{\otimes}_\varepsilon \ell_1^{(4)}\right) = n\left(\ell_{\infty}^{(4)} \tilde{\otimes}_\pi \ell_{\infty}^{(4)}\right) < 1. \)
Numerical index and duality

**Proposition**

Let $X$ be a Banach space, $T \in L(X)$. Then

- $\sup \text{Re} \ V(T) = \lim_{\alpha \to 0^+} \frac{\| \text{Id} + \alpha T \| - 1}{\alpha}$.
- $\nu(T^*) = \nu(T)$ for every $T \in L(X)$.
- Therefore, $n(X^*) \leq n(X)$.

(Duncan–McGregor–Pryce–White, 1970)

**Question (From the 1970’s)**

Is $n(X) = n(X^*)$?

**Negative answer (Boyko–Kadets–M.–Werner, 2007)**

Consider the space

$$X = \left\{ (x, y, z) \in c \oplus c \oplus c : \lim x + \lim y + \lim z = 0 \right\}.$$ 

Then, $n(X) = 1$ but $n(X^*) < 1$. 


Numerical index and renorming

Proposition (Finet–M.–Payá, 2003)

\( X \) separable or reflexive, then

\[
\{ n(Y) : Y \cong X \} \supseteq \begin{cases} [0,1] & \text{real case} \\ [e^{-1},1] & \text{complex case} \end{cases}
\]


\( X \) real Banach space, \( n(X) = 1 \), \( \dim(X) = \infty \), \( \implies X^* \supset \ell_1 \).

Proposition (Boyko–Kadets–M.–Merí, 2009)

\( X \) separable, \( X \supset c_0 \) \( \implies \exists Y \cong X \) with \( n(Y) = 1 \).
Some interesting open problems

Open problems

1. Characterize (without operators) Banach spaces with numerical index 1.
2. $X$ with $n(X) = 1, \dim(X) = \infty$ $X \supset c_0$ or $X \supset \ell_1$?
3. Characterize those $X$ admitting a renorming with numerical index 1.
4. For instance, if $X \supset c_0$ or $\supset \ell_1$ can $X$ be renormed with numerical index 1?
5. Find isomorphic or isometric conditions assuring that $n(X) = n(X^*)$.

The oldest open problem

Calculate the numerical index of “classical” spaces.
- In particular, calculate $n(L_p(\mu))$. 
The 2000’s results on the numerical index on $L_p$-spaces

E. Ed-dari.
On the numerical index of Banach spaces.
Linear Algebra Appl. (2005)

E. Ed-dari and M. Khamsi.
The numerical index of the $L_p$ space.

E. Ed-dari, M. Khamsi, and A. Aksoy.
On the numerical index of vector-valued function spaces.
Linear Mult. Algebra (2007)

M. Martín, and J. Merí.
A note on the numerical index of the $L_p$-space of dimension two.
Linear Mult. Algebra (2009)

M. Martín, J. Merí, M. Popov, and B. Randrianantoanina.
Numerical index of absolute sums of Banach spaces.
**Numerical index of $L_p$-spaces**

### Known results on the numerical index of $L_p$-spaces

1. $n(\ell_p) \leq n(\ell_p^{(m+1)}) \leq n(\ell_p^{(m)})$ for $m \in \mathbb{N}$.
   (M.–Payá, 2000)

2. $n(L_p[0,1]) = n(\ell_p) = \lim_{m \to \infty} n(\ell_p^{(m)}) = \inf_{m \in \mathbb{N}} n(\ell_p^{(m)})$.

3. In the real case,
   $$\max \left\{ \frac{1}{2^{1/p}}, \frac{1}{2^{1/q}} \right\} v \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leq n(\ell_p^{(2)}) \leq v \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
   and
   $$v \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \max_{t \in [0,1]} \frac{|t^{p-1} - t|}{1 + t^p}$$
   (M.–Merí, 2009)
Ideas behind the proofs I

The numerical index decreases with the dimension

\[ n(\ell_p) \leq n(\ell_p^{(m+1)}) \leq n(\ell_p^{(m)}) \text{ for } m \in \mathbb{N}. \]

Proposition (M.–Payá, 2000)

\[ Z = U \oplus V \text{ with absolute sum (i.e. } \|u + v\| = f(\|u\|, \|v\|) \text{ for } u \in U, \ v \in V). \]
\[ \implies n(Z) \leq \min\{n(U), n(V)\}. \]

Proof of the decreasing

- \(\ell_p^{(m)}\) is an absolute summand in both \(\ell_p^{(m+1)}\) and in \(\ell_p\).
Ideas behind the proofs II

One inequality

\[ n(L_p[0,1]) \leq \lim_{m \to \infty} n(\ell_p^m). \]

Proposition (M.–Merí–Popov–Randrianantoanina, 2011)

\( E \) order continuous Köthe space, \( X \) Banach space

\[ \implies n(E(X)) \leq n(X). \]

Proof of the inequality

- \( E = L_p[0,1], \ X = \ell_p^m. \)
- \( E \equiv E(X) \) so \( n(E) \leq n(\ell_p^m). \)
Ideas behind the proofs III

The reversed inequality

\[ n(L_p[0,1]) \geq \lim_{m \to \infty} n(\ell_p^m) \quad \text{and} \quad n(\ell_p) \geq \lim_{m \to \infty} n(\ell_p^m). \]

Proposition (M.–Merí–Popov–Randrianantoanina, 2011)

\(Z\) Banach space, \(\{Z_i\}_{i \in I}\) increasing family of one-complemented subspaces whose union is dense. Then, \(\Rightarrow n(Z) \geq \limsup_{i \in I} n(Z_i).\)

Corollary

\(Z\) Banach space with monotone basis \((e_m)\), \(Z_m = \text{span}\{e_k : 1 \leq k \leq m\}\). \(\Rightarrow n(Z) \geq \limsup_{m \to \infty} n(Z_m).\)

Proof of the inequality

- \(Z = \ell_p, (e_m)\) canonical basis \(\Rightarrow Z_m \equiv \ell_p^m\) for all \(m \in \mathbb{N}\).
- \(E = L_p[0,1], (e_m)\) Haar system \(\Rightarrow Z_m \equiv \ell_p^m\) for \(m = 2^k\) \((k \in \mathbb{N})\).
The two-dimensional case

In the real case,

\[
\max \left\{ \frac{1}{2^{1/p}}, \frac{1}{2^{1/q}} \right\} M_p \leq n(\ell_p^{(2)}) \leq M_p \quad \text{where} \quad M_p = \max_{t \in [0,1]} \frac{|t^{p-1}-t|}{1+t^p}
\]

Proposition (Duncan-McGregor-Pryce-White, 1970)

\[
T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
operator in \(\ell_p^{(2)}\). Then

\[
v(T) = \max \left\{ \max_{t \in [0,1]} \frac{|a + dt^p| + |bt + ct^{p-1}|}{1+t^p}, \max_{t \in [0,1]} \frac{|d + at^p| + |ct + bt^{p-1}|}{1+t^p} \right\}.
\]

Proof of the result

- \(n(\ell_p^{(2)}) \leq M_p\) since \(\left\| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\| = 1\) and \(v\left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = M_p\).

- We compare \(v(T)\) with \(M_p\), but we use \(\|T\|_1\) and \(\|T\|_\infty\) instead of \(\|T\|_p\).
Questions

1. Is $n(\ell_p^{(m+1)}) = n(\ell_p^{(m)})$ for $m \geq 2$ ?
2. In the real case, is $n(L_p[0,1])$ positive ?
3. We do not have results for the complex case, even for dimension two.

The 2010's results

- We left the finite-dimensional approach and introduce the **absolute numerical radius**.
- This allows to show that $n(L_p[0,1]) > 0$ in the real case.
The new results on the numerical index of $L_p$-spaces

M. Martín, J. Merí, M. Popov.
On the numerical index of real $L_p(\mu)$-spaces.

M. Martín, J. Merí, M. Popov.
On the numerical radius of operators in Lebesgue spaces.
J. Funct. Anal. (to appear)
The absolute numerical radius in $L_p$

**The numerical radius in $L_p$**

- For $x \in L_p(\mu)$, write $x^\# = |x|^{p-1} \text{sign}(x)$.
- It is the unique element in $L_q(\mu)$ such that
  
  $$\|x\|_p^p = \|x^\#\|_q^q \quad \text{and} \quad \int x x^\# d\mu = \|x\|_p \|x^\#\|_q = \|x\|_p^p.$$  
- Therefore, for $T \in L(L_p(\mu))$ one has
  
  $$v(T) = \sup \left\{ \left\| \int x^\# T x \, d\mu \right\| : x \in L_p(\mu), \|x\|_p = 1 \right\}$$  
  
  $$= \sup \left\{ \left\| \int |x|^{p-1} \text{sign}(x) T x \, d\mu \right\| : x \in L_p(\mu), \|x\|_p = 1 \right\}.$$  

**Absolute numerical radius**

For $T \in L(L_p(\mu))$,

$$|v|(T) := \sup \left\{ \int |x^\# T x| \, d\mu : x \in L_p(\mu), \|x\|_p = 1 \right\}$$  

$$= \sup \left\{ \int |x|^{p-1} |T x| \, d\mu : x \in L_p(\mu), \|x\|_p = 1 \right\}.$$
The absolute numerical index of $L_p$

**Obvious remark**

$$v(T) \leq |v|(T) \leq \|T\| \text{ for every } T \in L(L_p(\mu)).$$

**Absolute numerical index**

$$|n|(L_p(\mu)) = \inf \left\{ |v|(T) : T \in L(L_p(\mu)), \|T\| = 1 \right\}$$

$$= \max \left\{ k \geq 0 : k\|T\| \leq |v|(T) \quad \forall T \in L(L_p(\mu)) \right\}.$$ 

- $n(L_p(\mu))$ is the greatest constant $M \geq 0$ such that

  $$\sup \left\{ \left| \int |x|^{p-1} \text{sign}(\bar{x})Tx \, d\mu \right| : x \in L_p(\mu), \|x\|_p = 1 \right\} \geq M \|T\|$$

  for every $T \in L(L_p(\mu)).$

- $|n|(L_p(\mu))$ is the greatest constant $K \geq 0$ such that

  $$\sup \left\{ \left| \int |x|^{p-1}|Tx| \, d\mu \right| : x \in L_p(\mu), \|x\|_p = 1 \right\} \geq K \|T\|$$

  for every $T \in L(L_p(\mu)).$
The first results

**Proposition 1**

Write $M_p = \max_{t \in [0,1]} \frac{|t^{p-1} - t|}{1 + t^p}$. Then, in the real case,

$$v(T) \geq \frac{M_p}{6} |v|(T) \quad (T \in L(L_p(\mu))).$$

**Proposition 2**

In the real case,

$$|v|(T) \geq \frac{1}{2} v(T_C) \geq \frac{n(L_p^C(\mu))}{2} \|T\| \quad (T \in L(L_p(\mu))).$$

We do not know the value of $n(L_p^C(\mu))$, but $n(X) \geq 1/e$ for complex spaces, so

**Theorem**

In the real case, $n(L_p(\mu)) \geq \frac{M_p}{12e} > 0$ for $1 < p < \infty$, $p \neq 2$.

We improved **Proposition 2** calculating $|n|(L_p(\mu))$ for real and complex spaces.
Calculating $|n|(L_p(\mu))$  

The constant

Set $\kappa_p := \max_{\tau > 0} \frac{\tau^{p-1}}{1 + \tau^p} = \max_{\lambda \in [0,1]} \frac{1}{q} \left(1 - \lambda\right)^{-\frac{1}{p}} = \frac{1}{p^{1/p} q^{1/q}}$.

The best possibility for $|n|(L_p(\mu))$

If $\dim(L_p(\mu)) \geq 2$, then there is a (positive) operator $T \in L(L_p(\mu))$ with

$$\|T\| = 1, \quad |v|(T) = \kappa_p.$$ 

The examples for $\ell_p$ and $L_p[0,1]$:

- For $\ell_p$: consider the extension by zero of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
- For $L_p[0,1]$:

$$Tf = 2 \left[ \int_0^{1/2} f(s) \, ds \right] \chi_{[\frac{1}{2},1]} \quad (f \in L_p[0,1]).$$
Calculating $|n|(L_p(\mu)) \geq \kappa_p$

**Proof for positive operators:**

- Fix $T \in L(L_p(\mu))$ positive with $\|T\| = 1$, $\tau > 0$ and $\varepsilon > 0$.
- Find $x \geq 0$ with $\|x\| = 1$ and $\|Tx\|^p > 1 - \varepsilon$, set
  
  $$y = x \lor \tau Tx \quad \text{and} \quad A = \{\omega \in \Omega : x(\omega) \geq \tau(Tx)(\omega)\},$$

  and observe that

  $$\|y\|^p = \int_A x^p \, d\mu + \int_{\Omega \setminus A} (\tau Tx)^p \, d\mu \leq 1 + \tau^p \quad \text{and} \quad y^\# = x^{p-1} \lor (\tau Tx)^{p-1}.$$

- Now,

  $$|v|(T) \geq \frac{1}{\|y\|^p} \int_{\Omega} y^\# Ty \, d\mu \geq \frac{1}{1 + \tau^p} \int_{\Omega} y^\# Ty \, d\mu \geq \frac{1}{1 + \tau^p} \int_{\Omega} (\tau Tx)^{p-1} Tx \, d\mu = \frac{\tau^{p-1}}{1 + \tau^p} \int_{\Omega} (Tx)^p \, d\mu \geq \frac{\tau^{p-1}}{1 + \tau^p} (1 - \varepsilon).$$

- Taking supremum on $\tau > 0$ and $\varepsilon > 0$, we get $|v|(T) \geq \kappa_p$. 
One consequence and further results

**Corollary**

\[ n(L_p(\mu)) \geq \frac{M_p \kappa_p}{6} \text{ in the real case.} \]

**More results**

- If \( T \in L(L_p[0,1]) \) is rank-one \( \implies v(T) \geq \kappa_p^2 \|T\|. \)
- If \( T \in L(L_p[0,1]) \) is compact, then
  \[
  v(T) \geq \kappa_p^2 \|T\| \text{ (complex case), } \quad v(T) \geq \max_{\tau > 0} \frac{\kappa_p \tau^{p-1} - \tau}{1 + \tau^p} \|T\| \text{ (real case).}
  \]

**Open problems with conjectures**

- Is \( n(L_p(\mu)) = M_p \) (dim \( \geq 2 \)) in the real case?
  - It is enough to prove that \( n(L_p[0,1]) \geq M_p \) or \( n(\ell_p) \geq M_p. \)
- Is \( n(L_p(\mu)) = \kappa_p \) (dim \( \geq 2 \)) in the complex case?
  - It is enough to prove that \( n(L_p[0,1]) \geq \kappa_p \) or \( n(\ell_p) \geq \kappa_p. \)