Numerical index theory

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Advanced Training School in Mathematics

Workshop on Geometry of Banach spaces and its Applications

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Schedule of the talk

1. Basic notation
2. Numerical range of operators
3. Two results on surjective isometries
4. Numerical index of Banach spaces
5. The alternative Daugavet property
6. Lush spaces
7. Slicely countably determined spaces
8. Remarks on two recent results
9. Extremely non-complex Banach spaces
Basic notation

- $K$ base field ($\mathbb{R}$ or $\mathbb{C}$):
  - $T$ modulus-one scalars,
  - $\text{Re} z$ real part of $z$ ($\text{Re} z = z$ if $K = \mathbb{R}$).
- $H$ Hilbert space: $\langle \cdot | \cdot \rangle$ denotes the inner product.
- $X$ Banach space:
  - $S_X$ unit sphere, $B_X$ unit ball,
  - $X^*$ dual space,
  - $L(X)$ bounded linear operators,
  - $W(X)$ weakly compact linear operators,
  - $\text{Iso}(X)$ surjective linear isometries,
- $X$ Banach space, $T \in L(X)$:
  - $\text{Sp}(T)$ spectrum of $T$.
  - $T^* \in L(X^*)$ adjoint operator of $T$. 

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**Basic notation**

X Banach space, $B \subset X$, $C$ convex subset of $X$:

- $B$ is *rounded* if $TB = B$,
- $\text{co}(B)$ convex hull of $B$,
- $\text{co}(B)$ closed convex hull of $B$,
- $\text{aconv}(B) = \text{co}(T B)$ absolutely convex hull of $B$,
- $\text{ext}(C)$ extreme points of $C$,
- *slice* of $C$:

$$S(C, x^*, \alpha) = \{ x \in C : \text{Re} x^*(x) > \sup \text{Re} x^*(C) - \alpha \}$$

where $x^* \in X^*$ and $0 < \alpha < \sup \text{Re} x^*(C)$. 
Numerical range of operators

- Definitions and first properties
- The exponential function
- Numerical ranges and isometries

F. F. Bonsall and J. Duncan

_numerical ranges. Vol I and II._

Numerical range: Hilbert spaces

Hilbert space numerical range (Toeplitz, 1918)

- A $n \times n$ real or complex matrix
  
  $$W(A) = \{ (Ax \mid x) : x \in \mathbb{K}^n, (x \mid x) = 1 \}.$$  

- $H$ real or complex Hilbert space, $T \in L(H)$,
  
  $$W(T) = \{ (Tx \mid x) : x \in H, \|x\| = 1 \}.$$
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**Remark**

- Given $T \in L(H)$ we associate
  - a sesquilinear form $\varphi_T(x, y) = (Tx \mid y)$ ($x, y \in H$),
  - a quadratic form $\widehat{\varphi}_T(x) = \varphi_T(x, x) = (Tx \mid x)$ ($x \in H$).

- Then, $W(T) = \widehat{\varphi}_T(S_H)$. 

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- Then, \( W(T) = \widehat{\varphi}_T(S_H) \). Therefore:
  - \( \widehat{\varphi}_T(B_H) = [0, 1]W(T) \),
  - \( \widehat{\varphi}_T(H) = \mathbb{R}^+W(T) \).
  - But we cannot get $W(T)$ from $\widehat{\varphi}_T(B_H)$!

Some properties

Hilbert space, $T \in L(H)$:

- $(\text{Toeplitz-Hausdorff})$ $W(T)$ is convex.

- For $T, S \in L(H)$, $\alpha, \beta \in K$:
  - $W(\alpha T + \beta S) \subseteq \alpha W(T) + \beta W(S)$;
  - $W(\alpha \text{Id} + S) = \alpha + W(S)$.

- $W(U^*TU) = W(T)$ for every $T \in L(H)$ and every $U$ unitary.

- $\text{Sp}(T) \subseteq W(T)$.

- If $T$ is normal, then $W(T) = \text{co} \text{Sp}(T)$.

- In the real case ($\dim(H) > 1$), there is $T \in L(H)$, $T \neq 0$ with $W(T) = \{0\}$.

- In the complex case, $\sup \{|(Tx)| : x \in \mathcal{S}H\} \geq \frac{1}{2} \|T\|$. If $T$ is actually self-adjoint, then $\sup \{|(Tx)| : x \in \mathcal{S}H\} = \|T\|$.
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$H$ complex Hilbert space, $T \in L(H)$, then

$$M := \sup\{(Tx \mid x) : x \in S_H\} \geq \frac{1}{2} \|T\|.$$
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For \( x, y \in S_H \) fixed, use the polarization formula:

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(Tx \mid y) = \frac{1}{4} \left[ (T(x + y) \mid x + y) - (T(x - y) \mid x - y) \right. \\
\left. + \ i \ (T(x + iy) \mid x + iy) - i \ (T(x - iy) \mid x - iy) \right].
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- $| (Tx \mid y) | \leq \frac{1}{4} M [ \|x + y\|^2 + \|x - y\|^2 + \|x + iy\|^2 + \|x - iy\|^2 ]$.

- By the parallelogram’s law:

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| (Tx \mid y) | \leq \frac{1}{4} M [ 2\|x\|^2 + 2\|y\|^2 + 2\|x\|^2 + 2\|iy\|^2 ] = 2M.
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- We just take supremum on $x, y \in S_H$. 

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Some reasons to study numerical ranges

- It gives a "picture" of the matrix/operator which allows to "see" many properties (algebraic or geometrical) of the matrix/operator.
- It is a comfortable way to study the spectrum.
- It is useful to estimate spectral radii of small perturbations of matrices.
- It is useful to work with some concepts like hermitian operator, skew-hermitian operator, dissipative operator, etc.

Example

Consider

\[ A = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \] and

\[ B = \begin{pmatrix} 0 & 0 \\ \varepsilon & 0 \end{pmatrix} \].

\[ \text{Sp}(A) = \{0\}, \quad \text{Sp}(B) = \{0\}. \]

\[ \text{Sp}(A + B) = \{\pm \sqrt{M \varepsilon}\} \subseteq W(A + B) \subseteq W(A) + W(B), \]

so the spectral radius of \( A + B \) is bounded above by \( \frac{1}{2} (|M| + |\varepsilon|) \).

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Numerical range of operators  Definitions and first properties

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### Numerical range: Banach spaces (I)

**Banach spaces numerical range (Bauer 1962; Lumer, 1961)**

Let $X$ be a Banach space, $T \in L(X)$,

$$V(T) = \{ x^*(Tx) : x^* \in S_{X^*}, \ x \in S_X, \ x^*(x) = 1 \}$$
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- (Zenger–Crabb) Actually, \( \overline{\text{co}}(\text{Sp}(T)) \subseteq \overline{V(T)}. \)

- \( \overline{\text{co}}\text{Sp}(T) = \bigcap\{V_p(T) : p \text{ equivalent norm}\} \) where \( V_p(T) \) is the numerical range of \( T \) in the Banach space \( (X,p) \).
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- $V(U^{-1}TU) = V(T)$ for every $T \in L(X)$ and every $U \in \text{Iso}(X)$.
- $V(T) \subseteq V(T^*) \subseteq \overline{V(T)}$. 
Observation

The numerical range depends on the base field:

- $X$ complex Banach space $\Rightarrow X_{\mathbb{R}}$ real space underlying $X$.
- $T \in \mathcal{L}(X) = \Rightarrow T_{\mathbb{R}} \in \mathcal{L}(X_{\mathbb{R}})$ is $T$ view as a real operator.

Then $V(T_{\mathbb{R}}) = \text{Re} V(T)$.

Consequence:

- $X$ complex, then there is $S \in \mathcal{L}(X_{\mathbb{R}})$ with $\|S\| = 1$ and $V(S) = \{0\}$.

Some motivation for the numerical range

It allows to carry to the general case the concepts of hermitian operator, skew-hermitian operator, dissipative operators. . .

It gives a description of the Lie algebra corresponding to the Lie group of all onto isometries on the space.

It gives an easy and quantitative proof of the fact that $\text{Id}$ is an strongly extreme point of $\mathcal{B}\mathcal{L}(X)$ (MLUR point).
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Some motivation for the numerical range

It allows to carry to the general case the concepts of hermitian operator, skew-hermitian operator, dissipative operators...

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Numerical radius: definition and properties
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**Numerical radius**

Let $X$ be a real or complex Banach space, $T \in L(X)$,

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v(T) = \sup \{ |\lambda| : \lambda \in V(T) \}
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= \sup \{ |x^*(Tx)| : x^* \in S_{X^*}, x \in S_X, x^*(x) = 1 \}
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Let $X$ be a Banach space, $T \in L(X)$

- $v(\cdot)$ is a seminorm, i.e.
  - $v(T + S) \leq v(T) + v(S)$ for every $T, S \in L(X)$. 
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Numerical radius: definition and properties

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In a real or complex Banach space $X$, let $T \in \mathcal{L}(X)$.

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Numerical radius: examples
Numerical range of operators Definitions and first properties

Numerical radius: examples

Some examples

1. $H$ real Hilbert space $\dim(H) > 1$ implies that there exists $T \in L(X)$ with $v(T) = 0$ and $\|T\| = 1$. 

2. $X = L^1(\mu) \Rightarrow v(T) = \|T\|$ for every $T \in L(X)$. 

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In particular, this is the case for $X = C(K)$. 

Miguel Martín (University of Granada (Spain)) Numerical index theory Bangalore, June 2009 13 / 136
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Proving a result

\[ X = C(K) \implies \nu(T) = \|T\| \text{ for every } T \in L(X). \]
Fix $T \in L(C(K))$. Find $f_0 \in X(E)$ and $\xi_0 \in K$ such that $|Tf_0(\xi_0)| \sim \|T\|$. 

$X = C(K) \implies v(T) = \|T\|$ for every $T \in L(X)$. 

If $f_0(\xi_0) \sim 1$, then we were done. This is our goal. Consider the non-empty open set $V = \{\xi \in \mathbb{R} : f_0(\xi) \sim f_0(\xi_0)\}$ and find $\varphi : [0,1] \times [0,1] \to [0,1]$ continuous with $\text{supp}(\varphi) \subset V$ and $\varphi(\xi_0) = 1$.

Write $f_0(\xi_0) = \lambda \omega_1 + (1 - \lambda) \omega_2$ with $|\omega_i| = 1$, and consider the functions $f_i = (1 - \varphi) f_0 + \varphi \omega_i$ for $i = 1, 2$.

Then, $f_i \in C(K)$, $\|f_i\| \leq 1$, and $\|f_0 - (\lambda f_1 + (1 - \lambda) f_2)\| \sim 0$. Therefore, there is $i \in \{1, 2\}$ such that $|f_i(\xi_0)| \sim \|T\|$, but now $|f_i(\xi_0)| = 1$.

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If \( X = L_1(\mu) \), then \( X^* \equiv C(K_\mu) \). Therefore, \( v(T) = v(T^*) = \|T^*\| = \|T\|. \)
Numerical radius: real and complex spaces

Example

$X$ complex Banach space, define $T \in L(X) \mapsto \mathbb{R}$ by $T(x) = ix$ ($x \in X$).

$\|T\| = 1$ and $v(T) = 0$ if viewed in $X \mathbb{R}$.

$\|T\| = 1$ and $V(T) = \{i\}$, so $v(T) = 1$ if viewed in (complex) $X$.

Theorem (Bohnenblust-Karlin; Glickfeld)

$X$ complex Banach space, $T \in L(X)$:

$v(T) \geq 1 e^{\|T\|}$.

The constant $1 e^{\|T\|}$ is optimal: $\exists X$ two-dimensional complex, $\exists T \in L(X)$ with $\|T\| = e$ and $v(T) = 1$.
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$X$ complex Banach space, define $T \in L(X_{\mathbb{R}})$ by

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Numerical radius: real and complex spaces
Numerical index: definition and properties

**Numerical index**

\[ n(X) = \max \left\{ k \geq 0 : \|K\|T\| \leq v(T) \quad \forall T \in L(X) \right\} = \inf \left\{ v(T) : T \in L(X), \|T\| = 1 \right\}. \]

**Elementary properties**

- In the real case, \( 0 \leq n(X) \leq 1 \).
- In the complex case, \( \frac{1}{e} \leq n(X) \leq 1 \).
- Actually, the above inequalities are best possible:
  \[ \left\{ n(X) : X \text{ complex Banach space} \right\} = \left[ e^{-1}, 1 \right], \left\{ n(X) : X \text{ real Banach space} \right\} = \left[ 0, 1 \right]. \]

**\( v \)** norm on \( L(X) \) equivalent to the given norm \( \iff \) \( n(X) > 0 \).

**\( v(T) = \|T\| \)** for every \( T \in L(X) \) \( \iff \) \( n(X) = 1 \).

\( n(X^*) \leq n(X) \).
Numerical index: definition and properties

Numerical index

\( X \) real or complex Banach space

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**Numerical index**

Given a real or complex Banach space $X$, the numerical index $n(X)$ is defined as:

$$n(X) = \max \{ k \geq 0 : K \|T\| \leq \nu(T) \quad \forall T \in L(X) \}$$

where $K$ is a constant, $\|T\|$ is the norm of $T$, and $\nu(T)$ is an additional norm on $L(X)$.

This definition can be equivalently written as:

$$n(X) = \inf \{ \nu(T) : T \in L(X), \|T\| = 1 \}.$$

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- $\nu$ norm on $L(X)$ equivalent to the given norm $\iff n(X) > 0$. 
Numerical index: definition and properties

**Numerical index**

Let $X$ be a real or complex Banach space.

$$n(X) = \max\{k \geq 0 : K \|T\| \leq \nu(T) \quad \forall T \in L(X)\}$$

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**Elementary properties**

Let $X$ be a Banach space.

- In the real case, $0 \leq n(X) \leq 1$.
- In the complex case, $1/e \leq n(X) \leq 1$.
- Actually, the above inequalities are best possible:

  $$\{n(X) : X \text{ complex Banach space}\} = [e^{-1}, 1],$$
  $$\{n(X) : X \text{ real Banach space}\} = [0, 1].$$

- $\nu$ norm on $L(X)$ equivalent to the given norm $\iff n(X) > 0$.
- $\nu(T) = \|T\|$ for every $T \in L(X) \iff n(X) = 1$. 
Numerical index:

### X real or complex Banach space

\[
n(X) = \max\{k \geq 0 : K \|T\| \leq v(T) \quad \forall T \in L(X)\} = \inf \{v(T) : T \in L(X), \|T\| = 1\}.
\]

### Elementary properties

- \(X\) Banach space.
  - In the real case, \(0 \leq n(X) \leq 1\).
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    \{n(X) : X \text{ complex Banach space}\} = [e^{-1}, 1],
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  - \(v\) norm on \(L(X)\) equivalent to the given norm \(\iff n(X) > 0\).
  - \(v(T) = \|T\|\) for every \(T \in L(X)\) \(\iff n(X) = 1\).
  - \(n(X^*) \leq n(X)\).
Numerical range of operators  
Definitions and first properties

Numerical index: examples

Some examples

1. Hilbert, \(\dim(H) > 1\):
   - \(n(H) = \begin{cases} 0 & \text{real case,} \\ 1 & \text{complex case.} \end{cases}\)

2. Complex space \(X\):
   - \(n(X_R) = 0\).

3. \(n(L_1(\mu)) = 1\), \(\mu\) positive measure.

4. \(X^* \equiv L_1(\mu) \Rightarrow n(X) = 1\).

5. In particular,
   - \(n(C(K)) = 1\),
   - \(n(C_0(L)) = 1\),
   - \(n(L_\infty(\mu)) = 1\).

6. \(n(A(D)) = 1\) and \(n(H_\infty) = 1\).
Some examples

1. $H$ Hilbert, $\dim(H) > 1$:

$$n(H) = \begin{cases} 
0 & \text{real case,} \\
\frac{1}{2} & \text{complex case.}
\end{cases}$$
### Some examples

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2. **$X$ complex space $\implies n(X_\mathbb{R}) = 0$.**
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Numerical index: examples

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The exponential function. Definition

\begin{equation}
\exp(T) = \sum_{n=0}^{\infty} \frac{T^n}{n!}
\end{equation}

where

\begin{align*}
T_0 &= \text{Id} \\
T_n &= T \circ \cdots \circ T
\end{align*}

It is well-defined since the series is absolutely convergent.

\[ \|\exp(T)\| \leq e \|T\| \]

We will improve this inequality in the sequel.
The exponential function.

Definition

For a Banach space $X$ and $T \in \mathcal{L}(X)$, the exponential function is defined as

$$\exp(T) = \sum_{n=0}^{\infty} \frac{1}{n!} T^n$$

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The exponential function

Let $X$ be a Banach space, $T \in L(X)$:

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The exponential function: properties

Let $X$ be a Banach space, $T, S \in \mathcal{L}(X)$. Then

\[ TS = ST \Rightarrow \exp(T + S) = \exp(T) \exp(S) \]

\[ \exp(T) \exp(-T) = \exp(0) = \text{Id} \Rightarrow \exp(T) \text{ is surjective isomorphism.} \]

The exponential formula

Let $X$ be a Banach space, $T \in \mathcal{L}(X)$:

\[ \sup \text{Re} V(T) = \sup \alpha > 0 \log \| \exp(\alpha T) \|_{\alpha} = \lim_{\alpha \to 0} \alpha \log \| \exp(\alpha T) \|_{\alpha}. \]

Consequence

Let $X$ be a Banach space, $T \in \mathcal{L}(X)$:

\[ \| \exp(\lambda T) \| \leq e |\lambda| v(T) (\lambda \in K) \]

$v(T)$ is the best possible constant.
Properties

\( X \) Banach space, \( T, S \in L(X) \).

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The exponential function: properties

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A motivating example

A real or complex $n \times n$ matrix. TFAE:

- $A$ is skew-adjoint (i.e. $A^* = -A$).

- $B = \exp(\rho A)$ is unitary for every $\rho \in \mathbb{R}$ (i.e. $B^*B = BB^* = \text{Id}$).
### A motivating example

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### In term of Hilbert spaces

$H$ ($n$-dimensional) Hilbert space, $T \in L(H)$. TFAE:

- $\operatorname{Re} W(T) = \{0\}$.
- $\exp(\rho T) \in \operatorname{Iso}(H)$ for every $\rho \in \mathbb{R}$. 
Semigroups of isometries: motivating example

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For general Banach spaces

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Semigroups of isometries: characterization

**Theorem (Bonsall-Duncan, 1970’s; Rosenthal, 1984)**

Let $X$ be a real or complex Banach space, $T \in L(X)$. TFAE:

- $\text{Re } V(T) = \{0\}$ (\textit{T} is \textbf{skew-hermitian}).
- $\|\exp(\rho T)\| \leq 1$ for every $\rho \in \mathbb{R}$.
- $\{ \exp(\rho T) : \rho \in \mathbb{R}_0^+ \} \subset \text{Iso}(X)$.
- $T$ belongs to the tangent space to $\text{Iso}(X)$ at $\text{Id}$.
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This follows from the exponential formula

$$\sup \text{Re} V(T) = \lim_{\beta \downarrow 0} \frac{\| \text{Id} + \beta T \| - 1}{\beta} = \sup_{\alpha > 0} \frac{\log \| \exp(\alpha T) \|}{\alpha}.$$
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Remark

If $X$ is complex, there always exists exponential one-parameter semigroups of surjective isometries:

$$ t \mapsto e^{it} \text{Id} \quad \text{generator: } i \text{Id}. $$
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Let $X$ be a real or complex Banach space, $T \in L(X)$. TFAE:

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**Main consequence**

If $X$ is a real Banach space such that $V(T) = \{0\}$, then $\text{Iso}(X)$ is “small”:

- it does not contain any exponential one-parameter semigroup,
- the tangent space of $\text{Iso}(X)$ at $\text{Id}$ is zero.
Two results on surjective isometries

- Isometries on finite-dimensional spaces
- Isometries and duality

M. Martín
The group of isometries of a Banach space and duality.

M. Martín, J. Merí, and A. Rodríguez-Palacios.
Finite-dimensional spaces with numerical index zero.

H. P. Rosenthal
The Lie algebra of a Banach space.
Isometries in finite-dimensional spaces

Theorem

\[ \text{Iso}(X) \text{ is infinite.} \]
\[ n(X) = 0. \]

There is \( T \in L(X) \), \( T \neq 0 \), with \( v(T) = 0 \).

Examples of spaces of this kind

2. \( X \mathbb{R} \), the real space subjacent to any complex space \( X \).
3. An absolute sum of any real space and one of the above.
4. Moreover, if \( X = X_0 \oplus X_1 \) where \( X_1 \) is complex and
\[ \| x_0 + x_1 \| = \| x_0 \| + |e^{i\theta} x_1| \] (Note that the other 3 cases are included here)

Question

Can every Banach space \( X \) with \( n(X) = 0 \) be decomposed as in \( \)?
Theorem

$X$ finite-dimensional \underline{real} space. TFAE:

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Isometries in finite-dimensional spaces

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- Hilbert spaces.
Two results on surjective isometries

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Two results on surjective isometries  Isometries on finite-dimensional spaces

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2. \( X_{\mathbb{R}} \), the real space subjacent to any complex space \( X \).
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Question

Can every Banach space \( X \) with \( n(X) = 0 \) be decomposed as in 4?
Negative answer

Infinite-dimensional case

There is an infinite-dimensional real Banach space $X$ with $n(X) = 0$ but $X$ is polyhedral. In particular, $X$ does not contain $c_0$ isometrically.

An easy example is $X = \bigoplus_{n \geq 2} X_n$, where $X_n$ is the two-dimensional space whose unit ball is the regular polygon of $2^n$ vertices.

Note that such an example is not possible in the finite-dimensional case.
Infinite-dimensional case

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Note
Such an example is not possible in the finite-dimensional case.
Quasi affirmative answer

Finite-dimensional case

Let $X$ be a finite-dimensional real space. TFAE:

1. $(X) = 0$
2. $X = X_0 \oplus X_1 \oplus \cdots \oplus X_n$
   such that $X_0$ is a (possible null) real space, $X_1, \ldots, X_n$ are non-null complex spaces,
   there are $\rho_1, \ldots, \rho_n$ rational numbers, such that

   $\|x_0 + e^{i\rho_1 \theta} x_1 + \cdots + e^{i\rho_n \theta} x_n\| = \|x_0 + x_1 + \cdots + x_n\|

   for every $x_i \in X_i$ and every $\theta \in \mathbb{R}$.

Remark

The theorem is due to Rosenthal, but with real $\rho_i$'s.

The fact that the $\rho_i$'s may be chosen as rational numbers is due to M.-Merí–Rodríguez-Palacios.
Two results on surjective isometries  
Isometries on finite-dimensional spaces

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- The fact that the $\rho$’s may be chosen as rational numbers is due to M.–Merí–Rodríguez-Palacios.
Sketch of the proof

Fix $T \in L(X)$ with $\|T\| = 1$ and $v(T) = 0$. We get that $\|\exp(\rho T)\| = 1$ for every $\rho \in \mathbb{R}$.

A Theorem by Auerbach: there exists a Hilbert space $H$ with $\dim(H) = \dim(X)$ such that every surjective isometry in $L(X)$ remains isometry in $L(H)$.

Apply the above to $\exp(\rho T)$ for every $\rho \in \mathbb{R}$. You get that $T$ is skew-hermitian in $L(H)$, so $T^* = -T$ and $T^2$ is self-adjoint. The $X_j$'s are the eigenspaces of $T^2$.

Use Kronecker's Approximation Theorem to change the eigenvalues of $T^2$ by rational numbers.
Sketch of the proof

- Fix $T \in L(X)$ with $\|T\| = 1$ and $v(T) = 0$. 

A Theorem by Auerbach: there exists a Hilbert space $H$ with $\dim(H) = \dim(X)$ such that every surjective isometry in $L(X)$ remains an isometry in $L(H)$. Apply the above to $\exp(\rho T)$ for every $\rho \in \mathbb{R}$. You get that $T$ is skew-hermitian in $L(H)$, so $T^* = -T$ and $T^2$ is self-adjoint. The $X_j$'s are the eigenspaces of $T^2$. Use Kronecker's Approximation Theorem to change the eigenvalues of $T^2$ by rational numbers.
**Sketch of the proof**

- Fix $T \in L(X)$ with $\|T\| = 1$ and $v(T) = 0$.
- We get that $\|\exp(\rho T)\| = 1$ for every $\rho \in \mathbb{R}$.
Sketch of the proof

- Fix $T \in L(X)$ with $\|T\| = 1$ and $\nu(T) = 0$.

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Sketch of the proof

- Fix \( T \in L(X) \) with \( \|T\| = 1 \) and \( v(T) = 0 \).

- We get that \( \| \exp(\rho T) \| = 1 \) for every \( \rho \in \mathbb{R} \).

- A Theorem by Auerbach: there exists a Hilbert space \( H \) with \( \dim(H) = \dim(X) \) such that every surjective isometry in \( L(X) \) remains isometry in \( L(H) \).

- Apply the above to \( \exp(\rho T) \) for every \( \rho \in \mathbb{R} \).
Sketch of the proof

- Fix $T \in L(X)$ with $\|T\| = 1$ and $v(T) = 0$.
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Sketch of the proof

- Fix $T \in L(X)$ with $\|T\| = 1$ and $\nu(T) = 0$.
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- Use Kronecker’s Approximation Theorem to change the eigenvalues of $T^2$ by rational numbers.
A simple case of getting rational numbers

Let $X = X_0 \oplus X_1 \oplus X_2$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $\|x_0 + e^{i\rho x_1} + e^{i\alpha \rho x_2}\| = \|x_0 + x_1 + x_2\|$ $\forall \rho$, $\forall x_0, x_1, x_2$.

Then $\|x_0 + x_1 + x_2\| = \|x_0 + e^{i\rho} (x_1 + e^{i(\alpha - 1)\rho} x_2)\|$ $\forall \rho$.

Take $\rho = \frac{2\pi k}{\alpha - 1}$ with $k \in \mathbb{Z}$.

Then $\|x_0 + (x_1 + x_2)\| = \|x_0 + e^{i2\pi k} (x_1 + x_2)\|$ $\forall k \in \mathbb{Z}$.

But $\{2\pi k/(\alpha - 1) : k \in \mathbb{Z}\}$ is dense in $\mathbb{T}$, so $\|x_0 + (x_1 + x_2)\| = \|x_0 + e^{i\rho} (x_1 + x_2)\|$ $\forall \rho \in \mathbb{R}$ and $X = X_0 \oplus \mathbb{Z}$ where $\mathbb{Z} = \mathbb{Z}_1 \oplus \mathbb{Z}_2$ is a complex space.
A simple case of getting rational numbers

Let $X = X_0 \oplus X_1 \oplus X_2$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ s.t.

$$\|x_0 + e^{i\rho}x_1 + e^{i\alpha \rho}x_2\| = \|x_0 + x_1 + x_2\| \quad \forall \rho, \forall x_0, x_1, x_2.$$
A simple case of getting rational numbers

- Let $X = X_0 \oplus X_1 \oplus X_2$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ s.t.

$$\left\| x_0 + e^{i\rho} x_1 + e^{i\alpha \rho} x_2 \right\| = \left\| x_0 + x_1 + x_2 \right\| \forall \rho, \forall x_0, x_1, x_2.$$

- Then $\left\| x_0 + x_1 + x_2 \right\| = \left\| x_0 + e^{i\rho} \left( x_1 + e^{i(\alpha-1)\rho} x_2 \right) \right\| \forall \rho$. 
A simple case of getting rational numbers

- Let $X = X_0 \oplus X_1 \oplus X_2$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ s.t.
  \[ \|x_0 + e^{i\rho}x_1 + e^{i\alpha\rho}x_2\| = \|x_0 + x_1 + x_2\| \quad \forall \rho, \forall x_0, x_1, x_2. \]

- Then $\|x_0 + x_1 + x_2\| = \|x_0 + e^{i\rho}(x_1 + e^{i(\alpha - 1)\rho}x_2)\| \quad \forall \rho.$

- Take $\rho = \frac{2\pi k}{\alpha - 1}$ with $k \in \mathbb{Z}.$
A simple case of getting rational numbers

- Let $X = X_0 \oplus X_1 \oplus X_2$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ s.t.
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- Then $\|x_0 + x_1 + x_2\| = \|x_0 + e^{i\rho} \left(x_1 + e^{i(\alpha - 1)\rho} x_2\right)\| \ \forall \rho.$

- Take $\rho = \frac{2\pi k}{\alpha - 1}$ with $k \in \mathbb{Z}$.

- Then $\|x_0 + (x_1 + x_2)\| = \|x_0 + e^{i \frac{2\pi k}{\alpha - 1}} (x_1 + x_2)\| \ \forall k \in \mathbb{Z}$
A simple case of getting rational numbers

Let \( X = X_0 \oplus X_1 \oplus X_2 \) and \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) s.t.
\[
\| x_0 + e^{i\rho} x_1 + e^{i\alpha \rho} x_2 \| = \| x_0 + x_1 + x_2 \| \quad \forall \rho, \quad \forall x_0, x_1, x_2.
\]

Then \( \| x_0 + x_1 + x_2 \| = \| x_0 + e^{i\rho} \left( x_1 + e^{i(\alpha-1)\rho} x_2 \right) \| \quad \forall \rho. \)

Take \( \rho = \frac{2\pi k}{\alpha - 1} \) with \( k \in \mathbb{Z}. \)

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But \( \left\{ \frac{2\pi k}{\alpha - 1} : k \in \mathbb{Z} \right\} \) is dense in \( \mathbb{T} \), so
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and \( X = X_0 \oplus Z \) where \( Z = X_1 \oplus X_2 \) is a complex space
Consequences

If $\dim(X) = 2$, then $X \equiv \mathbb{C}$.

If $\dim(X) = 3$, then $X \equiv \mathbb{R} \oplus \mathbb{C}$ (absolute sum).

Natural question: Are all finite-dimensional $X$'s with $n(X) = 0$ of the form $X = X_0 \oplus X_1$?

Answer: No.

Example: $X = (\mathbb{R}^4, \| \cdot \|)$, 

$$
\| (a, b, c, d) \| = \frac{1}{4} \int_0^{2\pi} \left| \Re \left( e^{2it}(a + ib) + e^{it}(c + id) \right) \right| dt.
$$

Then $n(X) = 0$ but the unique possible decomposition is $X = \mathbb{C} \oplus \mathbb{C}$ with 

$$
\| e^{it}x_1 + e^{2it}x_2 \| = \| x_1 + x_2 \|.
$$
Consequences

Corollary

$X$ real space with $n(X) = 0$.

- If $\dim(X) = 2$, then $X \equiv \mathbb{C}$.
- If $\dim(X) = 3$, then $X \equiv \mathbb{R} \oplus \mathbb{C}$ (absolute sum).
Consequences

Corollary

X real space with \( n(X) = 0 \).
- If \( \dim(X) = 2 \), then \( X \cong \mathbb{C} \).
- If \( \dim(X) = 3 \), then \( X \cong \mathbb{R} \oplus \mathbb{C} \) (absolute sum).

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Are all finite-dimensional X’s with \( n(X) = 0 \) of the form \( X = X_0 \oplus X_1 \) ?
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No.
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Let $X$ be a real space with $n(X) = 0$. Then:

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Answer

No.

Example

Let $X = (\mathbb{R}^4, \| \cdot \|)$, where $\| (a, b, c, d) \| = \frac{1}{4} \int_0^{2\pi} \left| \text{Re} \left( e^{2it} (a + ib) + e^{it} (c + id) \right) \right| \, dt$. Then $n(X) = 0$ but the unique possible decomposition is $X = \mathbb{C} \oplus \mathbb{C}$ with

$$\left\| e^{it} x_1 + e^{2it} x_2 \right\| = \| x_1 + x_2 \|.$$
The Lie-algebra of a Banach space

The Lie-algebra of a Banach space is defined as the set of all linear operators \( T \) on the Banach space \( X \) such that \( v(T) = 0 \) for all \( v \) in the dual space \( X' \). When \( X \) is finite-dimensional, \( \text{Iso}(X) \) is a Lie-group and \( Z(X) \) is its tangent space (i.e., its Lie-algebra). 

\[ \dim(X) = n \Rightarrow \dim(Z(X)) \leq n(n-1)/2. \]

Equality holds if and only if \( X \) is a Hilbert space.

An open problem: Given \( n \geq 3 \), which are the possible \( \dim(Z(X)) \) over all \( n \)-dimensional \( X \)’s?

Observation (Javier Merí, PhD): When \( \dim(X) = 3 \), \( \dim(Z(X)) \) cannot be 2.
The Lie-algebra of a Banach space

**Lie-algebra**

- Let $X$ be a real Banach space, then $\mathcal{Z}(X) = \{ T \in L(X) : v(T) = 0 \}$.
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The Lie-algebra of a Banach space

Lie-algebra

$X$ real Banach space, $Z(X) = \{ T \in L(X) : v(T) = 0 \}$.

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Remark

- $\dim(X) = n \implies \dim(Z(X)) \leq \frac{n(n-1)}{2}$.
- Equality holds $\iff H$ Hilbert space.
# The Lie-algebra of a Banach space

## Lie-algebra

An $X$ real Banach space, $\mathcal{Z}(X) = \{ T \in L(X) : v(T) = 0 \}$.

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## Remark

- $\dim(X) = n \implies \dim(\mathcal{Z}(X)) \leq \frac{n(n-1)}{2}$.
- Equality holds $\iff H$ Hilbert space.

## An open problem

Given $n \geq 3$, which are the possible $\dim(\mathcal{Z}(X))$ over all $n$-dimensional $X$’s?
The Lie-algebra of a Banach space

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The Lie-algebra of a Banach space

### Lie-algebra

Let $X$ be a real Banach space, $\mathcal{Z}(X) = \{ T \in L(X) : v(T) = 0 \}$.

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### Proof

If $\dim(X) = 3$, $n(X) = 0$, then $X = \mathbb{C} \oplus \mathbb{R}$ (absolute sum).

### Remark

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### Lie-algebra

Let $X$ be a real Banach space, then 
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#### Proof

If $\dim(X) = 3$, $n(X) = 0$, then $X = \mathbb{C} \oplus \mathbb{R}$ (absolute sum).
- If $\oplus = \oplus_2$, then $X$ is a Hilbert space and $\dim(\mathcal{Z}(X)) = 3$. ✓

#### Remark

- If $\oplus \neq \oplus_2$, then isometries respect summands and $\dim(\mathcal{Z}(X)) = 1$. ✓

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The Lie-algebra of a Banach space

**Lie-algebra**

Let $X$ be a real Banach space, then $\mathcal{Z}(X) = \{T \in L(X) : v(T) = 0\}$.

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If $\dim(X) = 3$, $n(X) = 0$, then $X = \mathbb{C} \oplus \mathbb{R}$ (absolute sum).

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Given $n \geq 3$, which are the possible $\dim(\mathcal{Z}(X))$ over all $n$-dimensional $X$’s?

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When $\dim(X) = 3$, $\dim(\mathcal{Z}(X))$ cannot be 2.
Remark

$X$ Banach space.

$T \in \text{Iso}(X) \Rightarrow T^* \in \text{Iso}(X^*)$.

$\text{Iso}(X^*)$ can be bigger than $\text{Iso}(X)$.

The problem

How much bigger can be $\text{Iso}(X^*)$ than $\text{Iso}(X)$?

Is it possible that $\mathbb{Z}(\text{Iso}(X^*))$ is big while $\mathbb{Z}(\text{Iso}(X))$ is trivial?

The answer is yes. This is what we are going to present next.
Semigroups of surjective isometries and duality

Remark

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Semigroups of surjective isometries and duality

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The answer is yes. This is what we are going to present next.
Semigroups of surjective isometries and duality

\[ \mathcal{E}(K \parallel L) = \{ f \in \mathcal{C}(K) : f|_L \in E \} \]

Theorem \[ \mathcal{E}(K \parallel L)^* \equiv E^* \oplus 1 \mathcal{C}_0(K \parallel L)^* \]
Spaces $C_E(K\|L)$

$K$ compact, $L \subset K$ closed nowhere dense, $E \subset C(L)$.

$$C_E(K\|L) = \{f \in C(K) : f|_L \in E\}.$$
Spaces $C_E(K\|L)$

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$C_E(K\|L) = \{ f \in C(K) : f|_L \in E \}$.

Theorem

$C_E(K\|L)^* \cong E^* \oplus_1 C_0(K\|L)^*$ \quad \& \quad n(C_E(K\|L)) = 1.$
Semigroups of surjective isometries and duality

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Theorem

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Proof.
Semigroups of surjective isometries and duality

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$$C_E(K\|L)^* \equiv E^* \oplus_1 C_0(K\|L)^* \quad \& \quad n(C_E(K\|L)) = 1.$$  

Proof.

- $C_0(K\|L)$ is an $M$-ideal of $C(K)$
  $$\implies C_0(K\|L) \text{ is an } M\text{-ideal of } C_E(K\|L).$$
Semigroups of surjective isometries and duality

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- $C_0(K\|L)$ is an $M$-ideal of $C(K)$
  $$\implies C_0(K\|L)$$ is an $M$-ideal of $C_E(K\|L)$.

- Meaning that $C_E(K\|L)^* \cong C_0(K\|L)^\perp \oplus_1 C_0(K\|L)^*$. 

\[\square\]
Spaces $C_E(K\parallel L)$

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$$C_E(K\parallel L) = \{ f \in C(K) : f|_L \in E \}.$$ 

Theorem

$$C_E(K\parallel L)^* \equiv E^* \oplus_1 C_0(K\parallel L)^* \quad \& \quad n(C_E(K\parallel L)) = 1.$$ 

Proof.

- $C_0(K\parallel L)$ is an $M$-ideal of $C(K)$
  $$\implies C_0(K\parallel L)$$

- Meaning that $C_E(K\parallel L)^* \equiv C_0(K\parallel L)^\perp \oplus_1 C_0(K\parallel L)^*$.

- $C_0(K\parallel L)^\perp \equiv (C_E(K\parallel L)/C_0(K\parallel L))^* \equiv E^*$.
Two results on surjective isometries

Isometries and duality

Semigroups of surjective isometries and duality

Spaces \( C_E(K\|L) \)

- \( K \) compact, \( L \subset K \) closed nowhere dense, \( E \subset C(L) \).
- \( C_E(K\|L) = \{ f \in C(K) : f|_L \in E \} \).

Theorem

\[ C_E(K\|L)^\ast \equiv E^\ast \oplus_1 C_0(K\|L)^\ast \quad \& \quad n(C_E(K\|L)) = 1. \]

Proof.

- \( C_0(K\|L) \) is an \( M \)-ideal of \( C(K) \)
  \[ \implies C_0(K\|L) \text{ is an } M \text{-ideal of } C_E(K\|L). \]

- Meaning that \( C_E(K\|L)^\ast \equiv C_0(K\|L)^\perp \oplus_1 C_0(K\|L)^\ast \).

- \( C_0(K\|L)^\perp \equiv (C_E(K\|L)/C_0(K\|L))^\ast \equiv E^\ast : \)

- \( \Phi : C_E(K\|L) \longrightarrow E \), \( \Phi(f) = f|_L \).
  - \( \|\Phi\| \leq 1 \) and \( \ker \Phi = C_0(K\|L) \).
  - \( \tilde{\Phi} : C_E(K\|L)/C_0(K\|E) \longrightarrow E \) onto isometry:
    \[ \{ g \in E : \|g\| < 1 \} \subseteq \Phi(\{ f \in C_E(K\|L) : \|f\| < 1 \}). \]
Semigroups of surjective isometries and duality

**Spaces** $C_E(K\|L)$

$K$ compact, $L \subset K$ closed nowhere dense, $E \subset C(L)$.

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**Theorem**

$$C_E(K\|L)^* \equiv E^* \oplus_1 C_0(K\|L)^* \quad \& \quad n(C_E(K\|L)) = 1.$$ 

**Proof.**

- $C_0(K\|L)$ is an $M$-ideal of $C(K)$
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- **Meaning that** $C_E(K\|L)^* \equiv C_0(K\|L)^\perp \oplus_1 C_0(K\|L)^*$

- $C_0(K\|L)^\perp \equiv (C_E(K\|L)/C_0(K\|L))^* \equiv E^*$:

- $\Phi : C_E(K\|L) \longrightarrow E$, $\Phi(f) = f|_L$.
  - $\|\Phi\| \leq 1$ and $\ker \Phi = C_0(K\|L)$.
  - $\widetilde{\Phi} : C_E(K\|L)/C_0(K\|L) \longrightarrow E$ onto isometry:
    - $\{g \in E : \|g\| < 1\} \subseteq \Phi(\{f \in C_E(K\|L) : \|f\| < 1\}).$
### Semigroups of surjective isometries and duality

#### Spaces $C_E(K\|L)$

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#### Proof.

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Semigroups of surjective isometries and duality

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Proof.

- $A = \{ (0, \delta_t) : t \in K \setminus L \} \subset S_{C_E(K\|L)^*}$ is norming for $X = C_E(K\|L)$. 

Semigroups of surjective isometries and duality

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Theorem

$C_E(K\|L)^* \equiv E^* \oplus_1 C_0(K\|L)^*$ and $n(C_E(K\|L)) = 1$.

Proof.

- $A = \{(0, \delta_t) : t \in K \setminus L\} \subset S_{C_E(K\|L)^*}$ is norming for $X = C_E(K\|L)$.
- $|x^{**}(a^*)| = 1$ for every $x^{**} \in \text{ext}(B_{X^{**}})$ and every $a^* \in A$. 
Semigroups of surjective isometries and duality

**Spaces** $C_E(K\|L)$

$K$ compact, $L \subset K$ closed nowhere dense, $E \subset C(L)$.

$$C_E(K\|L) = \{ f \in C(K) : f|_L \in E \}.$$  

**Theorem**

$$C_E(K\|L)^* \equiv E^* \oplus_1 C_0(K\|L)^* \quad \& \quad n(C_E(K\|L)) = 1.$$  

**Proof.**

- $\mathcal{A} = \{(0, \delta_t) : t \in K \setminus L\} \subset S_{C_E(K\|L)^*}$ is norming for $X = C_E(K\|L)$.
- $|x^{**}(a^*)| = 1$ for every $x^{**} \in \text{ext}(B_{X^{**}})$ and every $a^* \in \mathcal{A}$.
- This gives $n(C_E(K\|L)) = 1$:
  - $T \in L(X), \varepsilon > 0$, take $a^* \in \mathcal{A}$ with $\|T^*(a^*)\| > \|T\| - \varepsilon$,
  - Take $x^{**} \in \text{ext}(B_{X^{**}})$ with $|x^{**}(T^*(a^*))| > \|T\| - \varepsilon$,
  - Since $|x^{**}(a^*)| = 1$, we have
    $$\nu(T) = \nu(T^*) \geq |x^{**}(T^*(a^*))| > \|T\| - \varepsilon. \checkmark$$
# Semigroups of surjective isometries and duality

## Spaces $C_E(K\|L)$

$K$ compact, $L \subset K$ closed nowhere dense, $E \subset C(L)$.

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## Theorem

\[ C_E(K\|L)^* \equiv E^* \oplus_1 C_0(K\|L)^* \quad \& \quad n(C_E(K\|L)) = 1. \]

## Consequence: the example

Take $K = [0, 1]$, $L = \Delta$ (Cantor set), $E = \ell_2 \subset C(\Delta)$.

- $\text{Iso}(C_{\ell_2}([0,1]\|\Delta))$ has no exponential one-parameter semigroups.
- $C_{\ell_2}([0,1]\|\Delta)^* \equiv \ell_2 \oplus_1 C_0([0,1]\|\Delta)^*$, so taken $S \in \text{Iso}(\ell_2)$.

\[ \implies T = \begin{pmatrix} S & 0 \\ 0 & \text{Id} \end{pmatrix} \in \text{Iso}(C_{\ell_2}([0,1]\|\Delta)^*) \]

Then, $\text{Iso}(C_{\ell_2}([0,1]\|\Delta)^*)$ contains infinitely many exponential one-parameter semigroups.
Some comments
Some comments

In terms of linear dynamical systems

In $C_\ell^2(\mathbb{R}_+ \to C_\ell^2(\mathbb{R}_0,1\|\Delta))$ there is no $A \in \mathcal{L}(X)$ such that the solution to the linear dynamical system $x' = A x$ ($x: \mathbb{R}_+ \to C_\ell^2(\mathbb{R}_0,1\|\Delta)$) is given by a semigroup of isometries. There are infinitely many such $A$'s in $C_\ell^2(\mathbb{R}_0,1\|\Delta)^*$, in $C_\ell^2(\mathbb{R}_0,1\|\Delta)^{**}$.

Further results (Koszmider–M.–Merí., 2009)

There are unbounded $A$'s on $C_\ell^2(\mathbb{R}_0,1\|\Delta)$ such that the solution to the linear dynamical system $x'(t) = A x(t)$ is a one-parameter $C_0$ semigroup of isometries. There is $X$ such that $\text{Iso}(X) = \{-\text{Id}, \text{Id}\}$ and $X^* = \ell^2 \oplus 1 L^1(\nu)$.

Therefore, there is no semigroups in $\text{Iso}(X)$, but there are infinitely many exponential one-parameter semigroups in $\text{Iso}(X^*)$. 
Some comments

In terms of linear dynamical systems

- In $C_{\ell_2}([0, 1]\|\Delta)$ there is no $A \in L(X)$ such that the solution to the linear dynamical system

$$x' = Ax \quad (x : \mathbb{R}_0^+ \longrightarrow C_{\ell_2}([0, 1]\|\Delta))$$

(which is $x(t) = \exp(t A)(x(0))$) is given by a semigroup of isometries.
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In terms of linear dynamical systems

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- There is $X$ such that

$\text{Iso}(X) = \{-\text{Id}, \text{Id}\}$ and $X^* = \ell_2 \oplus_1 L_1(\nu)$.

- Therefore, there is no semigroups in $\text{Iso}(X)$, but there are infinitely many exponential one-parameter semigroups in $\text{Iso}(X^*)$. 
Numerical index of Banach spaces

- Basic definitions and examples
- Stability properties
- Duality
- The isomorphic point of view
- Banach spaces with numerical index one
  - Isomorphic properties
  - Isometric properties
  - Asymptotic behavior
- How to deal with numerical index 1 property?

V. Kadets, M. Martín, and R. Payá.
Recent progress and open questions on the numerical index of Banach spaces.
*RACSAM* (2006)
Numerical radius

Let $X$ be a Banach space, $T \in L(X)$. The numerical radius of $T$ is

$$v(T) = \sup \{|x^*(Tx)| : x^* \in S_{X^*}, x \in S_X, x^*(x) = 1\}$$
Numerical index of Banach spaces: definitions

**Numerical radius**

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**Remark**

The numerical radius is a continuous seminorm in $L(X)$. Actually, $v(\cdot) \leq \|\cdot\|$
Numerical index of Banach spaces: definitions

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\]

**Remark**

The numerical radius is a continuous seminorm in \( L(X) \). Actually, \( v(\cdot) \leq \|\cdot\| \)

**Numerical index (Lumer, 1968)**

\( X \) Banach space, the **numerical index** of \( X \) is

\[
n(X) = \inf \{ v(T) : T \in L(X), \|T\| = 1 \}
\]

\[
= \max \{ k \geq 0 : k \|T\| \leq v(T) \ \forall \ T \in L(X) \}
\]

\[
= \inf \left\{ M \geq 0 : \exists T \in L(X), \|T\| = 1, \| \exp(\rho T) \| \leq e^{\rho M} \ \forall \rho \in \mathbb{R} \right\}
\]
Recalling some basic properties

\[ n(X) = 1 \iff v \text{ and } \|\cdot\| \text{ coincide.} \]

\[ n(X) = 0 \iff v \text{ is not an equivalent norm in } L(X). \]

\[ X \text{ complex} \Rightarrow n(X) \geq \frac{1}{e}. \]

(Bohnenblust–Karlin, 1955; Glickfeld, 1970)

Actually,

\[ \{ n(X): X \text{ complex, dim}(X) = 2 \} = \left[ e^{-1}, 1 \right] \]

\[ \{ n(X): X \text{ real, dim}(X) = 2 \} = [0, 1] \]

(Duncan–McGregor–Pryce–White, 1970)
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Recalling some basic properties

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- Actually,

\[
\{ n(X) : X \text{ complex, } \dim(X) = 2 \} = [e^{-1}, 1] \\
\{ n(X) : X \text{ real, } \dim(X) = 2 \} = [0, 1]
\]

(Duncan–McGregor–Pryce–White, 1970)
Some examples

1. $H$ Hilbert space, $\dim(H) > 1$,

\[
\begin{align*}
n(H) &= 0 \quad \text{if } H \text{ is real} \\
n(H) &= 1/2 \quad \text{if } H \text{ is complex}
\end{align*}
\]
Numerical index of Banach spaces: examples (I)

Some examples

1. $H$ Hilbert space, $\dim(H) > 1$, 
   
   $n(H) = 0$ if $H$ is real  
   $n(H) = 1/2$ if $H$ is complex

2. $n(L_1(\mu)) = 1$  $\mu$ positive measure  
   $n(C(K)) = 1$  $K$ compact Hausdorff space

   (Duncan et al., 1970)
Some examples

1. $H$ Hilbert space, $\dim(H) > 1$,
   
   \[ n(H) = 0 \quad \text{if } H \text{ is real} \]
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2. $n(L_1(\mu)) = 1$ \quad $\mu$ positive measure
   \[ n(C(K)) = 1 \quad K \text{ compact Hausdorff space} \]
   (Duncan et al., 1970)

3. If $A$ is a $C^*$-algebra \( \Rightarrow \)
   
   \[ \begin{aligned}
   n(A) &= 1 \quad &\text{A commutative} \\
   n(A) &= 1/2 \quad &\text{A not commutative}
   \end{aligned} \]

   (Huruya, 1977; Kaidi–Morales–Rodríguez, 2000)
Numerical index of Banach spaces: examples (I)

Some examples

1. $H$ Hilbert space, $\dim(H) > 1$,

   $n(H) = 0$ if $H$ is real
   $n(H) = 1/2$ if $H$ is complex

2. $n(L_1(\mu)) = 1$ $\mu$ positive measure
   $n(C(K)) = 1$ $K$ compact Hausdorff space

   (Duncan et al., 1970)

3. If $A$ is a $C^*$-algebra $\Rightarrow \begin{cases} n(A) = 1 & A$ commutative \\ n(A) = 1/2 & A$ not commutative \end{cases}$

   (Huruya, 1977; Kaidi–Morales–Rodríguez, 2000)

4. If $A$ is a function algebra $\Rightarrow n(A) = 1$

   (Werner, 1997)
For \( n \geq 2 \), the unit ball of \( X_n \) is a \( 2n \) regular polygon:

\[
\begin{align*}
 n(X_n) &= \begin{cases} 
 \tan \left( \frac{\pi}{2n} \right) & \text{if } n \text{ is even}, \\
 \sin \left( \frac{\pi}{2n} \right) & \text{if } n \text{ is odd}.
\end{cases}
\end{align*}
\]

(M.–Merí, 2007)
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\end{cases}
\]

(M.–Merí, 2007)

6 Every finite-codimensional subspace of \( C[0,1] \) has numerical index 1

(Boyko–Kadets–M.–Werner, 2007)
Even more examples

Numerical index of $L_p$-spaces, $1 < p < \infty$: 

\[ n(\ell_p(\mathbb{2})) = \lim_{m \to \infty} n(\ell_p(m)) \leq M_p \leq n(\ell_2(\mathbb{2})) \]

(M.–Merí, 2009)

In the real case, 
\[ \max\{\frac{1}{2}, \frac{1}{2} \cdot \frac{1}{p}\} M_p \leq n(\ell_p(\mathbb{2})) \leq M_p = \max(t \in [0,1] | t^p - (1-t)^p) \] 

(M.–Merí–Popov, 2009)

In particular, 
\[ n(\ell_p(\mathbb{2})) > 0 \] 

for $p \neq 2$.

(M.–Merí–Popov, 2009)
Numerical index of Banach spaces: some examples (III)

Even more examples

- **Numerical index of $L_p$-spaces, $1 < p < \infty$:**
  
  \[ n(L_p[0,1]) = n(\ell_p) = \lim_{m \to \infty} n(\ell_p^m). \]

  (Ed-Dari, 2005 & Ed-Dari-Khamsi, 2006)
Even more examples

- Numerical index of $L_p$-spaces, $1 < p < \infty$:
  - $n(L_p[0,1]) = n(\ell_p) = \lim_{m \to \infty} n(\ell_p^{(m)})$.
    (Ed-Dari, 2005 & Ed-Dari-Khamsi, 2006)
  - $n(\ell_p^{(2)})$ ?
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   (Ed-Dari, 2005 & Ed-Dari-Khamsi, 2006)

2. $n(\ell_p^{(2)})$?

3. In the real case,

   $$\max \left\{ \frac{1}{2^{1/p}}, \frac{1}{2^{1/q}} \right\} M_p \leq n(\ell_p^{(2)}) \leq M_p$$

   and $M_p = \nu \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \max_{t \in [0,1]} \frac{|t^{p-1} - t|}{1 + t^p}$

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Even more examples

- Numerical index of $L_p$-spaces, $1 < p < \infty$:
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    $\frac{1 + t^p}{1 + t^p}$
    (M.–Merí, 2009)
  - In the real case, $n(L_p(\mu)) \geq \frac{M_p}{8e}$.
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Even more examples

- **Numerical index of $L_p$-spaces, $1 < p < \infty$:**
  - $n(L_p[0, 1]) = n(\ell_p) = \lim_{m \to \infty} n(\ell_p^{(m)})$.
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- $n(\ell_p^{(2)})$?

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  $(\text{M.--Merí, 2009})$

- In the real case, $n(L_p(\mu)) \geq \frac{M_p}{8e}$.

- In particular, $n(L_p(\mu)) > 0$ for $p \neq 2$.

  $(\text{M.--Merí--Popov, 2009})$
Numerical index: open problems on computing

1. Compute $n(L_p[0,1])$ for $1 < p < \infty$, $p \neq 2$.

2. Is $n(\ell(2)_p) = \text{M}_p$ (real case)?

3. Is $n(\ell(2)_p) = \left(\frac{1}{p} + \frac{1}{q}\right)^{-1}$ (complex case)?

4. Compute the numerical index of real $C^*$-algebras.

5. Compute the numerical index of more classical Banach spaces: $C_m[0,1]$, $\text{Lip}(K)$, Lorentz spaces, Orlicz spaces, ...
Open problems

1. Compute $n(L_p[0,1])$ for $1 < p < \infty$, $p \neq 2$. 

2. Is $n(\ell^p(2)) = M_p$ (real case)?

3. Is $n(\ell^p(2)) = (p_1^{1/p_2} + q_1^{1/q_2})^{-1}$ (complex case)?

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Open problems

1. Compute $n(L_p[0,1])$ for $1 < p < \infty$, $p \neq 2$.

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Numerical index: open problems on computing

Open problems

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Open problems

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Stability properties

Direct sums of Banach spaces (M.–Payá, 2000)

\[ n\left(\bigoplus_{\lambda \in \Lambda} X_{\lambda}\right)_{c_0} = n\left(\bigoplus_{\lambda \in \Lambda} X_{\lambda}\right)_{\ell_1} = n\left(\bigoplus_{\lambda \in \Lambda} X_{\lambda}\right)_{\ell_\infty} = \inf_{\lambda} n\left(X_{\lambda}\right) \]
Stability properties

Direct sums of Banach spaces (M.–Payá, 2000)

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n\left(\bigoplus_{\lambda \in \Lambda} X_\lambda\right)_{c_0} = n\left(\bigoplus_{\lambda \in \Lambda} X_\lambda\right)_{\ell_1} = n\left(\bigoplus_{\lambda \in \Lambda} X_\lambda\right)_{\ell_\infty} = \inf_{\lambda} n(X_\lambda)
\]

Consequences

- There is a real Banach space \( X \) such that

  \[\nu(T) > 0 \quad \text{when} \quad T \neq 0,\]

  but \( n(X) = 0 \)

  (i.e. \( \nu(\cdot) \) is a norm on \( L(X) \) which is not equivalent to the operator norm).
Stability properties

Direct sums of Banach spaces (M.–Payá, 2000)

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Consequences

- There is a real Banach space \( X \) such that
  \[ v(T) > 0 \quad \text{when} \ T \neq 0, \]
  but \( n(X) = 0 \)
  (i.e. \( v(\cdot) \) is a norm on \( L(X) \) which is not equivalent to the operator norm).

- For every \( t \in [0, 1] \), there exist a real \( X_t \) isomorphic to \( c_0 \) (or \( \ell_1 \) or \( \ell_\infty \)) with \( n(X_t) = t \).

- For every \( t \in [e^{-1}, 1] \), there exist a complex \( Y_t \) isomorphic to \( c_0 \) (or \( \ell_1 \) or \( \ell_\infty \)) with \( n(Y_t) = t \).
Vector-valued function spaces (López-M.-Merí-Payá-Villena, 2000's)

$E$ Banach space, $\mu$ positive $\sigma$-finite measure, $K$ compact space. Then

$$n(C(K, E)) = n(C_w(K, E)) = n(L_1(\mu, E)) = n(L_\infty(\mu, E)) = n(E),$$

and $n(C_w^*(K, E^*)) \leq n(E)$
Numerical index Stability properties

Stability properties (II)

Vector-valued function spaces (López-M.-Merí-Payá-Villena, 2000's)

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and $n(C_w^*(K, E^*)) \leq n(E)$

Tensor products (Lima, 1980)

There is no general formula for $n(X\tilde{\otimes}_\varepsilon Y)$ nor for $n(X\tilde{\otimes}_\pi Y)$:

- $n(\ell_1^{(4)} \tilde{\otimes}_\pi \ell_1^{(4)}) = n(\ell_\infty^{(4)} \tilde{\otimes}_\varepsilon \ell_\infty^{(4)}) = 1.$
- $n(\ell_1^{(4)} \tilde{\otimes}_\varepsilon \ell_1^{(4)}) = n(\ell_\infty^{(4)} \tilde{\otimes}_\pi \ell_\infty^{(4)}) < 1.$
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- $n(\ell_1^{(4)} \tilde{\otimes}_\varepsilon \ell_1^{(4)}) = n(\ell_{\infty}^{(4)} \tilde{\otimes}_\pi \ell_{\infty}^{(4)}) < 1$.


$$n(L_p([0,1],E)) = n(\ell_p(E)) = \lim_{m \to \infty} n(E \oplus_p^m \oplus_p E).$$
Proposition

Let \( X \) be a Banach space, \( T \in \mathcal{L}(X) \). Then

\[
\sup \text{Re} \, V(T) = \lim_{\alpha \to 0^+} \|\text{Id} + \alpha T\| - 1/\alpha.
\]

Then, \( v(T^*) = v(T) \) for every \( T \in \mathcal{L}(X) \).

Therefore, \( n(X^*) \leq n(X) \).

(Duncan–McGregor–Pryce–White, 1970)

Question (From the 1970's)

Is \( n(X) = n(X^*) \)?

Negative answer (Boyko–Kadets–M.–Werner, 2007)

Consider the space \( X = \{(x, y, z) \in c \oplus \ell^\infty \oplus \ell^\infty : \lim x + \lim y + \lim z = 0 \} \).

Then, \( n(X) = 1 \) but \( n(X^*) < 1 \).
Proposition

Let $X$ be a Banach space, $T \in L(X)$. Then

$$\sup \Re V(T) = \lim_{\alpha \to 0^+} \frac{\|\text{Id} + \alpha T\| - 1}{\alpha}.$$ 

(Duncan–McGregor–Pryce–White, 1970)
Proposition

$X$ Banach space, $T \in L(X)$. Then

- $\sup \text{Re } V(T) = \lim_{\alpha \to 0^+} \frac{\|\text{Id} + \alpha T\| - 1}{\alpha}$.

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Is \( n(X) = n(X^*) \) ?
**Proposition**

Let $X$ be a Banach space, $T \in L(X)$. Then

1. \[ \sup \text{Re} V(T) = \lim_{\alpha \to 0^+} \frac{\|\text{Id} + \alpha T\| - 1}{\alpha}. \]

2. Then, $v(T^*) = v(T)$ for every $T \in L(X)$.

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**Question (From the 1970's)**

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Consider the space

\[ X = \{(x, y, z) \in c \oplus_{\infty} c \oplus_{\infty} c : \lim x + \lim y + \lim z = 0\}. \]

Then, $n(X) = 1$ but $n(X^*) < 1$. 
Numerical index and duality. Proof of main example

\[ X = \{ (x, y, z) \in \mathbb{C} \oplus_{\infty} \mathbb{C} \oplus_{\infty} \mathbb{C} : \lim x + \lim y + \lim z = 0 \} : \]

\[ n(X) = 1 \quad \text{but} \quad n(X^*) < 1. \]
Numerical index and duality. Proof of main example

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Proof

- \( c^* = \ell_1 \oplus_1 K \lim \implies X^* = [c^* \oplus_1 c^* \oplus_1 c^*] / (\lim, \lim, \lim). \)
Numerical index and duality. Proof of main example

\[ X = \{ (x, y, z) \in c \oplus_\infty c \oplus_\infty c : \lim x + \lim y + \lim z = 0 \} : \]

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**Proof**

- \( c^* = \ell_1 \oplus_1 K \lim \implies X^* = [c^* \oplus_1 c^* \oplus_1 c^*] / (\lim, \lim, \lim). \)
- Then, writing \( Z = \ell_1^{(3)} / (1, 1, 1), \) we can identify
  \[ X^* \equiv \ell_1 \oplus_1 \ell_1 \oplus_1 \ell_1 \oplus_1 Z, \quad X^{**} \equiv \ell_\infty \oplus_\infty \ell_\infty \oplus_\infty \ell_\infty \oplus_\infty Z^*. \]
**Numerical index and duality. Proof of main example**

\[
X = \{ (x, y, z) \in c \oplus_{\infty} c \oplus_{\infty} c : \lim x + \lim y + \lim z = 0 \}:
\]

\[
n(X) = 1 \quad \text{but} \quad n(X^*) < 1.
\]

**Proof**

- \( c^* = \ell_1 \oplus_1 \mathbb{K} \lim \quad \Rightarrow \quad X^* = [c^* \oplus_1 c^* \oplus_1 c^*] / (\lim, \lim, \lim). \)

- Then, writing \( Z = \ell_1^{(3)} / (1, 1, 1) \), we can identify
  \[
  X^* \equiv \ell_1 \oplus_1 \ell_1 \oplus_1 \ell_1 \oplus_1 Z, \quad X^{**} \equiv \ell_{\infty} \oplus_{\infty} \ell_{\infty} \oplus_{\infty} \ell_{\infty} \oplus_{\infty} Z^*.
  \]

- \( A = \{(e_n, 0, 0, 0) : n \in \mathbb{N}\} \cup \{(0, e_n, 0, 0) : n \in \mathbb{N}\} \cup \{(0, 0, e_n, 0) : n \in \mathbb{N}\} \subset X^*. \)
Numerical index and duality. Proof of main example

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- Then \( B_{X^*} = \overline{\text{aco}}{(w^*) (A)} \) and

\[ |x^{**}(a)| = 1 \quad \forall \ x^{**} \in \text{ext}(B_{X^{**}}) \ \forall \ a \in A. \]
Numerical index theory

Numerical index and duality. Proof of main example

\[ X = \{ (x, y, z) \in c \oplus_{\infty} c \oplus_{\infty} c : \lim x + \lim y + \lim z = 0 \} : \]

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Proof

- \[ c^* = \ell_1 \oplus_1 \mathbb{K} \lim \implies X^* = \left[ c^* \oplus_1 c^* \oplus_1 c^* \right] / (\lim, \lim, \lim). \]

- Then, writing \( Z = \ell_1^{(3)} / (1, 1, 1) \), we can identify
  \[ X^* \equiv \ell_1 \oplus_1 \ell_1 \oplus_1 \ell_1 \oplus_1 Z, \quad X^{**} \equiv \ell_\infty \oplus_{\infty} \ell_\infty \oplus_{\infty} \ell_\infty \oplus_{\infty} Z^* . \]

- \[ A = \{ (e_n, 0, 0, 0) : n \in \mathbb{N} \} \cup \{ (0, e_n, 0, 0) : n \in \mathbb{N} \} \cup \{ (0, 0, e_n, 0) : n \in \mathbb{N} \} \subset X^*. \]

- Then \( B_{X^*} = \overline{\text{co}}^w (A) \) and
  \[ |x^{**}(a)| = 1 \quad \forall x^{**} \in \text{ext}(B_{X^{**}}) \forall a \in A. \]

- Fix \( T \in L(X), \varepsilon > 0. \) Find \( a \in A \) with \( \|T^*(a)\| > \|T^*\| - \varepsilon. \)
Numerical index and duality. Proof of main example

\[ X = \{ (x, y, z) \in c \oplus c \oplus c : \lim x + \lim y + \lim z = 0 \} : \]

\[ n(X) = 1 \quad \text{but} \quad n(X^*) < 1. \]

**Proof**

- \( c^* = \ell_1 \oplus \mathbb{K} \lim \Rightarrow X^* = [c^* \oplus c^* \oplus c^*] / (\lim, \lim, \lim). \)

- Then, writing \( Z = \ell_1^{(3)} / (1, 1, 1) \), we can identify

\[ X^* \equiv \ell_1 \oplus \ell_1 \oplus \ell_1 \oplus \ell_1 \oplus Z, \quad X^{**} \equiv \ell_\infty \oplus \ell_\infty \oplus \ell_\infty \oplus \ell_\infty \oplus Z^*. \]

- \( A = \{(e_n, 0, 0, 0) : n \in \mathbb{N}\} \cup \{(0, e_n, 0, 0) : n \in \mathbb{N}\} \cup \{(0, 0, e_n, 0) : n \in \mathbb{N}\} \subset X^*. \)

- Then \( B_{X^*} = \overline{\text{aco}}(A) \) and

\[ |x^{**}(a)| = 1 \quad \forall x^{**} \in \text{ext}(B_{X^{**}}) \quad \forall a \in A. \]

- Fix \( T \in L(X), \varepsilon > 0. \) Find \( a \in A \) with \( \|T^*(a)\| > \|T^*\| - \varepsilon. \)

- Then we find \( x^{**} \in \text{ext}(B_{X^{**}}) \) such that

\[ |x^{**}(T^*(a))| = \|T^*(a)\| > \|T^*\| - \varepsilon. \]
Numerical index and duality. Proof of main example

\begin{align*}
X &= \{(x, y, z) \in c \oplus_{\infty} c \oplus_{\infty} c : \lim x + \lim y + \lim z = 0\}:
\end{align*}

\begin{align*}
n(X) &= 1 \quad \text{but} \quad n(X^*) < 1.
\end{align*}

Proof

- \( c^* = \ell_1 \oplus_1 K \lim \rightarrow X^* = [c^* \oplus_1 c^* \oplus_1 c^*] / (\lim, \lim, \lim) \).
- Then, writing \( Z = \ell_1^{(3)} / (1, 1, 1) \), we can identify
  \begin{align*}
  X^* &\equiv \ell_1 \oplus_1 \ell_1 \oplus_1 \ell_1 \oplus_1 Z, \\
  X^{**} &\equiv \ell_\infty \oplus_\infty \ell_\infty \oplus_\infty \ell_\infty \oplus_\infty Z^*.
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- Then \( B_{X^*} = \overline{\text{aco } w^*}(A) \) and
  \begin{align*}
  |x^{**}(a)| = 1 \quad \forall x^{**} \in \text{ext}(B_{X^{**}}) \quad \forall a \in A.
  \end{align*}

- Fix \( T \in L(X) \), \( \varepsilon > 0 \). Find \( a \in A \) with \( \|T^*(a)\| > \|T^*\| - \varepsilon \).
- Then we find \( x^{**} \in \text{ext}(B_{X^{**}}) \) such that
  \begin{align*}
  |x^{**}(T^*(a))| = \|T^*(a)\| > \|T^*\| - \varepsilon.
  \end{align*}

- Since \( |x^{**}(a)| = 1 \), this gives that \( v(T^*) > \|T^*\| - \varepsilon \), so \( v(T) = \|T\| \) and \( n(X) = 1 \). \( \checkmark \)
Numerical index and duality. Proof of main example

\[ X = \{ (x, y, z) \in c \oplus c \oplus c : \lim x + \lim y + \lim z = 0 \} : \]
\[ \quad n(X) = 1 \quad \text{but} \quad n(X^*) < 1. \]

**Proof**

- \( c^* = \ell_1 \oplus_1 \mathbf{K} \lim \implies X^* = [c^* \oplus_1 c^* \oplus_1 c^*] / (\lim, \lim, \lim). \)
- Then, writing \( Z = \ell_1^{(3)} / (1, 1, 1), \) we can identify
  \[ X^* \equiv \ell_1 \oplus_1 \ell_1 \oplus_1 \ell_1 \oplus_1 Z, \quad X^{**} \equiv \ell_\infty \oplus_\infty \ell_\infty \oplus_\infty \ell_\infty \oplus_\infty Z^*. \]
- \( Z \) is an \( L \)-summand of \( X^* \) so
  \[ n(X^*) = n(Z). \]
Numerical index and duality. Proof of main example

\[ X = \{ (x, y, z) \in c \oplus_\infty c \oplus_\infty c : \lim x + \lim y + \lim z = 0 \} : \]

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- \( Z \) is an \( L \)-summand of \( X^* \) so

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- But \( n(Z) < 1 ! \) \( \checkmark \)
Numerical index and duality. Proof of main example

\[ X = \left\{ (x, y, z) \in c \oplus_\infty c \oplus_\infty c : \lim x + \lim y + \lim z = 0 \right\} : \]
\[ n(X) = 1 \quad \text{but} \quad n(X^*) < 1. \]

Proof

- \( c^* = \ell_1 \oplus_1 K \lim \implies X^* = [c^* \oplus_1 c^* \oplus_1 c^*] / (\lim, \lim, \lim). \)
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- \( Z \) is an \( L \)-summand of \( X^* \) so
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- But \( n(Z) < 1! \) ✓

\[ \text{Figure: } B_Z \]
The above example can be squeezed to get more counterexamples.
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**Example 1**

- Exists $X$ real with $n(X) = 1$ and $n(X^*) = 0$.
- Exists $X$ complex with $n(X) = 1$ and $n(X^*) = 1/e$. 
The above example can be squeezed to get more counterexamples.

Example 1

- Exists $X$ real with $n(X) = 1$ and $n(X^*) = 0$.
- Exists $X$ complex with $n(X) = 1$ and $n(X^*) = 1/e$.

Example 2

- Given $t \in ]0, 1]$, exists $X$ real with $n(X) = t$ and $n(X^*) = 0$.
- Given $t \in ]1/e, 1]$, exists $X$ complex with $n(X) = 1$ and $n(X^*) = 1/e$. 
Some positive partial answers

\[ n(X) = n(X^*) \]

when \( X \) is reflexive (evident).

\( X \) is a \( C^* \)-algebra or a von Neumann predual (1970’s – 2000’s).

\( X \) is \( L \)-embedded in \( X^{**} \) (M., 2009).

If \( X \) has RNP and \( n(X) = 1 \), then \( n(X^*) = 1 \) (M., 2002).

If \( X \) is \( M \)-embedded in \( X^{**} \) and \( n(X) = 1 \), then \( n(Y) = 1 \) for \( X \subseteq Y \subseteq X^{**} \).

Example \( X = C_K(\ell_2([0, 1])) \). Then \( n(X) = 1 \) and

\[ X^* \equiv K(\ell_2) \oplus 1 C_0(K \parallel \Delta) \]

and

\[ X^{**} \equiv L(\ell_2) \oplus \infty C_0(K \parallel \Delta) \]

Therefore, \( X^{**} \) is a \( C^* \)-algebra, but \( n(X^*) = 1/2 < n(X) = 1 \).
Some positive partial answers

One has \( n(X) = n(X^*) \) when

- \( X \) is reflexive (evident).
Some positive partial answers

One has $n(X) = n(X^*)$ when

- $X$ is reflexive (evident).
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Some positive partial answers

One has $n(X) = n(X^*)$ when

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Numerical index and duality (III)

Some positive partial answers

One has \( n(X) = n(X^*) \) when

- \( X \) is reflexive (evident).
- \( X \) is a \( C^* \)-algebra or a von Neumann predual (1970’s – 2000’s).
- \( X \) is \( L \)-embedded in \( X^{**} \) (M., 2009).
- If \( X \) has RNP and \( n(X) = 1 \), then \( n(X^*) = 1 \) (M., 2002).
Some positive partial answers

One has $n(X) = n(X^*)$ when

- $X$ is reflexive (evident).
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- If $X$ has RNP and $n(X) = 1$, then $n(X^*) = 1$ (M., 2002).
- If $X$ is $M$-embedded in $X^{**}$ and $n(X) = 1$
  \[ \implies n(Y) = 1 \text{ for } X \subseteq Y \subseteq X^{**}. \]
Some positive partial answers

One has $n(X) = n(X^*)$ when

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  $\implies n(Y) = 1$ for $X \subseteq Y \subseteq X^{**}$.

Example

$X = C_K(\ell_2)([0, 1] \Vert \Delta)$. Then $n(X) = 1$ and

$$X^* \equiv K(\ell_2)^* \oplus_1 C_0(K \Vert \Delta)^* \quad \text{and} \quad X^{**} \equiv L(\ell_2) \oplus_\infty C_0(K \Vert \Delta)^{**}.$$  

Therefore, $X^{**}$ is a $C^*$-algebra, but $n(X^*) = 1/2 < n(X) = 1$. 
Numerical index and duality: open problems

Main question

Find isometric or isomorphic properties assuring that $\pi(X) = \pi(X^*)$.

Question 1

If $Z$ has a unique predual $X$, does $\pi(X) = \pi(X^*)$?

Question 2

If $Z$ is a dual space, does there exist a predual $X$ such that $\pi(X) = \pi(X^*)$?

Question 4

If $X$ has the RNP, does $\pi(X) = \pi(X^*)$?
Main question

Find isometric or isomorphic properties assuring that $n(X) = n(X^*)$. 
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### Main question

Find isometric or isomorphic properties assuring that $n(X) = n(X^*)$.

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### Question 2

$Z$ dual space, does there exists a predual $X$ such that $n(X) = n(X^*)$?

### Question 4

If $X$ has the RNP, does $n(X) = n(X^*)$?
The isomorphic point of view

Renorming and numerical index (Finet–M.–Payà, 2003)

\( (X, \| \cdot \|) \) (separable or reflexive) Banach space. Then

Real case:

\[ [0, 1] \subseteq \{ n(\mathcal{N}(\mathcal{D})) : |\cdot| \simeq \| \cdot \| \} \]

Complex case:

\[ [e^{-1}, 1] \subseteq \{ n(\mathcal{N}(\mathcal{D})) : |\cdot| \simeq \| \cdot \| \} \]

Open question

The result is known to be true when \( X \) has a long biorthogonal system. Is it true in general?

Remark

In some sense, any other value of \( n(\mathcal{N}(\mathcal{D})) \) but 1 is isomorphically trivial.

⋆ What about the value 1?
Renorming and numerical index (Finet–M.–Payá, 2003)

\((X, \| \cdot \|)\) (separable or reflexive) Banach space. Then
The isomorphic point of view

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Remark

In some sense, any other value of \(n(X)\) but 1 is isomorphically trivial.

★ What about the value 1?
Numerical index

Recall that $X$ has **numerical index one** ($n(X) = 1$) iff

$$\|T\| = \sup \{|x^*(Tx)| : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}$$

(i.e. $v(T) = \|T\|$) for every $T \in L(X)$. 

Numerical index Banach spaces with numerical index one

Recall that $X$ has **numerical index one** ($n(X) = 1$) iff

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\|T\| = \sup \{|x^*(Tx)| : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}
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(i.e. $\nu(T) = \|T\|$) for every $T \in L(X)$.

**Observation**

For Hilbert spaces, the above formula is equivalent to

$$
\|T\| = \sup \{|\langle Tx, x \rangle| : x \in S_X\}
$$

which is known to be valid for every self-adjoint operator $T$. 
Recall that $X$ has numerical index one ($n(X) = 1$) iff

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Observation

For Hilbert spaces, the above formula is equivalent to

$$\|T\| = \sup \{ |\langle Tx, x \rangle| : x \in S_X \}$$

which is known to be valid for every self-adjoint operator $T$.

Examples

$C(K), L_1(\mu), A(\mathbb{D}), H^\infty, \text{finite-codimensional subspaces of } C[0,1] \ldots$
Isomorphic properties (prohibitive results)

Question

Does every Banach space admit an equivalent norm with numerical index 1?
Isomorphic properties (prohibitive results)

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Negative answer (López–M.–Payá, 1999)

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Not every real Banach space can be renormed to have numerical index 1. Concretely:

- If $X$ is real, reflexive, and $\dim(X) = \infty$, then $n(X) < 1$. 

More details on this later on.
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- Actually, if $X$ is real, $X^{**}/X$ separable and $n(X) = 1$, then $X$ is finite-dimensional.
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- Actually, if $X$ is real, $X^{**}/X$ separable and $n(X) = 1$, then $X$ is finite-dimensional.
- Moreover, if $X$ is real, RNP, $\dim(X) = \infty$, and $n(X) = 1$, then $X \supset \ell_1$.
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Not every real Banach space can be renormed to have numerical index 1. Concretely:

- If $X$ is real, reflexive, and $\dim(X) = \infty$, then $n(X) < 1$.
- Actually, if $X$ is real, $X^{**}/X$ separable and $n(X) = 1$, then $X$ is finite-dimensional.
- Moreover, if $X$ is real, RNP, $\dim(X) = \infty$, and $n(X) = 1$, then $X \supset \ell_1$.

**A very recent result (Avilés–Kadets–M.–Merí–Shepelska)**

If $X$ is real, $\dim(X) = \infty$ and $n(X) = 1$, then $X^* \supset \ell_1$.

**More details on this later on.**
Lemma

Let $\mathcal{B}X$ be a Banach space, and $n(\mathcal{B}X) = 1$. Then $\|x^* - 0(x_0)\| = 1$ for all $x^* \in \text{ext}(\mathcal{B}X^*)$ and all denting point $x_0$ of $\mathcal{B}X$.

Proof:

Fix $\varepsilon > 0$. As $x_0$ is a denting point, there exists $y^* \in S_{\mathcal{B}X^*}$ and $\alpha > 0$ such that $\|z - x_0\| < \varepsilon$ whenever $z \in \mathcal{B}X^*$ satisfies $\Re(y^*(z)) > 1 - \alpha$.

(Choquet's lemma): $x^* \in \text{ext}(\mathcal{B}X^*)$, there exists $y \in S_{\mathcal{B}X}$ and $\beta > 0$ such that $|z^*(x_0) - x^*_0(x_0)| < \varepsilon$ whenever $z^* \in \mathcal{B}X^*$ satisfies $\Re(z^*(y)) > 1 - \beta$.

Let $T = y^* \otimes y \in L(\mathcal{B}X)$. Then $\|T\| = 1 = \|v(T)\|$. We may find $x \in S_{\mathcal{B}X}$, $x^* \in S_{\mathcal{B}X^*}$ such that $x^*_0(x_0) = 1$ and $|x^*_0(Tx) - x^*_0(x_0)| > 1 - \min\{\alpha, \beta\}$.

By choosing suitable $s, t \in T$ we have $\Re(y^*(sx)) = |y^*(x_0)| > 1 - \alpha$ and $\Re(tx^* (y)) = |x^*_0(y)| > 1 - \beta$.

It follows that $\|sx - x_0\| < \varepsilon$ and $|tx^* (x_0) - x^*_0(x_0)| < \varepsilon$, and so $1 - |x^*_0(x_0)| \leq |tx^*_0(sx) - x^*_0(x_0)| \leq |tx^*_0(sx) - tx^*_0(x_0)| + |tx^*_0(x_0)| < 2\varepsilon$.\[\square\]
### Lemma

Let $X$ be a Banach space with $n(X) = 1$. Then for all $x^*_0 \in \text{ext}(B_{X^*})$ and all denting point $x_0$ of $B_X$, we have $|x^*_0(x_0)| = 1$. 

Proof: Fix $\varepsilon > 0$. As $x_0$ is a denting point, there exist $y^* \in S_{X^*}$ and $\alpha > 0$ such that $\|z - x_0\| < \varepsilon$ whenever $z \in B_{X^*}$ satisfies $\text{Re} y^*(z) > 1 - \alpha$. (Choquet's lemma: $x^*_0 \in \text{ext}(B_{X^*})$, then $\exists y \in S_X$ and $\beta > 0$ such that $|z^*(y) - x^*_0(y)| < \varepsilon$ whenever $z \in B_{X^*}$ satisfies $\text{Re} z^*(y) > 1 - \beta$.)

Let $T = y^* \otimes y \in L(X)$. Then $\|T\| = 1$ implies $v(T) = 1$. We may find $x \in S_X$, $x^* \in S_{X^*}$ such that $x^*(x) = 1$ and $|x^*(Tx)| = |y^*(x)| > 1 - \min\{\alpha, \beta\}$. By choosing suitable $s, t \in T$ we have $\text{Re} y^*(sx) = |y^*(x)| > 1 - \alpha$ and $\text{Re} tx^*(y) = |x^*(y)| > 1 - \beta$. It follows that $\|sx - x_0\| < \varepsilon$ and $|tx^*(x_0) - x^*_0(x_0)| < \varepsilon$, and so $1 - |x^*_0(x_0)| \leq |tx^*(sx) - x^*_0(x_0)| \leq |tx^*(sx)| + |tx^*(x_0)| < 2\varepsilon$. 

\hfill ✓
Lemma

Let \( X \) be a Banach space, \( n(X) = 1 \).

\[ |x^*_0(x_0)| = 1 \] for all \( x^*_0 \in \text{ext}(B_{X^*}) \) and all denting point \( x_0 \) of \( B_X \).

Proof:

Fix \( \varepsilon > 0 \).

As \( x_0 \) is a denting point, there exists \( y^* \in S_{X^*} \) and \( \alpha > 0 \) such that

\[ \|z - x_0\| < \varepsilon \] whenever \( z \in B_{X^*} \) satisfies

\[ \text{Re} y^*(z) > 1 - \alpha. \]

(Choquet's lemma):

\( x^*_0 \in \text{ext}(B_{X^*}) \),

there exists \( y \in S_X \) and \( \beta > 0 \) such that

\[ |z^*(x_0) - x^*_0(x_0)| < \varepsilon \] whenever \( z^* \in B_{X^*} \) satisfies

\[ \text{Re} z^*(y) > 1 - \beta. \]

Let \( T = y^* \otimes y \in L(X) \).

\[ \|T\| = 1 = \Rightarrow v(T) = 1. \]

We may find \( x \in S_X \), \( x^* \in S_{X^*} \), such that

\[ x^*(x) = 1 \] and

\[ |x^*_0(Tx)| > 1 - \min\{\alpha, \beta\}. \]

By choosing suitable \( s \), \( t \in T \) we have

\[ \text{Re} y^*(sx) = |y^*(x)| > 1 - \alpha \]

and

\[ \text{Re} tx^*(y) = |x^*(y)| > 1 - \beta. \]

It follows that

\[ \|sx - x_0\| < \varepsilon \] and

\[ |tx^*(x_0) - x^*_0(x_0)| < \varepsilon, \]

and so

\[ 1 - |x^*_0(x_0)| \leq |tx^*(sx) - x^*_0(x_0)(x_0)| \leq \varepsilon. \]

\[ \blacksquare \]
Lemma

Let $X$ be a Banach space, $n(X) = 1$.

Then $|x^*_0(x_0)| = 1$ for all $x^*_0 \in \text{ext}(B_{X^*})$ and all denting point $x_0$ of $B_X$.

Proof:

- Fix $\epsilon > 0$. AS $x_0$ denting point, $\exists y^* \in S_{X^*}$ and $\alpha > 0$ such that
  
  $$\|z - x_0\| < \epsilon \quad \text{whenever } z \in B_{X^*} \text{ satisfies } \Re y^*(z) > 1 - \alpha.$$
Proving the 1999 results (I)

Lemma

Let $X$ be a Banach space, $n(X) = 1$ implies $|x^*_0(x_0)| = 1$ for all $x^*_0 \in \text{ext}(B_{X^*})$ and all denting point $x_0$ of $B_X$.

Proof:

- Fix $\varepsilon > 0$. As $x_0$ denting point, $\exists y^* \in S_{X^*}$ and $\alpha > 0$ such that $\|z - x_0\| < \varepsilon$ whenever $z \in B_{X^*}$ satisfies $\text{Re} \ y^*(z) > 1 - \alpha$.

- (Choquet's lemma): $x^*_0 \in \text{ext}(B_{X^*})$, $\exists y \in S_X$ and $\beta > 0$ such that $|z^*(x_0) - x^*_0(x_0)| < \varepsilon$ whenever $z^* \in B_{X^*}$ satisfies $\text{Re} \ z^*(y) > 1 - \beta$. 

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Lemma

X Banach space, \( n(X) = 1 \)
\[ \implies |x_0^*(x_0)| = 1 \text{ for all } x_0^* \in \text{ext } (B_{X^*}) \text{ and all denting point } x_0 \text{ of } B_X. \]

Proof:
- Fix \( \varepsilon > 0 \). AS \( x_0 \) denting point, \( \exists y^* \in S_{X^*} \) and \( \alpha > 0 \) such that
  \[ \|z - x_0\| < \varepsilon \quad \text{whenever } z \in B_{X^*} \text{ satisfies } \text{Re } y^*(z) > 1 - \alpha. \]
- (Choquet’s lemma): \( x_0^* \in \text{ext } (B_{X^*}) \), \( \exists y \in S_X \) and \( \beta > 0 \) such that
  \[ |z^*(x_0) - x_0^*(x_0)| < \varepsilon \quad \text{whenever } z^* \in B_{X^*} \text{ satisfies } \text{Re } z^*(y) > 1 - \beta. \]
- Let \( T = y^* \otimes y \in L(X) \). \( \|T\| = 1 \implies v(T) = 1. \)
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\[ \implies |x_0^*(x_0)| = 1 \quad \text{for all} \quad x_0^* \in \text{ext} \,(B_{X^*}) \quad \text{and all denting point} \quad x_0 \quad \text{of} \quad B_X. \]

Proof:
- Fix \( \varepsilon > 0 \). AS \( x_0 \) denting point, \( \exists y^* \in S_{X^*} \) and \( \alpha > 0 \) such that
  \[ \|z - x_0\| < \varepsilon \quad \text{whenever} \quad z \in B_{X^*} \quad \text{satisfies} \quad \Re y^*(z) > 1 - \alpha. \]
- (Choquet’s lemma): \( x_0^* \in \text{ext} \,(B_{X^*}) \), \( \exists y \in S_X \) and \( \beta > 0 \) such that
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- Let \( T = y^* \otimes y \in L(X) \). \( \|T\| = 1 \implies v(T) = 1. \)
- We may find \( x \in S_X, \ x^* \in S_{X^*}, \) such that
  \[ x^*(x) = 1 \quad \text{and} \quad |x^*(Tx)| = |y^*(x)||x^*(y)| > 1 - \min\{\alpha, \beta\}. \]
Lemma

Let $X$ be a Banach space, $n(X) = 1$ implies $|x_0^*(x_0)| = 1$ for all $x_0^* \in \text{ext}(B_{X^*})$ and all denting point $x_0$ of $B_X$.

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- Fix $\varepsilon > 0$. AS $x_0$ denting point, $\exists y^* \in S_{X^*}$ and $\alpha > 0$ such that $\|z - x_0\| < \varepsilon$ whenever $z \in B_{X^*}$ satisfies $\text{Re} y^*(z) > 1 - \alpha$.
- (Choquet’s lemma): $x_0^* \in \text{ext}(B_{X^*})$, $\exists y \in S_X$ and $\beta > 0$ such that $|z^*(x_0) - x_0^*(x_0)| < \varepsilon$ whenever $z^* \in B_{X^*}$ satisfies $\text{Re} z^*(y) > 1 - \beta$.
- Let $T = y^* \otimes y \in L(X)$. $\|T\| = 1 \implies v(T) = 1$.
- We may find $x \in S_X$, $x^* \in S_{X^*}$, such that $x^*(x) = 1$ and $|x^*(Tx)| = |y^*(x)||x^*(y)| > 1 - \min\{\alpha, \beta\}$.
- By choosing suitable $s, t \in \mathbb{T}$ we have $\text{Re} y^*(sx) = |y^*(x)| > 1 - \alpha$ and $\text{Re} tx^*(y) = |x^*(y)| > 1 - \beta$. 

Miguel Martín (University of Granada (Spain)) Numerical index theory Bangalore, June 2009
Lemma

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- (Choquet’s lemma): $x^*_0 \in \text{ext } (B_{X^*})$, $\exists y \in S_X$ and $\beta > 0$ such that $|z^*(x_0) - x^*_0(x_0)| < \varepsilon$ whenever $z^* \in B_{X^*}$ satisfies $\text{Re } z^*(y) > 1 - \beta$.
- Let $T = y^* \otimes y \in L(X)$. $\|T\| = 1 \implies v(T) = 1$.
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Let $X$ be a Banach space, $n(X) = 1$.

$\implies |x_0^*(x_0)| = 1$ for all $x_0^* \in \text{ext}(B_{X^*})$ and all denting point $x_0$ of $B_X$.

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- (Choquet’s lemma): $x_0^* \in \text{ext}(B_{X^*})$, $\exists y \in S_X$ and $\beta > 0$ such that $|z^*(x_0) - x_0^*(x_0)| < \varepsilon$ whenever $z^* \in B_{X^*}$ satisfies $\text{Re} z^*(y) > 1 - \beta$.

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- It follows that $\|sx - x_0\| < \varepsilon$ and $|tx^*(x_0) - x_0^*(x_0)| < \varepsilon$, and so

$$1 - |x_0^*(x_0)| \leq |tx^*(sx) - x_0^*(x_0)| \leq |tx^*(sx) - tx^*(x_0)| + |tx^*(x_0) - x_0^*(x_0)| < 2\varepsilon. \checkmark$$
Numerical index Banach spaces with numerical index one

Proving the 1999 results (II)

**Proposition**

Let $X$ be a real Banach space, $A \subset S_X$ be infinite with $|x^*| = 1$ for all $x^* \in \text{ext}(B_{X^*})$, and $a \in A$. Then $X \supseteq c_0$ or $X \supseteq \ell_1$.

**Proof:**

$X \supseteq \ell_1 \checkmark$ (Rosenthal $\ell_1$-theorem): Otherwise, $\exists \{a_n\} \subseteq A$ non-trivial weak Cauchy. Consider $Y$ the closed linear span of $\{a_n : n \in \mathbb{N}\}$.

$$\|a_n - a_m\| = 2$$ if $n \neq m$ $\Rightarrow \dim(Y) = \infty$.

(Krein-Milman theorem): every $y^* \in \text{ext}(B_{Y^*})$ has an extension which belongs to $\text{ext}(B_{X^*})$.

So, $|y^*(a_n)| = 1$ for all $y^* \in \text{ext}(B_{Y^*})$, $n \in \mathbb{N}$.

$\{a_n\}$ weak Cauchy $\Rightarrow \{y^*(a_n)\}$ is eventually $1$ or $-1$.

Then $\text{ext}(B_{Y^*}) = \bigcup_{k \in \mathbb{N}} (E_k \cup -E_k)$ where $E_k = \{y^* \in \text{ext}(B_{Y^*}) : y^*(a_n) = 1 \text{ for } n \geq k\}$.

$\{a_n\}$ separates points of $Y^*$ $\Rightarrow E_k$ finite, so $\text{ext}(B_{Y^*})$ countable.

(Fonf): $Y \supseteq c_0$. So, $X \supseteq c_0$. $\checkmark$
Proposition

\( X \) real, \( A \subset S_X \) infinite with \( |x^*(a)| = 1 \ \forall x^* \in \text{ext} (B_X^*) \), \( \forall a \in A \).

\( \Rightarrow \quad X \supseteq c_0 \) or \( X \supseteq \ell_1 \).
Proposition

\[ X \text{ real}, \ A \subset S_X \text{ infinite with } |x^*(a)| = 1 \ \forall x^* \in \text{ext} (B_{X^*}), \ \forall a \in A. \]

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$X$ real, $A \subset S_X$ infinite with $|x^*(a)| = 1 \ \forall x^* \in \text{ext } (B_{X^*})$, $\forall a \in A$.

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Proof:

- $X \supseteq \ell_1$ ✔
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Let $X$ be real, $A \subset S_X$ infinite such that $|x^*(a)| = 1 \ \forall x^* \in \text{ext} \ (B_X^*)$, $\forall a \in A$.

$$\implies X \supseteq c_0 \text{ or } X \supseteq \ell_1.$$ 

**Proof:**

- $X \supseteq \ell_1$ ✔️

- (Rosenthal $\ell_1$-theorem): Otherwise, $\exists \{a_n\} \subset A$ non-trivial weak Cauchy.
Proposition

\[ X \text{ real, } A \subset S_X \text{ infinite with } |x^*(a)| = 1 \quad \forall x^* \in \text{ext} (B_{X^*}), \quad \forall a \in A. \]

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Proof:

- \( X \supseteq \ell_1 \checkmark \)
- **(Rosenthal \( \ell_1 \)-theorem):** Otherwise, \( \exists \{a_n\} \subset A \) non-trivial weak Cauchy.
- Consider \( Y \) the closed linear span of \( \{a_n : n \in \mathbb{N}\} \).
Proposition

\( \text{real, } A \subset S_X \text{ infinite with } |x^*(a)| = 1 \ \forall x^* \in \text{ext (}B_{X^*}\text{)}, \ \forall a \in A. \)
\( \implies X \supseteq c_0 \text{ or } X \supseteq \ell_1. \)

Proof:

- \( X \supseteq \ell_1 \checkmark \)
- \( (\text{Rosenthal } \ell_1\text{-theorem}): \) Otherwise, \( \exists \ \{a_n\} \subseteq A \) non-trivial weak Cauchy.
- Consider \( Y \) the closed linear span of \( \{a_n \mid n \in \mathbb{N}\} \).
- \( \|a_n - a_m\| = 2 \text{ if } n \neq m \implies \dim(Y) = \infty. \)
Proposition

$X$ real, $A \subset S_X$ infinite with $|x^*(a)| = 1 \ \forall x^* \in \text{ext} \ (B_{X^*}), \ \forall a \in A.$

$\implies X \supseteq c_0 \text{ or } X \supseteq \ell_1.$

Proof:

- $X \supseteq \ell_1 \ \checkmark$
- (Rosenthal $\ell_1$-theorem): Otherwise, $\exists \ \{a_n\} \subseteq A$ non-trivial weak Cauchy.
- Consider $Y$ the closed linear span of $\{a_n : n \in \mathbb{N}\}$.
- $\|a_n - a_m\| = 2 \text{ if } n \neq m \implies \dim(Y) = \infty.$
- (Krein-Milman theorem): every $y^* \in \text{ext} \ (B_{Y^*})$ has an extension which belongs to $\text{ext} \ (B_{X^*})$. 

\[\text{(Fonf)}: Y \supseteq c_0. \text{ So, } X \supseteq c_0. \ \checkmark\]
Proposition

\( X \) real, \( A \subset S_X \) infinite with \( |x^*(a)| = 1 \ \forall x^* \in \text{ext} (B_{X^*}), \ \forall a \in A. \)
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Proof:

- \( X \supseteq \ell_1 \)
- (Rosenthal \( \ell_1 \)-theorem): Otherwise, \( \exists \{a_n\} \subseteq A \) non-trivial weak Cauchy.
- Consider \( Y \) the closed linear span of \( \{a_n \ : \ n \in \mathbb{N}\} \).
- \( \|a_n - a_m\| = 2 \text{ if } n \neq m \implies \dim(Y) = \infty. \)
- (Krein-Milman theorem): every \( y^* \in \text{ext} (B_{Y^*}) \) has an extension which belongs to \( \text{ext} (B_{X^*}) \).
- So, \( |y^*(a_n)| = 1 \ \forall y^* \in \text{ext} (B_{Y^*}), \ \forall n \in \mathbb{N}. \)
Proposition

$X$ real, $A \subset S_X$ infinite with $|x^*(a)| = 1 \ \forall x^* \in \text{ext}(B_{X^*}), \ \forall a \in A.$

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- $\{a_n\}$ weak Cauchy $\implies \{y^*(a_n)\}$ is eventually 1 or $-1.$
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**Proof:**

- $X \supseteq \ell_1$ ✓
- (Rosenthal $\ell_1$-theorem): Otherwise, $\exists \{a_n\} \subset A$ non-trivial weak Cauchy.
- Consider $Y$ the closed linear span of $\{a_n : n \in \mathbb{N}\}$.
- $\|a_n - a_m\| = 2$ if $n \neq m \implies \dim(Y) = \infty$.
- (Krein-Milman theorem): every $y^* \in \text{ext} (B_Y^*)$ has an extension which belongs to $\text{ext} (B_X^*)$.
- So, $|y^*(a_n)| = 1 \ \forall y^* \in \text{ext} (B_Y^*)$, $\forall n \in \mathbb{N}$.
- $\{a_n\}$ weak Cauchy $\implies \{y^*(a_n)\}$ is eventually $1$ or $-1$.
- Then $\text{ext} (B_Y^*) = \bigcup_{k \in \mathbb{N}} (E_k \cup -E_k)$ where
  $$E_k = \{y^* \in \text{ext} (B_Y^*) : y^*(a_n) = 1 \text{ for } n \geq k\}.$$
Proposition

\[ X \text{ real, } A \subset S_X \text{ infinite with } |x^*(a)| = 1 \ \forall x^* \in \text{ext}\ (B_{X^*}), \ \forall a \in A. \]
\[ \implies X \supseteq c_0 \text{ or } X \supseteq \ell_1. \]

Proof:

- \[ X \supseteq \ell_1 \] ✓

- (Rosenthal \( \ell_1 \)-theorem): Otherwise, \( \exists \ \{a_n\} \subseteq A \) non-trivial weak Cauchy.
- Consider \( Y \) the closed linear span of \( \{a_n : n \in \mathbb{N}\} \).
- \( \|a_n - a_m\| = 2 \) if \( n \neq m \) \( \implies \dim(Y) = \infty. \)

- (Krein-Milman theorem): every \( y^* \in \text{ext}\ (B_{Y^*}) \) has an extension which belongs to \( \text{ext}\ (B_{X^*}) \).
- So, \( |y^*(a_n)| = 1 \ \forall y^* \in \text{ext}\ (B_{Y^*}), \forall n \in \mathbb{N}. \)
- \( \{a_n\} \) weak Cauchy \( \implies \{y^*(a_n)\} \) is eventually 1 or \(-1.\)
- Then \( \text{ext}\ (B_{Y^*}) = \bigcup_{k \in \mathbb{N}} (E_k \cup -E_k) \) where
  \[ E_k = \{y^* \in \text{ext}\ (B_{Y^*}) : y^*(a_n) = 1 \text{ for } n \geq k\}. \]
  \( \{a_n\} \) separates points of \( Y^* \) \( \implies E_k \) finite, so \( \text{ext}\ (B_{Y^*}) \) countable.
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\( X \) real, \( A \subset S_X \) infinite with \( |x^*(a)| = 1 \ \forall x^* \in \text{ext} (B_{X^*}), \forall a \in A. \)

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Proof:

- \( X \supseteq \ell_1 \) ✅
- (Rosenthal \( \ell_1 \)-theorem): Otherwise, \( \exists \{a_n\} \subseteq A \) non-trivial weak Cauchy.
- Consider \( Y \) the closed linear span of \( \{a_n : n \in \mathbb{N}\} \).
- \( \|a_n - a_m\| = 2 \) if \( n \neq m \) \( \implies \dim(Y) = \infty. \)
- (Krein-Milman theorem): every \( y^* \in \text{ext} (B_{Y^*}) \) has an extension which belongs to \( \text{ext} (B_{X^*}). \)
- So, \( |y^*(a_n)| = 1 \ \forall y^* \in \text{ext} (B_{Y^*}), \forall n \in \mathbb{N}. \)
- \( \{a_n\} \) weak Cauchy \( \implies \{y^*(a_n)\} \) is eventually 1 or \(-1.\)
- Then \( \text{ext} (B_{Y^*}) = \bigcup_{k \in \mathbb{N}} (E_k \cup -E_k) \) where

\[ E_k = \{y^* \in \text{ext} (B_{Y^*}) : y^*(a_n) = 1 \text{ for } n \geq k\}. \]

- \( \{a_n\} \) separates points of \( Y^* \) \( \implies E_k \) finite, so \( \text{ext} (B_{Y^*}) \) countable.
- (Fonf): \( Y \supseteq c_0. \) So, \( X \supseteq c_0. \) ✅
**Lemma**

Let $X$ be a Banach space, $n(X) = 1$.

\[ |x_0^*(x_0)| = 1 \quad \text{for all } x_0^* \in \text{ext}(B_{X^*}) \text{ and all denting point } x_0 \text{ of } B_X. \]

**Proposition**

Let $X$ be real, $A \subset S_X$ infinite with $|x^*(a)| = 1 \quad \forall x^* \in \text{ext}(B_{X^*}), \forall a \in A$.

\[ X \supseteq c_0 \text{ or } X \supseteq \ell_1. \]
Proving the 1999 results (III)

**Lemma**

$X$ Banach space, $n(X) = 1$  
$\implies |x^*_0(x_0)| = 1$ for all $x^*_0 \in \text{ext}(B_{X^*})$ and all denting point $x_0$ of $B_X$.

**Proposition**

$X$ real, $A \subset S_X$ infinite with $|x^*(a)| = 1 \ \forall x^* \in \text{ext}(B_{X^*}), \forall a \in A$.  
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**Main consequence**

$X$ real, RNP, $\text{dim}(X) = \infty$, and $n(X) = 1$  
$\implies X \supseteq \ell_1$. 
Proving the 1999 results (III)

Lemma

$X$ Banach space, $n(X) = 1$
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Proposition

$X$ real, $A \subset S_X$ infinite with $|x^*(a)| = 1 \ \forall x^* \in \text{ext}(B_{X^*}), \forall a \in A$.
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Main consequence

$X$ real, RNP, $\dim(X) = \infty$, and $n(X) = 1$ $\implies X \supseteq \ell_1$.

Proof.
### Lemma

Let $X$ be a Banach space with $n(X) = 1$. Then for all $x_0^* \in \text{ext} (B_{X^*})$ and all denting point $x_0$ of $B_X$,

$$|x_0^*(x_0)| = 1$$

### Proposition

Let $X$ be a real Banach space, $A \subset S_X$ infinite, and for all $x^* \in \text{ext} (B_{X^*})$ and all $a \in A$,

$$|x^*(a)| = 1$$

Then $X \supseteq c_0$ or $X \supseteq \ell_1$.

### Main consequence

If $X$ is a real Banach space, RNP, has infinite dimension, and $n(X) = 1$, then $X \supseteq \ell_1$.

**Proof.**

- If $X$ is RNP, then $X \not\supseteq c_0$. 

Proof completed.
Proving the 1999 results (III)

Lemma

Let $X$ be a Banach space, and $n(X) = 1$. Then $|x_0^*(x_0)| = 1$ for all $x_0^* \in \text{ext}(B_{X^*})$ and all denting point $x_0$ of $B_X$.

Proposition

If $X$ is real, $A \subset S_X$ is infinite with $|x^*(a)| = 1$ for all $x^* \in \text{ext}(B_{X^*})$ and $a \in A$. Then $X \supseteq c_0$ or $X \supseteq \ell_1$.

Main consequence

If $X$ is real, RNP, and $n(X) = 1$, then $X \supseteq \ell_1$.

Proof.

- If $X$ is RNP, and $\dim(X) = \infty$, then there exist infinitely many denting points of $B_X$.
- Therefore, $X \supseteq c_0$ or $X \supseteq \ell_1$. 
Lemma

$X$ Banach space, $n(X) = 1$ \[\implies |x^*_0(x_0)| = 1 \text{ for all } x^*_0 \in \text{ext } (B_{X^*}) \text{ and all denting point } x_0 \text{ of } B_X.\]

Proposition

$X$ real, $A \subset S_X$ infinite with $|x^*(a)| = 1 \ \forall x^* \in \text{ext } (B_{X^*}), \ \forall a \in A$. \[\implies X \supseteq c_0 \text{ or } X \supseteq \ell_1.\]

Main consequence

$X$ real, RNP, $\dim(X) = \infty$, and $n(X) = 1$ \[\implies X \supseteq \ell_1.\]

Proof.

- $X$ RNP, $\dim(X) = \infty \implies \exists$ infinitely many denting points of $B_X$.
- Therefore, $X \supseteq c_0$ or $X \supseteq \ell_1$.
- If $X$ RNP, then $X \not\supseteq c_0$. ✓
Proving the 1999 results (III)

**Lemma**

Let $X$ be a Banach space, $n(X) = 1$  
\[ \Rightarrow |x_0^*(x_0)| = 1 \text{ for all } x_0^* \in \text{ext}(B_{X^*}) \text{ and all denting point } x_0 \text{ of } B_X. \]

**Proposition**

Let $X$ be real, $A \subset S_X$ infinite with $|x^*(a)| = 1 \ \forall x^* \in \text{ext}(B_{X^*}), \ \forall a \in A$.  
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**Main consequence**

If $X$ is real, RNP, $\dim(X) = \infty$, and $n(X) = 1$  
\[ \Rightarrow X \supseteq \ell_1. \]

**Corollary**

If $X$ is real, $\dim(X) = \infty$, $n(X) = 1$.  
- $X$ is not reflexive.  
- $X^{**}/X$ is non-separable.
### Lemma

Let $X$ be a Banach space with $n(X) = 1$. Then $|x_0^*(x_0)| = 1$ for all $x_0^* \in \text{ext}(B_{X^*})$ and all denting point $x_0$ of $B_X$.

### Proposition

Let $X$ be real, and $A \subset S_X$ be infinite with $|x^*(a)| = 1$ for all $x^* \in \text{ext}(B_{X^*})$, $a \in A$. Then $X \supseteq c_0$ or $X \supseteq \ell_1$.

### Main consequence

If $X$ is real, reflexive, and $n(X) = 1$, then $X \supseteq \ell_1$.

### Corollary

Let $X$ be real, and $\dim(X) = \infty$. If $n(X) = 1$, then $X$ is not reflexive and $X^{**} / X$ is non-separable.
Lemma

$X$ Banach space, $n(X) = 1$ 
$\implies |x_0^*(x_0)| = 1$ for all $x_0^* \in \text{ext } (B_{X^*})$ and all denting point $x_0$ of $B_X$.

Proposition

$X$ real, $A \subset S_X$ infinite with $|x^*(a)| = 1 \forall x^* \in \text{ext } (B_{X^*}), \forall a \in A$. 
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$X$ real, RNP, $\text{dim}(X) = \infty$, and $n(X) = 1$ $\implies X \supseteq \ell_1$.

Corollary

$X$ real, $\text{dim}(X) = \infty$, $n(X) = 1$.
- $X$ is not reflexive.
- $X^{**}/X$ is non-separable.
Isomorphic properties (positive results)

- If $X$ is separable, $X \supset c_0$, then $X$ can be renormed to have numerical index 1.

Consequence:
If $X$ is separable containing $c_0$, then there is $Z \cong X$ such that $n(Z) = 1$ and
- $n(Z^*) = 0$ in the real case,
- $n(Z^*) = e^{-1}$ in the complex case.

Open questions:
- Find isomorphic properties which assure renorming with numerical index 1.
- In particular, if $X \supset \ell_1$, can $X$ be renormed to have numerical index 1?

Negative result (Bourgain–Delbaen, 1980):
There is $X$ such that $X^* \cong \ell_1$ and $X$ has the RNP. Then, $X$ cannot be renormed with numerical index 1 (in such a case, $X \supset \ell_1$!)

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Isomorphic properties (positive results)

A renorming result (Boyko–Kadets–M.–Merí, 2009)

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Isometric properties: finite-dimensional spaces

Finite-dimensional spaces (McGregor, 1971; Lima, 1978)

$x$ real or complex finite-dimensional space. TFAE:

1. $X = 1$.
2. $|x^* (x)| = 1$ for every $x^* \in \text{ext} (B_X^*)$, $x \in \text{ext} (B_X)$.

$B_X = aconv (F)$ for every maximal convex subset $F$ of $S_X$ ($X$ is a CL-space).

Remark

This shows a rough behavior of the norm of a finite-dimensional space with numerical index 1:

- The space is not smooth.
- The space is not strictly convex.

Question

What is the situation in the infinite-dimensional case?
Finite-dimensional spaces (McGregor, 1971; Lima, 1978)

Let $X$ be a real or complex finite-dimensional space. Then the following are TFAE:

- $n(X) = 1.$

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What is the situation in the infinite-dimensional case?
Theorem (Kadets–M.–Merí–Payá, 2009)\n\[ X \text{ infinite-dimensional Banach space, } n(X) = 1. \] Then $X^*$ is neither smooth nor strictly convex. The norm of $X$ cannot be Fréchet-smooth. There is no WLUR points in $S_X$. 

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Proving that $X^*$ is not smooth:
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**Proving that $X^*$ is not smooth:**

- If $X$ is smooth, then
  
  $T^*_{n(X)}(x^*_{n(X)}) = x^*_{n(X)}(x) = 0$. Thus,
  
  $\|T^*_{n(X)}\| = 1$.

  But, since $T_{n(X)} \rightarrow T$ and $T^2 = 0$, then
  
  $T_{n(X)}^2 \rightarrow 0$.
Theorem (Kadets–M.–Merí–Payá, 2009)

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- The norm of \( X \) cannot be Fréchet-smooth.
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Proving that \( X^* \) is not smooth:

- \( \dim(X) > 1 \), exists \( x_0 \in S_X \) and \( x_0^* \in S_{X^*} \) such that \( x_0^*(x_0) = 0 \). Then, consider \( T = x_0^* \otimes x_0 \) which satisfies \( T^2 = 0 \), \( \|T\| = 1 \).

- \textbf{(AcostaPayá1993)}: exists \( \{T_n\} \longrightarrow T \) such that
  \( \|T_n\| = 1 \), \( T_n^* \) attains its numerical radius \( v(T_n^*) = v(T_n) = \|T_n\| = 1 \).
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- (AcostaPayá1993): exists $\{T_n\} \rightarrow T$ such that $\|T_n\| = 1$, $T_n^*$ attains its numerical radius $v(T_n^*) = v(T_n) = \|T_n\| = 1$.
- We may find $\lambda_n \in \mathbb{T}$ and $(x_n^*, x_n^{**}) \in S_{X^*} \times S_{X^{**}}$ such that
  \[
  \lambda_n x_n^{**}(x_n^*) = 1 \quad \text{and} \quad [T_n^{**}(x_n^{**})](x_n^*) = x_n^{**}(T_n^*(x_n^*)) = 1.
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- If \( X^* \) is smooth: \( T_n^{**}(x_n^{**}) = \lambda_n x_n^{**} \). Thus,
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Isometric properties: infinite-dimensional spaces

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Corollary

$X = C(T)/A(D)$. $X^* = H^1$ is smooth $\implies n(X) < 1 \& n(H^1) < 1$. 
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**Corollary**

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**Example without completeness**

- There is $X$ (non-complete) strictly convex with $X^* \equiv L_1(\mu)$, so $n(X) = 1$.
- $\tilde{X}$ completion of $X$. For $F \subseteq S_{\tilde{X}}$ maximal face, $B_{\tilde{X}} = \overline{\text{aconv}}(F)$. 

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**Theorem (Kadets–M.–Merí–Payá, 2009)**

\( X \) infinite-dimensional Banach space, \( n(X) = 1 \). Then

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\( X = C(T)/A(D) \). \( X^* = H^1 \) is smooth \( \implies n(X) < 1 \) & \( n(H^1) < 1 \).

**Example without completeness**

- There is \( X \) (non-complete) strictly convex with \( X^* \equiv L_1(\mu) \), so \( n(X) = 1 \).
- \( \tilde{X} \) completion of \( X \). For \( F \subseteq S_{\tilde{X}} \) maximal face, \( B_{\tilde{X}} = \overline{\text{aconv}}(F) \).

**Open question**

Is there \( X \) with \( n(X) = 1 \) which is smooth or strictly convex?
Asymptotic behavior of the set of spaces with numerical index one

Theorem (Oikhberg, 2005)

There is a universal constant $c$ such that

$$\text{dist}(X, \ell^2(m)) \geq c m^{1/4}$$

for every $m \in \mathbb{N}$ and every $m$-dimensional $X$ with $n(X) = 1$.

Old examples

$$\text{dist}(\ell^1(m), \ell^2(m)) = \text{dist}(\ell^\infty(m), \ell^2(m)) = m^{1/2}$$

Open questions

Is there a universal constant $\tilde{c}$ such that

$$\text{dist}(X, \ell^2(m)) \geq \tilde{c} m^{1/2}$$

for every $m \in \mathbb{N}$ and every $m$-dimensional $X$’s with $n(X) = 1$?

What is the diameter of the set of all $m$-dimensional $X$’s with $n(X) = 1$?
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Theorem (Oikhberg, 2005)

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Numerical index Banach spaces with numerical index one

Asymptotic behavior of the set of spaces with numerical index one

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- What is the diameter of the set of all $m$-dimensional $X$'s with $n(X) = 1$?
How to deal with numerical index 1 property?

One the one hand: weaker properties

In a general Banach space, we only can construct compact (actually, finite-rank) operators. Actually, we only may easily calculate the norm of rank-one operators. All the results given before for Banach spaces in which we use numerical index 1 only need \( v(T) = \|T\| \) for every rank-one operator \( T \). This is called the alternative Daugavet property (ADP) and we will present it in the next section.

One the other hand: stronger properties

We do not know any operator-free characterization of Banach spaces with numerical index 1. When we know that a Banach space has numerical index 1 (or that it can be renormed with numerical index 1), we actually prove more. Later we will study sufficient geometrical conditions. The weakest property is called lushness.
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One of the key ideas to get interesting results for Banach spaces with numerical index 1 is to study when the three properties below are equivalent.

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- Lushness
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- ADP

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Relationship between the properties

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Relationship between the properties

- One of the key ideas to get interesting results for Banach spaces with numerical index 1 is to study when the three properties below are equivalent.
- A very interesting property appears: the *slicely countably determination*.
- We will study this property later on.

![Diagram showing the relationship between lushness, numerical index 1, and ADP properties](attachment:image.png)
The alternative Daugavet property

- The Daugavet property
- The alternative Daugavet property
  - Geometric characterizations
  - $C^*$-algebras and preduals
  - Some results

M. Martín and T. Oikberg

An alternative Daugavet property

M. Martín

The alternative Daugavet property of $C^*$-algebras and $JB^*$-triples
The alternative Daugavet property

The Daugavet property: motivation

- In a Banach space $X$ with the \textbf{Radon-Nikodým property} the unit ball has many denting points.
In a Banach space $X$ with the 
Radon-Nikodým property the unit ball has many denting points.

$x \in S_X$ is a denting point of $B_X$ if for every $\varepsilon > 0$ one has

$$x \notin \overline{co}(B_X \setminus (x + \varepsilon B_X)).$$

$B_X \setminus (x + \varepsilon B_X)$
In a Banach space \( X \) with the **Radon-Nikodým property** the unit ball has many denting points.

- \( x \in S_X \) is a **denting point** of \( B_X \) if for every \( \varepsilon > 0 \) one has
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  \]

- \( C[0,1] \) and \( L_1[0,1] \) have an extremely opposite property: for every \( x \in S_X \) and every \( \varepsilon > 0 \)
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  \overline{co} \left( B_X \setminus (x + (2 - \varepsilon)B_X) \right) = B_X.
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The alternative Daugavet property

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- $C[0,1]$ and $L_1[0,1]$ have an extremely opposite property: for every $x \in S_X$ and every $\varepsilon > 0$
  \[ \overline{\text{co}} \left( B_X \setminus (x + (2 - \varepsilon) B_X) \right) = B_X . \]
- This geometric property is equivalent to a property of operators on the space.
The Daugavet property: definition

The Daugavet equation

\( X \) Banach space, \( T \in L(X) \)

\[ \|\text{Id} + T\| = 1 + \|T\| \quad \text{(DE)} \]
The Daugavet property: definition

The Daugavet equation

$X$ Banach space, $T \in L(X)$

$$\|\text{Id} + T\| = 1 + \|T\| \quad \text{(DE)}$$

Classical examples

1. **Daugavet, 1963:**
   Every compact operator on $C[0,1]$ satisfies (DE).

2. **Lozanoskii, 1966:**
   Every compact operator on $L_1[0,1]$ satisfies (DE).

3. **Abramovich, Holub, and more, 80’s:**
   $X = C(K)$, $K$ perfect compact space
   or $X = L_1(\mu)$, $\mu$ atomless measure
   $\implies$ every weakly compact $T \in L(X)$ satisfies (DE).
The Daugavet property: definition

**The Daugavet equation**

A Banach space $X$ is said to have the *Daugavet property* iff every rank-one operator on $X$ satisfies (DE).

★ Then, every weakly compact operator on $X$ satisfies (DE).

The Daugavet property: geometric characterizations

**Theorem [KSSW]**

Let $X$ be a Banach space. The following are equivalent (TFAE):

- $X$ has the Daugavet property.

Every rank-one operator $T \in L(X)$ satisfies

\[ \|\text{Id} + T\| = 1 + \|T\|. \]
The Daugavet property: geometric characterizations

**Theorem [KSSW]**

* X Banach space. TFAE:
  1. X has the Daugavet property.
  2. For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that
     \[ \text{Re } x^*(y) > 1 - \varepsilon \quad \text{and} \quad \|x - y\| \geq 2 - \varepsilon. \]
  3. For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y^* \in S_{X^*}$ such that
     \[ \text{Re } y^*(x) > 1 - \varepsilon \quad \text{and} \quad \|x^* - y^*\| \geq 2 - \varepsilon. \]
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- For every $x \in S_X$ and every $\varepsilon > 0$, we have
  \[ \overline{co} \left( B_X \setminus (x + (2 - \varepsilon)B_X) \right) = B_X. \]
Some propaganda

\( X \) with the Daugavet property. Then:

- \( X \) does not have the Radon-Nikodým property.

\( (\text{Wojtaszczyk, 1992}) \)
The Daugavet property: some results

Some propaganda

$X$ with the Daugavet property. Then:

- $X$ does not have the Radon-Nikodým property.  
  \((Wojtaszczyk, 1992)\)

- Every weakly-open subset of $B_X$ has diameter 2.  
  \((Shvidkoy, 2000)\)
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- $X$ contains a copy of $\ell_1$. $X^*$ contains a copy of $L_1[0,1]$.
  
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The Daugavet property: some results

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- $X$ does not have unconditional basis.
  \[(Kadets, 1996)\]

- $X$ does not embed into a unconditional sum of Banach spaces without a copy of $\ell_1$.
  \[(Shvidkoy, 2000)\]
The alternative Daugavet property

The DPr, the ADP and numerical index 1

Observation (Duncan-McGregor-Price-White, 1970)

For a Banach space $X$, $T \in \mathbb{L}(X)$:

$$\sup \Re \nu(T) = \|T\| \iff \|\text{Id} + T\| = 1 + \|T\|.$$  

$X$ Banach space: Daugavet property (DPr): every rank-one $T$ satisfies $\|\text{Id} + T\| = 1 + \|T\|$ (DE).

Numerical index $1$: every $T$ satisfies $\max_{\theta \in T} \|\text{Id} + \theta T\| = 1 + \|T\|$ (aDE).

The alternative Daugavet property (M.–Oikhberg, 2004)

Alternative Daugavet property (ADP): every rank-one $T \in \mathbb{L}(X)$ satisfies (aDE).

Then, every weakly compact operator satisfies (aDE).
Observation (Duncan-McGregor-Price-White, 1970)

$X$ Banach space, $T \in L(X)$:
Observation (Duncan-McGregor-Price-White, 1970)

Let $X$ be a Banach space, $T \in L(X)$:
\[ \sup \Re V(T) = \|T\| \iff \|\text{Id} + T\| = 1 + \|T\|. \]
The DPr, the ADP and numerical index 1

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\( X \) Banach space, \( T \in L(X) \):

- \( \sup \Re V(T) = \|T\| \iff \|\text{Id} + T\| = 1 + \|T\| \).
- \( v(T) = \|T\| \iff \max_{\theta \in T} \|\text{Id} + \theta T\| = 1 + \|T\| \).
The DPr, the ADP and numerical index 1

Observation (Duncan-McGregor-Price-White, 1970)

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X Banach space:

- **Daugavet property (DPr)**: every rank-one $T$ satisfies

$$\|\text{Id} + T\| = 1 + \|T\| \quad (\text{DE})$$
**The DPr, the ADP and numerical index 1**

**Observation (Duncan-McGregor-Price-White, 1970)**

\[ X \text{ Banach space, } T \in L(X): \]
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**X Banach space:**
- **Daugavet property (DPr):** every rank-one \( T \) satisfies
  \[ \|\text{Id} + T\| = 1 + \|T\| \] (DE)
- **numerical index 1:** every \( T \) satisfies
  \[ \max_{\theta \in T} \|\text{Id} + \theta T\| = 1 + \|T\| \] (aDE)
The DPr, the ADP and numerical index $1$

**Observation (Duncan-McGregor-Price-White, 1970)**

Let $X$ be a Banach space, $T \in L(X)$:

- $\sup \Re V(T) = \|T\| \iff \|\Id + T\| = 1 + \|T\|.$
- $v(T) = \|T\| \iff \max_{\theta \in \mathbb{T}} \|\Id + \theta T\| = 1 + \|T\|.$

**X Banach space:**

- **Daugavet property (DPr):** every rank-one $T$ satisfies
  \[ \|\Id + T\| = 1 + \|T\| \quad \text{(DE)} \]
- **Numerical index 1:** every $T$ satisfies
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**The alternative Daugavet property (M.–Oikhberg, 2004)**

- **Alternative Daugavet property (ADP):** every rank-one $T \in L(X)$ satisfies (aDE).
  - Then, every weakly compact operator satisfies (aDE).
Relations between the properties

Example: $C([0,1], K(\ell_2))$ has DPr, but has not numerical index 1. $c_0$ has numerical index 1, but has not DPr.

Remark: For RNP or Asplund spaces, ADP implies numerical index 1. Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.
Relations between the properties

- Daugavet property \iff Numerical index 1

Examples

- \( C([0,1], K(\ell_2)) \) has DPr, but has not numerical index 1
- \( c_0 \) has numerical index 1, but has not DPr
- \( c_0 \oplus_\infty C([0,1], K(\ell_2)) \) has ADP, neither DPr nor numerical index 1

Remarks

For RNP or Asplund spaces, ADP \( \Rightarrow \) numerical index 1.

Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.
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Remarks
- For RNP or Asplund spaces, ADP \implies numerical index 1.
- Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.
Geometric characterizations of the ADP

**Theorem**

X Banach space. TFAE:
- X has the ADP.

Every rank-one operator $T \in L(X)$ (equivalently, every weakly compact operator) satisfies

$$\max_{|\omega|=1} \|\text{Id} + \omega T\| = 1 + \|T\|.$$
The alternative Daugavet property

Geometric characterizations of the ADP

**Theorem**

* X Banach space. TFAE:
  - X has the ADP.
  - For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that
    $$|x^*(y)| > 1 - \varepsilon \quad \text{and} \quad \|x - y\| \geq 2 - \varepsilon.$$
  - For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y^* \in S_{X^*}$ such that
    $$|y^*(x)| > 1 - \varepsilon \quad \text{and} \quad \|x^* - y^*\| \geq 2 - \varepsilon.$$
Geometric characterizations of the ADP

Theorem

Let $X$ be a Banach space. TFAE:

1. $X$ has the ADP.
2. For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that
   \[ |x^*(y)| > 1 - \varepsilon \quad \text{and} \quad \|x - y\| \geq 2 - \varepsilon. \]
3. For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y^* \in S_{X^*}$ such that
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Geometric characterizations of the ADP

Theorem

\(X\) Banach space. TFAE:

- \(X\) has the ADP.
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- For every \(x \in S_X\), \(x^* \in S_{X^*}\), and \(\varepsilon > 0\), there exists \(y^* \in S_{X^*}\) such that
  \[|y^*(x)| > 1 - \varepsilon \quad \text{and} \quad \|x^* - y^*\| \geq 2 - \varepsilon.\]
- For every \(x \in S_X\) and every \(\varepsilon > 0\), we have
  \[B_X = \overline{\text{co}}\left(\mathbb{T} \{y \in B_X : \|x - y\| \geq 2 - \varepsilon\}\right).\]
Let $V_*$ be the predual of the von Neumann algebra $V$. 
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The Daugavet property of $V_*$ is equivalent to:

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The Daugavet property of $V_*$ is equivalent to:
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$V_*$ has numerical index 1 iff:
- $V$ is commutative, or
- $|v^*(v)| = 1$ for $v \in \text{ext}(B_V)$ and $v^* \in \text{ext}(B_{V^*})$. 

$V = C \oplus \infty N$, where $C$ is commutative and $N$ has no atomic projections.
Let $V_*$ be the predual of the von Neumann algebra $V$.

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- $V$ is commutative, or
- $|v^*(v)| = 1$ for $v \in \text{ext}(B_V)$ and $v^* \in \text{ext}(B_{V^*})$.

The alternative Daugavet property of $V_*$ is equivalent to:
- the atomic projections of $V$ are central, or
- $|v(v_*)| = 1$ for $v \in \text{ext}(B_V)$ and $v_* \in \text{ext}(B_{V^*})$, or
- $V = C \oplus_{\infty} N$, where $C$ is commutative and $N$ has no atomic projections.
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Some results on the ADP: isomorphic properties

Remark

Since when we use the numerical index 1 only rank-one operators may be used, most of the known results are valid for the ADP.

Theorem (L´opez–M.–Pay´a, 1999)

Not every real Banach space can be renormed with the ADP.

\[ X \text{ real reflexive with ADP} \implies X \text{ finite-dimensional.} \]

Moreover, \( X \text{ real, RNP, } \dim(X) = \infty, \text{ and ADP, then } X \supset \ell_1. \)

A very recent result (Avil ´es–Kadets–M.–Mer´ı–Shepelska)

If \( X \) is real, \( \dim(X) = \infty \) and \( X \) has the ADP, then \( X^* \supset \ell_1. \)

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If \( X \) is separable, \( X \supset c_0, \) then \( X \) can be renormed with the ADP.
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Remark

Also some isometric properties of Banach spaces with numerical index 1 are actually true for ADP.

Theorem (Kadets–M.–Merlí–Paya, 2009)

\( X \) infinite-dimensional with the ADP. Then \( X^* \) is neither smooth nor strictly convex. The norm of \( X \) cannot be Fréchet-smooth. There is no WLUR points in \( S_X \).

Corollary

\( X = C(T)/A(D) \). Since \( X^* = H_1 \) is smooth \( \Rightarrow \) nor \( X \) nor \( H_1 \) have the ADP.

Open question

Is there \( X \) with the ADP which is smooth or strictly convex?
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**Lush spaces**

- Definition and examples
- Lush renorming
- Reformulations of lushness and applications
- Lushness is not equivalent to numerical index one

K. Boyko, V. Kadets, M. Martín, and J. Merí.

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Motivation

Remark

Usually, when we show that a Banach space has numerical index 1, we actually prove more. We do not have an operator-free characterization of the spaces with numerical index 1. Hence, it makes sense to study geometrical sufficient conditions.

Some sufficient conditions

Let $X$ be a Banach space. Consider:

(a) Lindenstrauss, 1964: $X$ has the 3.2.I.P. if the intersection of every family of three mutually intersecting balls is not empty.

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$X$ is **lush** if given $x, y \in S_X$, $\varepsilon > 0$, there is $x^* \in S_{X^*}$ such that

$$x \in S(B_X, x^*, \varepsilon) \quad \text{and} \quad \text{dist}(y, a\text{conv}(S(B_X, x^*, \varepsilon))) < \varepsilon.$$
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v = \sum_{i=1}^{n} \lambda_i \theta_i x_i \quad \text{where} \quad x_i \in S(B_X, x^*, \varepsilon), \lambda_i \in [0,1], \sum \lambda_i = 1, \theta_i \in \mathbb{T},
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- Then \( |x^*(Tv)| = |x^*(x_0) - x^*(T(\frac{y_0}{\|Ty_0\|} - v))| \sim \|T\|. \)
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3. Then \(|x^*(Tv)| = \left| x^*(x_0) - x^* \left( T \left( \frac{y_0}{\|Ty_0\|} - v \right) \right) \right| \sim \|T\|\).
4. By a convexity argument, \(\exists i\) such that \(|x^*(Tx_i)| \sim \|T\|\) and \(\text{Re} x^*(x_i) \sim 1.\)
5. Then \(\max_{\omega \in \mathbb{T}} \|\text{Id} + \omega T\| \sim 1 + \|T\| \implies v(T) \sim \|T\|.\) \(\checkmark\)
Examples of lush spaces

   In particular, \( C(K), L^1(\mu), C_0(L) \), . . .

2. Preduals of \( L^1(\mu) \)-spaces.

3. C-rich subspaces
   \( K \) compact, \( X \) subspace of \( C(K) \) is C-rich iff
   \[ \forall U \text{ open nonempty and } \forall \epsilon > 0 \text{ exists } h : K \to [0, 1] \text{ continuous, } \text{supp}(h) \subseteq U \text{ such that } \text{dist}(h, X) < \epsilon. \]

4. More examples of lush spaces
   C-rich subspaces of \( C(K) \).

5. In particular, finite-codimensional subspaces of \( C[0, 1] \).

6. \( C_E(K \| L) \), where \( L \) nowhere dense in \( K \) and \( E \subseteq C(L) \).

7. \( Y \) if \( c_0 \subseteq Y \subseteq \ell_\infty \) (canonical copies).
Examples of lush spaces

- Almost-CL-spaces.
## Examples of lush spaces

2. In particular, \( C(K) \), \( L_1(\mu) \), \( C_0(L) \)…

Miguel Martín (University of Granada (Spain))

Numerical index theory

Bangalore, June 2009
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C-rich subspaces

Let $K$ be compact, $X$ a subspace of $C(K)$, and $X$ is called C-rich iff for every open nonempty set $U$ and every $\varepsilon > 0$ there exists a continuous function $h : K \to [0, 1]$ with support in $U$ such that $\text{dist}(h, X) < \varepsilon$. 

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6. $C_E(K\|L)$, where $L$ nowhere dense in $K$ and $E \subseteq C(L)$. 
Examples of lush spaces

2. In particular, \( C(K), L_1(\mu), C_0(L) \ldots \)
3. Preduals of \( L_1(\mu) \)-spaces.

C-rich subspaces

\( K \) compact, \( X \) subspace of \( C(K) \) is C-rich iff \( \forall U \) open nonempty and \( \forall \varepsilon > 0 \) exists \( h : K \rightarrow [0,1] \) continuous, \( \text{supp}(h) \subseteq U \) such that \( \text{dist}(h, X) < \varepsilon \).

More examples of lush spaces

4. C-rich subspaces of \( C(K) \).
5. In particular, finite-codimensional subspaces of \( C[0,1] \).
6. \( C_E(K\|L) \), where \( L \) nowhere dense in \( K \) and \( E \subseteq C(L) \).
7. \( Y \) if \( c_0 \subseteq Y \subseteq \ell_\infty \) (canonical copies).
Lush renorming

The goal

When we may get a lush equivalent norm?
Lush renorming

The goal

When we may get a lush equivalent norm?

Proposition

X separable, X ⊇ c₀ → exists ∥·∥ ≃ ∥·∥ and T : (X, ∥·∥) → ℓₘ with T isometric embedding & c₀ ⊆ T(X) (canonical copy).
The goal

When we may get a lush equivalent norm?

Proposition

\(X\) separable, \(X \supseteq c_0 \implies\) exists \(\|\cdot\| \simeq \|\cdot\|\) and \(T : (X, \|\cdot\|) \to \ell_\infty\) with \(T\) isometric embedding & \(c_0 \subseteq T(X)\) (canonical copy).

Recall this family of examples of lush spaces

\(Y\) if \(c_0 \subseteq Y \subseteq \ell_\infty\) (canonical copies).
The goal

When we may get a lush equivalent norm?

Proposition

\( X \) separable, \( X \supseteq c_0 \implies \text{exists } \| \cdot \| \simeq \| \cdot \| \text{ and } T : (X, \| \cdot \|) \to \ell_\infty \) with 
\( T \) isometric embedding & \( c_0 \subseteq T(X) \) (canonical copy).

Recall this family of examples of lush spaces

\( Y \) if \( c_0 \subseteq Y \subseteq \ell_\infty \) (canonical copies).

Theorem

\( X \) separable, \( X \supseteq c_0 \implies X \text{ admits an equivalent lush norm.} \)
Lush renorming

The goal

When we may get a lush equivalent norm?

Proposition

\[ X \text{ separable, } X \supseteq c_0 \implies \exists \| \cdot \| \simeq \| \cdot \| \text{ and } T : (X, \| \cdot \|) \to \ell_\infty \text{ with } T \text{ isometric embedding } \& \ c_0 \subseteq T(X) \text{ (canonical copy)}. \]

Recall this family of examples of lush spaces

\[ Y \text{ if } c_0 \subseteq Y \subseteq \ell_\infty \text{ (canonical copies)}. \]

Theorem

\[ X \text{ separable, } X \supseteq c_0 \implies X \text{ admits an equivalent lush norm}. \]

Corollary

Every closed subspace of \( c_0 \) admits an equivalent lush norm.
The goal

When we may get a lush equivalent norm?

Proposition

If $X$ separable, $X \supseteq c_0 = \ell_\infty$, then there exists an equivalent lush norm and $T$ isometric embedding such that $c_0 \subseteq T(X)$ (canonical copies).

Open problems

Recall this family of examples of lush spaces

Theorem

If $X$ separable, $X \supseteq c_0 \implies X$ admits an equivalent lush norm.

Corollary

Every closed subspace of $c_0$ admits an equivalent lush norm.
**Lush renorming**

**The goal**

When we may get a lush equivalent norm?

**Proposition**

When \( X \) separable, \( X \supseteq c_0 = \ell^\infty \Rightarrow \exists \| \cdot \| \simeq \| \cdot \| \) and \( T : (X, \| \cdot \|) \rightarrow \ell^\infty \) with \( T \) isometric embedding & \( c_0 \subseteq T(X) \) (canonical copy).

**Open problems**

- Find more sufficient conditions to get equivalent lush norms.

**Recall**

- \( Y \uparrow c_0 \subseteq Y \subseteq \ell^\infty \) (canonical copies).

**Theorem**

When \( X \) separable, \( X \supseteq c_0 \Rightarrow X \) admits an equivalent lush norm.

**Corollary**

Every closed subspace of \( c_0 \) admits an equivalent lush norm.
Lush renorming

The goal
When we may get a lush equivalent norm?

Proposition

- \( \text{X separable, } X \supseteq c_0 \Rightarrow \exists ||| \cdot ||| \cong \| \cdot \| \text{ and } T: (X, ||| \cdot |||) \to \ell_\infty \text{ with } T \text{ isometric embedding } \& c_0 \subseteq T(X) \) (canonical copy).

Recall this family of examples of lush spaces

- \( \text{Y if } c_0 \subseteq Y \subseteq \ell_\infty \) (canonical copies).

Open problems
- Find more sufficient conditions to get equivalent lush norms.
- When \( X \supseteq \ell_1 \)?

Theorem

- \( X \text{ separable, } X \supseteq c_0 \Rightarrow X \text{ admits an equivalent lush norm.} \)

Corollary

- Every closed subspace of \( c_0 \) admits an equivalent lush norm.
Lush renorming

The goal

When we may get a lush equivalent norm?

Proposition

$X$ separable, $X \supseteq c_0 = \Rightarrow \exists ||| \cdot ||| \simeq \| \cdot \|

and $T: (X, ||| \cdot |||) \to \ell_\infty$ with $T$ isometric embedding & $c_0 \subseteq T(X)$ (canonical copy).

Open problems

- Find more sufficient conditions to get equivalent lush norms.
- When $X \supseteq \ell_1$?
- When $X \supseteq \ell_\infty$?

Recall this family of examples of lush spaces $Y$ if $c_0 \subseteq Y \subseteq \ell_\infty$ (canonical copies).

Theorem

$X$ separable, $X \supseteq c_0 \Rightarrow X$ admits an equivalent lush norm.

Corollary

Every closed subspace of $c_0$ admits an equivalent lush norm.
Observation

(a) Exists \( A \subset B_{X^*} \) norming, \( |x^{**}(a^*)| = 1 \) \( \forall a^* \in A \) and \( \forall x^{**} \in \text{ext}(B_{X^{**}}) \).

(b) For \( x \in S_{X} \) and \( \varepsilon > 0 \), exists \( x^* \in S_{X^*} \) such that \( x \in S_{(B_{X}, x^*, \varepsilon)} \) and \( B_{X} = \text{aconv}(S_{(B_{X}, x^*, \varepsilon)}) \).

\[ a = = = = \]

\[ b = = = = \]

lushness

Definition (Werner, 1997)

\( X \) is nicely embedded in \( C_b(\Omega) \) if exists \( J: X \rightarrow C_b(\Omega) \) linear isometry with

1. \( \|J^* \delta_s\| = 1 \) \( \forall s \in \Omega \),
2. span\((J^* \delta_s)\) \( L \)-summand in \( X^* \) \( \forall s \in \Omega \).

Even more examples of lush spaces

Nicely embedded Banach spaces (they fulfil (a)).

In particular, function algebras (as \( A(D) \) and \( H_\infty \)).
Even more examples of lush spaces

Observation

$X$ Banach space. Consider the following assertions.

(a) Exists $A \subset B_{X^*}$ norming, $|x^{**}(a^*)| = 1$ $\forall a^* \in A$ and $\forall x^{**} \in \text{ext}(B_{X^{**}})$.

(b) For $x \in S_X$ and $\varepsilon > 0$, exists $x^* \in S_{X^*}$ such that

$$x \in S(B_X, x^*, \varepsilon) \quad \text{and} \quad B_X = \overline{\text{aconv}}(S(B_X, x^*, \varepsilon)).$$

(a) $\iff$ (b) $\iff$ lushness
### Observation

**X** Banach space. Consider the following assertions.

**(a)** Exists \( A \subset B_{X^*} \) norming, \( |x^{**}(a^*)| = 1 \ \forall a^* \in A \) and \( \forall x^{**} \in \text{ext}(B_{X^{**}}) \).

**(b)** For \( x \in S_X \) and \( \varepsilon > 0 \), exists \( x^* \in S_{X^*} \) such that

\[
x \in S(B_X, x^*, \varepsilon) \quad \text{and} \quad B_X = \overline{\text{aconv}}(S(B_X, x^*, \varepsilon)).
\]

\[\begin{align*}
(a) \quad \Longrightarrow \quad & (b) \quad \Longrightarrow \quad \text{lushness}
\end{align*}\]

### Definition (Werner, 1997)

**X** is **nicely embedded** in \( C_b(\Omega) \) if exists \( J : X \longrightarrow C_b(\Omega) \) linear isometry with

**(N1)** \( \|J^* \delta_s\| = 1 \ \forall s \in \Omega \),

**(N2)** \( \text{span}(J^* \delta_s) \) \( L \)-summand in \( X^* \) \( \forall s \in \Omega \).
Even more examples of lush spaces

Observation

$X$ Banach space. Consider the following assertions.

(a) Exists $A \subset B_{X^*}$ norming, $|x^{**}(a^*)| = 1 \ \forall a^* \in A$ and $\forall x^{**} \in \text{ext} \left( B_{X^{**}} \right)$.

(b) For $x \in S_X$ and $\varepsilon > 0$, exists $x^* \in S_{X^*}$ such that

$$x \in S(B_X, x^*, \varepsilon) \quad \text{and} \quad B_X = \overline{\text{aconv}}(S(B_X, x^*, \varepsilon)).$$

Definition (Werner, 1997)

$X$ is nicely embedded in $C_b(\Omega)$ if exists $J : X \longrightarrow C_b(\Omega)$ linear isometry with

\[(N1) \quad \|J^* \delta_s\| = 1 \ \forall s \in \Omega,\]

\[(N2) \quad \text{span}(J^* \delta_s) \ L\text{-summand in } X^* \ \forall s \in \Omega.\]

Even more examples of lush spaces
Even more examples of lush spaces

Observation

**X** Banach space. Consider the following assertions.

(a) Exists \( A \subset B_{X^*} \) norming, \( |x^{**}(a^*)| = 1 \ \forall a^* \in A \) and \( \forall x^{**} \in \text{ext}(B_{X^{**}}) \).

(b) For \( x \in S_X \) and \( \varepsilon > 0 \), exists \( x^* \in S_{X^*} \) such that

\[
x \in S(B_X, x^*, \varepsilon) \quad \text{and} \quad B_X = \overline{\text{aconv}}(S(B_X, x^*, \varepsilon)).
\]


Definition (Werner, 1997)

**X** is **nicely embedded** in \( C_b(\Omega) \) if exists \( J : X \rightarrow C_b(\Omega) \) linear isometry with

\[
(N1) \quad \|J^* \delta_s\| = 1 \ \forall s \in \Omega,
\]

\[
(N2) \quad \text{span}(J^* \delta_s) \text{ } L\text{-summand in } X^* \ \forall s \in \Omega.
\]


Even more examples of lush spaces

- Nicely embedded Banach spaces (they fulfil (a)).
Even more examples of lush spaces

Observation

Let $X$ be a Banach space. Consider the following assertions.

(a) Exists $A \subset B_{X^*}$ norming, $|x^{**}(a^*)| = 1 \forall a^* \in A$ and $\forall x^{**} \in \text{ext}(B_{X^{**}})$.

(b) For $x \in S_X$ and $\varepsilon > 0$, exists $x^* \in S_{X^*}$ such that

$$x \in S(B_X, x^*, \varepsilon) \quad \text{and} \quad B_X = \overline{\text{aconv}}(S(B_X, x^*, \varepsilon)).$$

Definition (Werner, 1997)

$X$ is nicely embedded in $C_b(\Omega)$ if there exists $J : X \longrightarrow C_b(\Omega)$ linear isometry with

1. $\|J^*\delta_s\| = 1 \forall s \in \Omega$,
2. span$(J^*\delta_s)$ $L$-summand in $X^* \forall s \in \Omega$.

Even more examples of lush spaces

- Nicely embedded Banach spaces (they fulfil (a)).
- In particular, function algebras (as $A(\mathbb{D})$ and $H^\infty$).
Some reformulations of lushness

Proposition

$X$ is lush, 

Every separable $E \subset X$ is contained in a separable lush $Y$ with $E \subset Y \subset X$. 
Some reformulations of lushness

Proposition

Let $X$ be a Banach space. TFAE:

- $X$ is lush,
- Every separable $E \subset X$ is contained in a separable lush $Y$ with $E \subset Y \subset X$. 
Proposition

\( X \) Banach space. TFAE:

- \( X \) is lush,
- Every separable \( E \subset X \) is contained in a separable lush \( Y \) with \( E \subset Y \subset X \).

Separable lush spaces

\( X \) separable. TFAE:

- \( X \) is lush.
- There is \( G \subseteq S_{X^*} \) norming such that
  \[
  B_X = \overline{aconv(S(B_X, x^*, \varepsilon))}
  \]
  for every \( \varepsilon > 0 \) and every \( x^* \in G \).
- There is \( G \subseteq \text{ext}(B_{X^*}) \) norming such that
  \[
  |x^{**}(x^*)| = 1 \quad (x^{**} \in \text{ext}(B_{X^{**}}), \ x^* \in G).
  \]
Some reformulations of lushness

**Proposition**

X Banach space. TFAE:

- X is lush,
- Every separable $E \subset X$ is contained in a **separable lush** $Y$ with $E \subset Y \subset X$.

**Separable lush spaces (real case)**

X real separable. TFAE:

- X is lush.
- There is $G \subseteq S_{X^*}$ **norming** such that
  \[ B_X = \text{aconv} \left( \{ x \in B_X : x^*(x) = 1 \} \right) \quad (x^* \in G). \]

Therefore, $|x^{**}(x^*)| = 1 \ \forall x^{**} \in \text{ext} \left( B_{X^{**}} \right) \ \forall x^* \in G.$
Some reformulations of lushness

**Proposition**

X Banach space. TFAE:

- X is lush,
- Every separable $E \subset X$ is contained in a separable lush $Y$ with $E \subset Y \subset X$.

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Therefore, $|x^{**}(x^*)| = 1 \ \forall x^{**} \in \text{ext} (B_{X^{**}}) \ \forall x^* \in G$.

We almost returned to the almost-CL-space definition !!
Some reformulations of lushness

**Proposition**

$X$ Banach space. TFAE:

- $X$ is lush,
- Every separable $E \subset X$ is contained in a separable lush $Y$ with $E \subset Y \subset X$.

**Separable lush spaces (real case)**

$X$ real separable. TFAE:

- $X$ is lush.
- There is $G \subseteq S_{X^*}$ norming such that

\[
B_X = \overline{\text{aconv}} \left( \{ x \in B_X : x^*(x) = 1 \} \right) \quad (x^* \in G).
\]

Therefore, $|x^{**}(x^*)| = 1 \ \forall x^{**} \in \text{ext} (B_{X^{**}}) \ \forall x^* \in G$.

**Consequence (real case)**

$X \subseteq C[0,1]$ strictly convex or smooth $\implies C[0,1]/X$ contains $C[0,1]$. 
An important consequence

Remark

X lush separable, \( \dim(X) = \infty \Rightarrow \) there is \( G \in S_X^* \) infinite such that

\[ |x^{**}(x^*)| = 1 \quad (x^{**} \in \text{ext}(B_X^{**}), x^* \in G). \]

Proposition (López–M.—Paya, 1999)

\( X \) real, \( A \subset S_X \) infinite such that

\[ |x^*(a)| = 1 \quad (x^* \in \text{ext}(B_X^*), a \in A). \]

Then, \( X \supseteq c_0 \) or \( X \supseteq \ell_1 \).

Main consequence

\( X \) real lush, \( \dim(X) = \infty \Rightarrow X^* \supseteq \ell_1. \)
An important consequence

**Remark**

If $X$ is lush separable, $\dim(X) = \infty \implies$ there is $G \in S_{X^*}$ infinite such that

$$|x^{**}(x^*)| = 1 \quad (x^{**} \in \text{ext}(B_{X^{**}}), \ x^* \in G).$$
An important consequence

Remark

\( X \) lush separable, \( \dim(X) = \infty \implies \) there is \( G \in S_{X^*} \) infinite such that

\[ |x^{**}(x^*)| = 1 \quad (x^{**} \in \text{ext}(B_{X^{**}}), \ x^* \in G). \]

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Then, \( X \supseteq c_0 \) or \( X \supseteq \ell_1 \).
An important consequence

**Remark**

\[ X \text{ lush separable, } \dim(X) = \infty \implies \text{ there is } G \subset S_{X^*} \text{ infinite such that} \]
\[ |x^{**}(x^*)| = 1 \quad (x^{**} \in \text{ext}(B_{X^{**}}), \ x^* \in G). \]

**Proposition (López–M.–Payá, 1999)**

\[ X \text{ real, } A \subset S_X \text{ infinite such that} \]
\[ |x^*(a)| = 1 \quad (x^* \in \text{ext}(B_{X^*}), \ a \in A). \]

Then, \( X \supseteq c_0 \) or \( X \supseteq \ell_1 \).

**Main consequence**

\[ X \text{ real lush, } \dim(X) = \infty \implies X^* \supseteq \ell_1. \]
An important consequence

Remark

$X$ lush separable, $\dim(X) = \infty \implies$ there is $G \in S_{X^*}$ infinite such that

$$|x^{**}(x^*)| = 1 \quad (x^{**} \in \text{ext}(B_{X^{**}}), \ x^* \in G).$$

Proposition (López–M.–Payá, 1999)

$X$ real, $A \subset S_X$ infinite such that

$$|x^*(a)| = 1 \quad (x^* \in \text{ext}(B_{X^*}), \ a \in A).$$

Then, $X \supseteq c_0$ or $X \supseteq \ell_1$.

Main consequence

$X$ real lush, $\dim(X) = \infty \implies X^* \supseteq \ell_1$.

Proof.
## An important consequence

### Remark

$x$ lush separable, $\dim(x) = \infty \implies$ there is $G \in S_{x^*}$ infinite such that

$$|x^{**}(x^*)| = 1 \quad (x^{**} \in \text{ext}(B_{x^{**}}), \ x^* \in G).$$

### Proposition (López–M.–Payá, 1999)

$x$ real, $A \subset S_x$ infinite such that

$$|x^*(a)| = 1 \quad (x^* \in \text{ext}(B_{x^*}), \ a \in A).$$

Then, $x \supseteq c_0$ or $x \supseteq \ell_1$.

### Main consequence

$x$ real lush, $\dim(x) = \infty \implies x^* \supseteq \ell_1$.

**Proof.**

- There is $E \subseteq x$ separable and lush.
An important consequence

Remark

\( X \) lush separable, \( \dim(X) = \infty \implies \) there is \( G \in S_{X^*} \) infinite such that

\[ |x^{**}(x^*)| = 1 \quad (x^{**} \in \text{ext}(B_{X^{**}}), \ x^* \in G). \]

Proposition (López–M.–Payá, 1999)

\( X \) real, \( A \subset S_X \) infinite such that

\[ |x^*(a)| = 1 \quad (x^* \in \text{ext}(B_{X^*}), \ a \in A). \]

Then, \( X \supseteq c_0 \) or \( X \supseteq \ell_1 \).

Main consequence

\( X \) real lush, \( \dim(X) = \infty \implies X^* \supseteq \ell_1. \)

Proof.

- There is \( E \subseteq X \) separable and lush.
- Then \( E^* \supseteq c_0 \) or \( E^* \supseteq \ell_1 \implies E^* \supseteq \ell_1. \)
An important consequence

**Remark**

\( X \) lush separable, \( \dim(X) = \infty \) \( \implies \) there is \( G \in S_{X^*} \) infinite such that

\[
|x^{**}(x^*)| = 1 \quad (x^{**} \in \text{ext}(B_{X^{**}}), \ x^* \in G).
\]

**Proposition (López–M.–Payá, 1999)**

\( X \) real, \( A \subset S_X \) infinite such that

\[
|x^*(a)| = 1 \quad (x^* \in \text{ext}(B_{X^*}), \ a \in A).
\]

Then, \( X \supseteq c_0 \) or \( X \supseteq \ell_1 \).

**Main consequence**

\( X \) real lush, \( \dim(X) = \infty \) \( \implies \) \( X^* \supseteq \ell_1 \).

Proof.

- There is \( E \subseteq X \) separable and lush.
- Then \( E^* \supseteq c_0 \) or \( E^* \supseteq \ell_1 \) \( \implies \) \( E^* \supseteq \ell_1 \).
- By “lifting” property of \( \ell_1 \) \( \implies \) \( X^* \supseteq \ell_1 \).
An important consequence

Remark

\( X \) lush separable, \( \dim(X) = \infty \implies \) there is \( G \in S_{X^*} \) infinite such that

\[ |x^{**}(x^*)| = 1 \quad (x^{**} \in \text{ext}(B_{X^{**}}), \ x^* \in G). \]

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Then, \( X \supseteq c_0 \) or \( X \supseteq \ell_1 \).

Main consequence

\( X \) real lush, \( \dim(X) = \infty \implies X^* \supseteq \ell_1. \)

Question

What happens if just \( n(X) = 1 \) ?
An important consequence

Remark

\( X \) lush separable, \( \dim(X) = \infty \implies \) there is \( G \in S_{X^*} \) infinite such that

\[ |x^{**}(x^*)| = 1 \quad (x^{**} \in \text{ext}(B_{X^{**}}), \ x^* \in G). \]

Proposition (López–M.–Payá, 1999)

\( X \) real, \( A \subset S_X \) infinite such that

\[ |x^*(a)| = 1 \quad (x^* \in \text{ext}(B_{X^*}), \ a \in A). \]

Then, \( X \supseteq c_0 \) or \( X \supseteq \ell_1 \).

Main consequence

\( X \) real lush, \( \dim(X) = \infty \implies X^* \supseteq \ell_1. \)

Question

What happens if just \( n(X) = 1 \)? The same, we will prove later.
Lush spaces

Lushness is not equivalent to numerical index one

Example

There is a separable Banach space $X$ such that $X^*$ is lush but $X$ is not lush. Since $n(X^*) = 1$, also $n(X) = 1$.

The set $\{x^* \in S_{X^*} : |x^{**}(x^*)| = 1 \text{ for every } x^{**} \in \text{ext}(B_{X^{**}})\}$ is empty.

Consequence

$X$ lush $\neq X^*$ lush

Proposition

$X^{**}$ lush $\neq X$ lush

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Lushness is not equivalent to numerical index one

Example

There is a separable Banach space $\mathcal{X}$ such that
- $\mathcal{X}^*$ is lush but $\mathcal{X}$ is not lush.
Lushness is not equivalent to numerical index one

Example

There is a separable Banach space $\mathcal{X}$ such that

- $\mathcal{X}^*$ is lush but $\mathcal{X}$ is not lush.
- Since $n(\mathcal{X}^*) = 1$, also $n(\mathcal{X}) = 1$. 
Example

There is a separable Banach space $\mathcal{X}$ such that

- $\mathcal{X}^*$ is lush but $\mathcal{X}$ is not lush.
- Since $n(\mathcal{X}^*) = 1$, also $n(\mathcal{X}) = 1$.
- The set

$$\{x^* \in S_{\mathcal{X}^*} : |x^{**}(x^*)| = 1 \text{ for every } x^{**} \in \text{ext}(B_{\mathcal{X}^{**}})\}$$

is empty.
Lushness is not equivalent to numerical index one

Example

There is a separable Banach space $\mathcal{X}$ such that

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is empty.

Consequence

$\text{X lush} \iff \text{X}^* \text{ lush}$
Lushness is not equivalent to numerical index one

Example

There is a separable Banach space $\mathcal{X}$ such that

- $\mathcal{X}^*$ is lush but $\mathcal{X}$ is not lush.
- Since $n(\mathcal{X}^*) = 1$, also $n(\mathcal{X}) = 1$.
- The set

$$\{x^* \in S_{\mathcal{X}^*} : |x^{**}(x^*)| = 1 \text{ for every } x^{**} \in \text{ext}(B_{\mathcal{X}^{**}})\}$$

is empty.

Consequence

$$X \text{ lush} \iff X^* \text{ lush}$$

Proposition

$$X^{**} \text{ lush} \iff X \text{ lush}$$
Slicely countably determined spaces

Slicely countably determined spaces
- Slicely Countably Determined sets and spaces
- Applications to numerical index 1 spaces
- SCD operators
- Open questions

A. Avilés, V. Kadets, M. Martín, J. Merí, and V. Shepelska
Slicely Countably Determined Banach spaces
SCD sets: Definitions and preliminary remarks

**SCD sets**
A subset $A$ of a Banach space $X$ is Slicely Countably Determined (SCD) if there is a sequence $\{S_n : n \in \mathbb{N}\}$ of slices of $A$ such that:

1. Every slice of $A$ contains one of the $S_n$'s.
2. $A \subseteq \text{conv}(B)$ if $B \subseteq A$ satisfies $B \cap S_n \neq \emptyset$ for all $n$.
3. Given a sequence $\{x_n : n \in \mathbb{N}\}$ with $x_n \in S_n$ for all $n$, $A \subseteq \text{conv}\{x_n : n \in \mathbb{N}\}$.

**Remarks**
- A set $A$ is SCD if and only if $A$ is SCD.
- If $A$ is SCD, then $A$ is separable.
X Banach space, $A \subset X$ bounded and convex.

**SCD sets**

$A$ is **Slicely Countably Determined (SCD)** if there is a sequence $\{S_n : n \in \mathbb{N}\}$ of slices of $A$ satisfying one of the following equivalent conditions:
X Banach space, $A \subset X$ bounded and convex.

**SCD sets**

$A$ is **Slicely Countably Determined (SCD)** if there is a sequence $\{S_n : n \in \mathbb{N}\}$ of slices of $A$ satisfying one of the following equivalent conditions:

- every slice of $A$ contains one of the $S_n$’s,
SCD sets: Definitions and preliminary remarks

$X$ Banach space, $A \subseteq X$ bounded and convex.

**SCD sets**

$A$ is **Slicely Countably Determined (SCD)** if there is a sequence $\{S_n : n \in \mathbb{N}\}$ of slices of $A$ satisfying one of the following equivalent conditions:

- every slice of $A$ contains one of the $S_n$’s,
- $A \subseteq \overline{\text{conv}}(B)$ if $B \subseteq A$ satisfies $B \cap S_n \neq \emptyset \ \forall n$, 

Remarks

$A$ is SCD iff $A$ is SCD. If $A$ is SCD, then it is separable.
SCD sets: Definitions and preliminary remarks

$X$ Banach space, $A \subset X$ bounded and convex.

SCD sets

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- every slice of $A$ contains one of the $S_n$’s,
- $A \subseteq \overline{\text{conv}}(B)$ if $B \subseteq A$ satisfies $B \cap S_n \neq \emptyset \ \forall n$,
- given $\{x_n\}_{n \in \mathbb{N}}$ with $x_n \in S_n \ \forall n \in \mathbb{N}$, $A \subseteq \overline{\text{conv}}(\{x_n : n \in \mathbb{N}\})$. 

Remarks

$A$ is SCD iff $A$ is SCD.

If $A$ is SCD, then it is separable.
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**SCD sets**

$A$ is **Slicely Countably Determined (SCD)** if there is a sequence $\{S_n : n \in \mathbb{N}\}$ of slices of $A$ satisfying one of the following equivalent conditions:

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**Remarks**

- $A$ is SCD iff $\overline{A}$ is SCD.
- If $A$ is SCD, then it is separable.
SCD sets: Elementary examples I

Example
A separable and \( A = \text{conv}(\text{dent}(A)) \Rightarrow A \) is SCD.

Proof.
Take \( \{a_n : n \in \mathbb{N}\} \) denting points with \( A = \text{conv}(\{a_n : n \in \mathbb{N}\}) \).

For every \( n, m \in \mathbb{N} \), take a slice \( S_{n,m} \) containing \( a_n \) and of diameter \( 1/m \).

If \( B \cap S_{n,m} \neq \emptyset \) for all \( n, m \in \mathbb{N} \), \( a_n \in B \) for all \( n \in \mathbb{N} \).

Therefore, \( A = \text{conv}(\{a_n : n \in \mathbb{N}\}) \subseteq \text{conv}(B) = \text{conv}(B) \).

✓

Example
In particular, \( A_{RNP} \) separable \( \Rightarrow A \) SCD.

Corollary
If \( X \) is separable LUR \( \Rightarrow B_X \) is SCD.

So, every separable space can be renormed such that \( B(X, |·|) \) is SCD.
Example

A separable and $A = \overline{\text{conv}}(\text{dent}(A)) \implies A$ is SCD.


Example

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Proof.
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- Take $\{a_n : n \in \mathbb{N}\}$ denting points with $A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\})$.
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Miguel Martín (University of Granada (Spain)) Numerical index theory Bangalore, June 2009 83 / 136
SCD sets: Elementary examples I

Example

A separable and $A = \overline{\text{conv}}(\text{dent}(A)) \implies A$ is SCD.

Proof.

- Take $\{a_n : n \in \mathbb{N}\}$ denting points with $A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\})$.
- For every $n, m \in \mathbb{N}$, take a slice $S_{n,m}$ containing $a_n$ and of diameter $1/m$.
- If $B \cap S_{n,m} \neq \emptyset \ \forall n, m \in \mathbb{N} \implies a_n \in \overline{B} \ \forall n \in \mathbb{N}$. 

Therefore, $A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\}) \subseteq \overline{\text{conv}}(B) = \overline{\text{conv}}(B) \implies A$ is SCD.

Example

In particular, $A_{\text{RNP}}$ separable $\Rightarrow A$ SCD.

Corollary

If $X$ is separable LUR $\Rightarrow B_X$ is SCD.

So, every separable space can be renormed such that $B_X$ is SCD.
Example

A separable and \( A = \overline{\text{conv}}(\text{dent}(A)) \implies A \) is SCD.

Proof.

- Take \( \{a_n : n \in \mathbb{N}\} \) denting points with \( A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\}) \).
- For every \( n, m \in \mathbb{N} \), take a slice \( S_{n,m} \) containing \( a_n \) and of diameter \( 1/m \).
- If \( B \cap S_{n,m} \neq \emptyset \ \forall n, m \in \mathbb{N} \implies a_n \in \overline{B} \ \forall n \in \mathbb{N} \).
- Therefore, \( A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\}) \subseteq \overline{\text{conv}(B)} = \overline{\text{conv}(B)}. \checkmark \)
SCD sets: Elementary examples I

**Example**

$A$ separable and $A = \overline{\text{conv}}(\text{dent}(A)) \implies A$ is SCD.

**Proof.**

- Take $\{a_n : n \in \mathbb{N}\}$ denting points with $A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\})$.
- For every $n, m \in \mathbb{N}$, take a slice $S_{n,m}$ containing $a_n$ and of diameter $1/m$.
- If $B \cap S_{n,m} \neq \emptyset \ \forall n, m \in \mathbb{N} \implies a_n \in \overline{B} \ \forall n \in \mathbb{N}$.
- Therefore, $A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\}) \subseteq \overline{\text{conv}(B)} = \overline{\text{conv}(B)}$. ✓

**Example**

In particular, $A$ RNP separable $\implies A$ SCD.
SCD sets: Elementary examples I

Example

A separable and $A = \overline{\text{conv}}(\text{dent}(A)) \implies A$ is SCD.

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- Take $\{a_n : n \in \mathbb{N}\}$ denting points with $A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\})$.
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- If $B \cap S_{n,m} \neq \emptyset \quad \forall n, m \in \mathbb{N} \implies a_n \in \overline{B} \quad \forall n \in \mathbb{N}$.
- Therefore, $A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\}) \subseteq \overline{\text{conv}}(\overline{B}) = \text{conv}(B)$. ✓

Example

In particular, $A$ RNP separable $\implies A$ SCD.

Corollary

- If $X$ is separable LUR $\implies B_X$ is SCD.
- So, every separable space can be renormed such that $B_{(X,||\cdot||)}$ is SCD.
Example

If $X^*$ is separable, then $A$ is SCD.

Proof.

Take \( \{x^*_n : n \in \mathbb{N} \} \) dense in $S_{X^*}$. For every $n, m \in \mathbb{N}$, consider $S_{n,m} = S(A, x^*_n, 1/m)$. It is easy to show that any slice of $A$ contains one of the $S_{n,m}$.

Negative example

If $X$ has the Daugavet property, then $B_X$ is not SCD. Therefore, $B_{C[0,1]}$, $B_{L_1[0,1]}$ are not SCD.

Proof.

Fix $x_0 \in B_X$ and a sequence of slices $\{S_n\}$ of $B_X$. By [KSSW] there is a sequence $(x_n) \subset B_X$ such that $x_n \in S_n$ for every $n \in \mathbb{N}$, $(x_n)_n \geq 0$ is equivalent to the basis of $\ell_1$, so $x_0 \not\in \text{lin} \{x_n : n \in \mathbb{N} \}$.

\( \Box \)
Example

If $X^*$ is separable $\implies A$ is SCD.
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SCD sets: Elementary examples II

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If $X^*$ is separable $\implies A$ is SCD.

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- Take $\{x^*_n : n \in \mathbb{N}\}$ dense in $S_{X^*}$.
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If $X$ has the Daugavet property $\implies B_X$ is not SCD.

Therefore, $B_{C[0,1]}$, $B_{L_1[0,1]}$ are not SCD.
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If $X^*$ is separable $\Rightarrow A$ is SCD.

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SCD sets: Elementary examples II

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If $X^*$ is separable $\implies A$ is SCD.

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- Fix $x_0 \in B_X$ and $\{S_n\}$ sequence of slices of $B_X$. 
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  - so $x_0 \not\in \text{lin}\{x_n : n \in \mathbb{N}\}$. ✓
SCD sets: Further examples I

Convex combination of slices

\[ W = \sum_{k=1}^{m} \lambda_k S_k \subset A \]

\[ \lambda_k \geq 0, \quad \sum \lambda_k = 1, \quad S_k \text{ slices.} \]

Proposition

In the definition of SCD we can use a sequence \( \{S_n : n \in \mathbb{N}\} \) of convex combination of slices.

Small combinations of slices

\( A \) has small combinations of slices iff every slice of \( A \) contains convex combinations of slices of \( A \) with arbitrary small diameter.

Example

If \( A \) has small combinations of slices + separable \( \Rightarrow A \) is SCD.

Particular case

\( A \) strongly regular + separable \( \Rightarrow A \) is SCD.
Convex combination of slices

\[ W = \sum_{k=1}^{m} \lambda_k S_k \subset A \text{ where } \lambda_k \geq 0, \sum \lambda_k = 1, S_k \text{ slices.} \]
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SCD sets: Further examples I

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A has small combinations of slices iff every slice of A contains convex combinations of slices of A with arbitrary small diameter.

Example

If A has small combinations of slices + separable \( \implies \) A is SCD.
### SCD sets: Further examples I

#### Convex combination of slices

\[ W = \sum_{k=1}^{m} \lambda_k S_k \subset A \text{ where } \lambda_k \geq 0, \sum \lambda_k = 1, S_k \text{ slices.} \]

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If \( A \) has small combinations of slices + separable \( \implies \) \( A \) is SCD.

#### Particular case

\( A \) strongly regular + separable \( \implies \) \( A \) is SCD.
SCD sets: Further examples II

Bourgain’s lemma

Every relative weak open subset of $A$ contains a convex combination of slices.

Corollary

In the definition of SCD we can use a sequence $\{S_n : n \in \mathbb{N}\}$ of relative weak open subsets.

$\pi$-bases

$A_{\pi}$-base of the weak topology of $A$ is a family $\{V_i : i \in I\}$ of weak open sets of $A$ such that every weak open subset of $A$ contains one of the $V_i$'s.

Proposition

If $(A, \sigma(X, X^*))$ has a countable $\pi$-base $\Rightarrow A$ is SCD.
Bourgain’s lemma
Every relative weak open subset of $A$ contains a convex combination of slices.
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Proposition
If $(A, \sigma(X, X^*))$ has a countable $\pi$-base $\implies A$ is SCD.
Theorem A separable without $\ell_1$-sequences $\Rightarrow (A, \sigma(X,X^*))$ has a countable $\pi$-base.

Proof. We see $(A, \sigma(X,X^*)) \subset C(T)$ where $T = (B_{X^*}, \sigma(X^*,X))$. By Rosenthal $\ell_1$ theorem, $(A, \sigma(X,X^*))$ is a relatively compact subset of the space of first Baire class functions on $T$. By a result of Todor ˇcevi´c, $(A, \sigma(X,X^*))$ has a $\sigma$-disjoint $\pi$-base.

$\{V_i : i \in I\}$ is $\sigma$-disjoint if $I = \bigcup_{n \in \mathbb{N}} I_n$ and each $\{V_i : i \in I_n\}$ is pairwise disjoint.

A $\sigma$-disjoint family of open subsets in a separable space is countable. ✓

Example A separable without $\ell_1$-sequences $\Rightarrow A$ is SCD.
Theorem

A separable without $\ell_1$-sequences $\implies (A, \sigma(X, X^*))$ has a countable $\pi$-base.
SCD sets: Further examples III

**Theorem**

A separable without $\ell_1$-sequences $\implies (A, \sigma(X, X^*))$ has a countable $\pi$-base.

**Proof.**

We see $(A, \sigma(X, X^*)) \subseteq C(T)$ where $T = (B_{X^*}, \sigma(X, X^*))$. By Rosenthal $\ell_1$ theorem, $(A, \sigma(X, X^*))$ is a relatively compact subset of the space of first Baire class functions on $T$. By a result of Todor ˇcevi´c, $(A, \sigma(X, X^*))$ has a $\sigma$-disjoint $\pi$-base.

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**Example**

A separable without $\ell_1$-sequences $\implies A$ is SCD.
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- We see $(A, \sigma(X, X^*)) \subset C(T)$ where $T = (B_{X^*}, \sigma(X^*, X))$.
- By Rosenthal $\ell_1$ theorem, $(A, \sigma(X, X^*))$ is a relatively compact subset of the space of first Baire class functions on $T$. 
Theorem

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**Theorem**

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- $\{V_i : i \in I\}$ is $\sigma$-disjoint if $I = \bigcup_{n \in \mathbb{N}} I_n$ and each $\{V_i : i \in I_n\}$ is pairwise disjoint.
SCD sets: Further examples III

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- A $\sigma$-disjoint family of open subsets in a separable space is countable. ✓
Theorem

A separable without \( \ell_1 \)-sequences \( \implies \) \((A, \sigma(X, X^*))\) has a countable \( \pi \)-base.

Proof.

- We see \((A, \sigma(X, X^*)) \subset C(T)\) where \(T = (B_{X^*}, \sigma(X^*, X))\).
- By Rosenthal \( \ell_1 \) theorem, \((A, \sigma(X, X^*))\) is a relatively compact subset of the space of first Baire class functions on \(T\).
- By a result of Todorčević, \((A, \sigma(X, X^*))\) has a \(\sigma\)-disjoint \(\pi\)-base.
- \(\{V_i : i \in I\}\) is \(\sigma\)-disjoint if \(I = \bigcup_{n \in \mathbb{N}} I_n\) and each \(\{V_i : i \in I_n\}\) is pairwise disjoint.
- A \(\sigma\)-disjoint family of open subsets in a separable space is countable. ✓

Example

A separable without \( \ell_1 \)-sequences \( \implies \) \(A\) is SCD.
SCD spaces: definition and examples

SCD space $X$ is Slicely Countably Determined (SCD) if so are its convex bounded subsets.

Examples of SCD spaces
1. $X$ separable strongly regular. In particular, RNP, CPCP spaces.
2. $X$ separable $X^* ⊉ \ell_1$. In particular, if $X^*$ is separable.

Examples of NOT SCD spaces
1. $X$ having the Daugavet property.
2. In particular, $C[0, 1]$, $L_1[0, 1]$.
3. There is $X$ with the Schur property which is not SCD.

Remark
Every subspace of a SCD space is SCD. This is false for quotients.
SCD space

X is **Slicely Countably Determined (SCD)** if so are its convex bounded subsets.
SCD spaces: definition and examples

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# SCD spaces: definition and examples

## SCD space

A **SCD** (Slicely Countably Determined) space is one where all its convex bounded subsets are SCD. Mathematically:

\[ X \text{ is Slicely Countably Determined (SCD) if so are its convex bounded subsets.} \]

## Examples of SCD spaces

1. **X** separable strongly regular. In particular, RNP, CPCP spaces.
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1. **X** having the Daugavet property.
2. In particular, \( C[0,1], \ L_1[0,1] \)
3. There is \( X \) with the Schur property which is not SCD.
SCD spaces: definition and examples

**SCD space**

X is **Slicely Countably Determined (SCD)** if so are its convex bounded subsets.

**Examples of SCD spaces**

1. X separable strongly regular. In particular, RNP, CPCP spaces.
2. X separable X $\not\supset \ell_1$. In particular, if $X^*$ is separable.

**Examples of NOT SCD spaces**

1. X having the Daugavet property.
2. In particular, $C[0,1]$, $L_1[0,1]$
3. There is X with the Schur property which is not SCD.

**Remark**

- Every subspace of a SCD space is SCD.
- This is false for quotients.
Theorem

\[ Z \subset X \]

If \( Z \) and \( X/Z \) are SCD \( \Rightarrow X \) is SCD.

Corollary

If \( \ell_1 \cong Y \subset X \Rightarrow X/Y \) contains a copy of \( \ell_1 \).

If \( \ell_1 \cong Y_1 \subset X \Rightarrow \) there is \( \ell_1 \cong Y_2 \subset X \) with \( Y_1 \cap Y_2 = 0 \).

Corollary

If \( X_1, \ldots, X_m \) SCD \( \Rightarrow X_1 \oplus \cdots \oplus X_m \) SCD.
SCD spaces: stability properties

**Theorem**

\[ Z \subset X. \text{ If } Z \text{ and } X/Z \text{ are SCD } \implies X \text{ is SCD.} \]
SCD spaces: stability properties

**Theorem**

\[ Z \subset X. \text{ If } Z \text{ and } X/Z \text{ are SCD } \implies X \text{ is SCD.} \]

**Corollary**

\[ X \text{ separable NOT SCD} \]
Theorem

\[ Z \subset X. \text{ If } Z \text{ and } X/Z \text{ are SCD } \implies X \text{ is SCD.} \]

Corollary

\[ X \text{ separable NOT SCD} \]

- If \( \ell_1 \simeq Y \subset X \implies X/Y \text{ contains a copy of } \ell_1. \]
SCD spaces: stability properties

**Theorem**

\[ Z \subseteq X. \text{ If } Z \text{ and } X/Z \text{ are SCD } \implies X \text{ is SCD.} \]

**Corollary**

\( X \) separable NOT SCD

- If \( \ell_1 \simeq Y \subseteq X \implies X/Y \text{ contains a copy of } \ell_1. \)
- If \( \ell_1 \simeq Y_1 \subseteq X \implies \text{ there is } \ell_1 \simeq Y_2 \subseteq X \text{ with } Y_1 \cap Y_2 = 0. \)
SCD spaces: stability properties

**Theorem**

$Z \subset X$. If $Z$ and $X/Z$ are SCD $\implies X$ is SCD.

**Corollary**

X separable NOT SCD

- If $\ell_1 \cong Y \subset X$ $\implies X/Y$ contains a copy of $\ell_1$.
- If $\ell_1 \cong Y_1 \subset X$ $\implies$ there is $\ell_1 \cong Y_2 \subset X$ with $Y_1 \cap Y_2 = 0$.

**Corollary**

$X_1, \ldots, X_m$ SCD $\implies X_1 \oplus \cdots \oplus X_m$ SCD.
Theorem

\[ X_1, X_2, \ldots \text{SCD, } E \text{ with unconditional basis.} \]

\[ E \not\subseteq c_0 \Rightarrow \bigoplus_{n \in \mathbb{N}} X_n \in SCD. \]

\[ E \not\subseteq \ell_1 \Rightarrow \bigoplus_{n \in \mathbb{N}} X_n \in SCD. \]

Examples

1. \( c_0 \) (\( \ell_1 \)) and \( \ell_1 \) (\( c_0 \)) are SCD.

2. \( c_0 \otimes \epsilon c_0, c_0 \otimes \pi c_0, c_0 \otimes \epsilon \ell_1, c_0 \otimes \pi \ell_1, \ell_1 \otimes \epsilon \ell_1, \) and \( \ell_1 \otimes \pi \ell_1 \) are SCD.

3. \( K(c_0) \) and \( K(c_0, \ell_1) \) are SCD.

4. \( \ell_2 \otimes \epsilon \ell_2 \equiv K(\ell_2) \) and \( \ell_2 \oplus \pi \ell_2 \equiv L_1(\ell_2) \) are SCD.
Theorem

\[ X_1, X_2, \ldots \text{ SCD, } E \text{ with unconditional basis.} \]

- \( E \not\subseteq c_0 \implies [\bigoplus_{n \in \mathbb{N}} X_n]_E \text{ SCD.} \)
- \( E \not\subseteq \ell_1 \implies [\bigoplus_{n \in \mathbb{N}} X_n]_E \text{ SCD.} \)
Theorem

X₁, X₂, … SCD, E with unconditional basis.

- E ⊈ c₀ ⇒ [⊔ₙ∈ℕ Xₙ]ₓ E SCD.
- E ⊈ ℓ₁ ⇒ [⊔ₙ∈ℕ Xₙ]ₓ E SCD.

Examples

1. c₀(ℓ₁) and ℓ₁(c₀) are SCD.
2. c₀ ⊗⁻ c₀, c₀ ⊗⁻ ℓ₁, and ℓ₁ ⊗⁻ ℓ₁ are SCD.
3. K(c₀) and K(c₀, ℓ₁) are SCD.
4. ℓ₂ ⊗⁻ ℓ₂ ≡ K(ℓ₂) and ℓ₂ ⊕⁻ ℓ₂ ≡ L₁(ℓ₂) are SCD.
Recalling the properties

Kadets-Shvidkoy-Sirotkin-Werner, 1997:

\[ \| \text{Id} + T \| = 1 + \| T \| \] (DE)

for every rank-one \( T \in L(X) \).

⋆ Then every weakly compact \( T \) also satisfies (DE).

Lumer, 1968:

\[ \max_{\theta \in T} \| \text{Id} + \theta T \| = 1 + \| T \| \] (aDE)

Equivalently, \( v(T) = \| T \| \) for every \( T \in L(X) \).

M.-Oikhberg, 2004:

\[ \text{X has the alternative Daugavet property (ADP) if} \]

\[ \text{every rank-one} \ T \in L(X) \text{ satisfies (aDE).} \]

⋆ Then every weakly compact \( T \) also satisfies (aDE).
Recalling the properties

Kadets-Shvidkoy-Sirotkin-Werner, 1997:
X has the Daugavet property (DPr) if

$$\|\text{Id} + T\| = 1 + \|T\|$$  \hspace{1cm} (DE)

for every rank-one $T \in L(X)$.
★ Then every weakly compact $T$ also satisfies (DE).
Recalling the properties

1. **Kadets-Shvidkoy-Sirotkin-Werner, 1997:**
   \( X \) has the **Daugavet property (DPr)** if
   \[
   \| \text{Id} + T \| = 1 + \| T \| \tag{DE}
   \]
   for every rank-one \( T \in L(X) \).
   ★ Then every weakly compact \( T \) also satisfies (DE).

2. **Lumer, 1968:** \( X \) has **numerical index 1** if \texttt{EVERY} operator on \( X \) satisfies
   \[
   \max_{\theta \in T} \| \text{Id} + \theta T \| = 1 + \| T \| \tag{aDE}
   \]
   ★ Equivalently, \( v(T) = \| T \| \) for \texttt{EVERY} \( T \in L(X) \).
The DPr, the ADP and numerical index 1

Recalling the properties

1. **Kadets-Shvidkoy-Sirotkin-Werner, 1997:**
   \( X \) has the **Daugavet property (DPr)** if
   \[
   \|\text{Id} + T\| = 1 + \|T\| \quad \text{(DE)}
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   for every rank-one \( T \in L(X) \).
   ★ Then every weakly compact \( T \) also satisfies (DE).

2. **Lumer, 1968:** \( X \) has **numerical index 1** if \textit{every} operator on \( X \) satisfies
   \[
   \max_{\theta \in \mathbb{T}} \|\text{Id} + \theta T\| = 1 + \|T\| \quad \text{(aDE)}
   \]
   ★ Equivalently, \( v(T) = \|T\| \) for \textit{every} \( T \in L(X) \).

3. **M.-Oikhberg, 2004:** \( X \) has the **alternative Daugavet property (ADP)** if
   every rank-one \( T \in L(X) \) satisfies (aDE).
   ★ Then every weakly compact \( T \) also satisfies (aDE).
Relations between these properties

Examples

- $C([0,1], K(\ell_2))$ has DPr, but has not numerical index $1$
- $c_0$ has numerical index $1$, but has not DPr
- $c_0 \oplus \infty C([0,1], K(\ell_2))$ has ADP, neither DPr nor numerical index $1$

Remarks

- For RNP or Asplund spaces, ADP implies numerical index $1$.
- Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.
Relations between these properties

Daugavet property $\rightarrow$ Numerical index 1 $\leftarrow$ ADP

Examples

- $C([0,1], K(\ell_2))$ has DPr, but has not numerical index 1
- $c_0$ has numerical index 1, but has not DPr
- $c_0 \oplus \infty C([0,1], K(\ell_2))$ has ADP, neither DPr nor numerical index 1
Relations between these properties

Daugavet property $\iff$ Numerical index 1

\[ \text{ADP} \]

Examples

- $C([0,1], K(\ell_2))$ has DPr, but has not numerical index 1
- $c_0$ has numerical index 1, but has not DPr
- $c_0 \oplus_\infty C([0,1], K(\ell_2))$ has ADP, neither DPr nor numerical index 1

Remarks

- For RNP or Asplund spaces, $\text{ADP} \implies \text{numerical index 1}$.  
- Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.
ADP + SCD $\iff$ numerical index 1
Characterizations of the ADP

Let $X$ be a Banach space. TFAE:

- $X$ has ADP (i.e. $\max_{\theta \in T} \|\text{Id} + \theta T\| = 1 + \|T\|$ for all $T$ rank-one).

This implies lushness and so, numerical index 1.
ADP + SCD $\implies$ numerical index 1

Characterizations of the ADP

Let $X$ be a Banach space. TFAE:

1. $X$ has ADP (i.e. $\max_{\theta \in \mathbb{T}} \|\text{Id} + \theta T\| = 1 + \|T\|$ for all $T$ rank-one).
2. Given $x \in S_X$, a slice $S$ of $B_X$ and $\varepsilon > 0$, there is $y \in S$ with
   \[\max_{\theta \in \mathbb{T}} \|x + \theta y\| > 2 - \varepsilon.\]
Characterizations of the ADP

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- **X** has ADP (i.e. \( \max_{\theta \in \mathbb{T}} \|\text{Id} + \theta T\| = 1 + \|T\| \) for all \( T \) rank-one).
- Given \( x \in S_X \), a slice \( S \) of \( B_X \) and \( \varepsilon > 0 \), there is \( y \in S \) with
  \[
  \max_{\theta \in \mathbb{T}} \|x + \theta y\| > 2 - \varepsilon.
  \]
- Given \( x \in S_X \), a sequence \( \{S_n\} \) of slices of \( B_X \), and \( \varepsilon > 0 \), there is \( y^* \in S_{X^*} \) such that \( x \in S(B_X, y^*, \varepsilon) \) and
  \[\text{conv}(T S(B_X, y^*, \varepsilon)) \cap S_n \neq \emptyset \quad (n \in \mathbb{N}).\]
Characterizations of the ADP

X Banach space. TFAE:

- X has ADP (i.e. \( \max_{\theta \in \mathbb{T}} \| \text{Id} + \theta T \| = 1 + \| T \| \) for all \( T \) rank-one).
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  \[
  \overline{\text{conv}}(T S(B_X, y^*, \varepsilon)) \cap S_n \neq \emptyset \quad (n \in \mathbb{N}).
  \]

Theorem

\( X \) ADP + \( B_X \) SCD \( \implies \) given \( x \in S_X \) and \( \varepsilon > 0 \), there is \( y^* \in S_{X^*} \) such that

\[
 x \in S(B_X, y^*, \varepsilon) \quad \text{and} \quad B_X = \overline{\text{conv}}(T S(B_X, y^*, \varepsilon)).
\]

⋆ This implies lushness and so, numerical index 1.
Some consequences
Some consequences

Corollary

1. $\text{ADP} + \text{strongly regular} \implies \text{numerical index } 1$ (actually, lushness).
2. $\text{ADP} + X \not\subseteq \ell_1 \implies \text{numerical index } 1$ (actually, lushness).
Some consequences

**Corollary**

- $\text{ADP} + \text{strongly regular} \implies \text{numerical index 1 (actually, lushness)}$.
- $\text{ADP} + X \not\subseteq \ell_1 \implies \text{numerical index 1 (actually, lushness)}$.

**Corollary**

$X \text{ real} + \dim(X) = \infty + \text{ADP} \implies X^* \supseteq \ell_1$. 
Some consequences

**Corollary**

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**Corollary**

$X \text{ real } + \dim(X) = \infty + \text{ADP} \implies X^* \supseteq \ell_1.$

Proof.
Some consequences

Corollary

- ADP + strongly regular $\implies$ numerical index 1 (actually, lushness).
- ADP + $X \not\supseteq \ell_1$ $\implies$ numerical index 1 (actually, lushness).

Corollary

$X$ real + dim$(X) = \infty$ + ADP $\implies$ $X^* \supseteq \ell_1$.

Proof.

- If $X \supseteq \ell_1$ $\implies$ $X^*$ contains $\ell_\infty$ as a quotient, so $X^*$ contains $\ell_1$ as a quotient, and the lifting property gives $X^* \supseteq \ell_1$. ✓
Some consequences

**Corollary**

- ADP + strongly regular $\implies$ numerical index 1 (actually, lushness).
- ADP + $X \not\supseteq \ell_1$ $\implies$ numerical index 1 (actually, lushness).

**Corollary**

$X$ real + $\dim(X) = \infty +$ ADP $\implies X^* \supseteq \ell_1$.

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- If $X \supseteq \ell_1$ $\implies X^*$ contains $\ell_\infty$ as a quotient, so $X^*$ contains $\ell_1$ as a quotient, and the lifting property gives $X^* \supseteq \ell_1$ $\checkmark$
- If $X \not\supseteq \ell_1$ $\implies X$ is SCD + ADP, so $X$ is lush.
Some consequences

**Corollary**

- ADP + strongly regular $\implies$ numerical index 1 (actually, lushness).
- ADP + $X \nsubseteq \ell_1$ $\implies$ numerical index 1 (actually, lushness).

**Corollary**

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X \text{ real } + \dim(X) = \infty + \text{ADP} \implies X^* \supseteq \ell_1.
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- If $X \nsubseteq \ell_1$ $\implies$ $X$ is SCD + ADP, so $X$ is lush.
- Lush + $\dim(X) = \infty$ $\implies$ $X^* \supseteq \ell_1$ \checkmark
Some consequences

**Corollary**

- $\text{ADP} + \text{strongly regular} \implies \text{numerical index} 1$ (actually, lushness).
- $\text{ADP} + X \not\ni \ell_1 \implies \text{numerical index} 1$ (actually, lushness).

**Corollary**

\[ X \text{ real} + \dim(X) = \infty + \text{ADP} \implies X^* \supseteq \ell_1. \]

In particular,
Some consequences

**Corollary**
- ADP + strongly regular $\implies$ numerical index 1 (actually, lushness).
- ADP + $X \not\supseteq \ell_1$ $\implies$ numerical index 1 (actually, lushness).

**Corollary**

\[
X \text{ real } + \dim(X) = \infty + \text{ADP} \implies X^* \supseteq \ell_1.
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In particular,

**Corollary**

\[
X \text{ real } + \dim(X) = \infty \text{ and numerical index } 1 \implies X^* \supseteq \ell_1.
\]
Some consequences

Corollary

- ADP + strongly regular $\implies$ numerical index 1 (actually, lushness).
- ADP + $X \not\subseteq \ell_1$ $\implies$ numerical index 1 (actually, lushness).

Corollary

$X$ real + $\dim(X) = \infty + \text{ADP} \implies X^* \supseteq \ell_1$.

In particular,

Corollary

$X$ real + $\dim(X) = \infty + \text{numerical index 1} \implies X^* \supseteq \ell_1$.

Open question

$X$ real, $\dim(X) = \infty$, $n(X) = 1 \implies X \supset c_0$ or $X \supset \ell_1$?
Slicely countably determined spaces SCD operators

SCD operators

$T \in \mathcal{L}(X)$ is an SCD-operator if $T(B_X)$ is an SCD-set.

Examples

1. $T(B_X)$ is separable and $\|T(B_X)\|$ is RPN,
2. $T(B_X)$ has no $\ell_1$ sequences,
3. $T$ does not fix copies of $\ell_1$.

Theorem

$X \text{ADP} + T \text{SCD-operator} \implies \max_{\theta \in T} \|\text{Id} + \theta T\| = 1 + \|T\|.$

$X \text{DPr} + T \text{SCD-operator} \implies \|\text{Id} + T\| = 1 + \|T\|.$

Main corollary

$X \text{ADP} + T \text{does not fix copies of} \ell_1 \implies \max_{\theta \in T} \|\text{Id} + \theta T\| = 1 + \|T\|.$
SCD operators

**SCD operator**

$T \in L(X)$ is an **SCD-operator** if $T(B_X)$ is an SCD-set.
SCD operators

**SCD operator**

\[ T \in L(X) \] is an **SCD-operator** if \( T(B_X) \) is an SCD-set.

**Examples**

\( T \) is an SCD-operator when \( T(B_X) \) is separable and

1. \( T(B_X) \) is RPN,
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SCD operators

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$T \in L(X)$ is an **SCD-operator** if $T(B_X)$ is an SCD-set.

Examples

$T$ is an SCD-operator when $T(B_X)$ is separable and
1. $T(B_X)$ is RPN,
2. $T(B_X)$ has no $\ell_1$ sequences,
3. $T$ does not fix copies of $\ell_1$

Theorem

- $X$ ADP + $T$ SCD-operator $\implies \max_{\theta \in T} \| \text{Id} + \theta T \| = 1 + \| T \|.$
- $X$ DPr + $T$ SCD-operator $\implies \| \text{Id} + T \| = 1 + \| T \|.$
SCD operators

**SCD operator**

\[ T \in L(X) \text{ is an SCD-operator if } T(B_X) \text{ is an SCD-set.} \]

**Examples**

\[ T \text{ is an SCD-operator when } T(B_X) \text{ is separable and } \]

1. \( T(B_X) \) is RPN,
2. \( T(B_X) \) has no \( \ell_1 \) sequences,
3. \( T \) does not fix copies of \( \ell_1 \)

**Theorem**

\[ X \text{ ADP } + T \text{ SCD-operator } \implies \max_{\theta \in T} \| \text{Id} + \theta T \| = 1 + \| T \|. \]

\[ X \text{ DPr } + T \text{ SCD-operator } \implies \| \text{Id} + T \| = 1 + \| T \|. \]

**Main corollary**

\[ X \text{ ADP } + T \text{ does not fix copies of } \ell_1 \implies \max_{\theta \in T} \| \text{Id} + \theta T \| = 1 + \| T \|. \]
SCD operators

SCD operator

\( T \in L(X) \) is an **SCD-operator** if \( T(B_X) \) is an SCD-set.

Examples

\( T \) is an SCD-operator when \( T(B_X) \) is separable and

1. \( T(B_X) \) is RPN,
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3. \( T \) does not fix copies of \( \ell_1 \)

Theorem

- \( X \text{ ADP} + T \text{ SCD-operator} \iff \max_{\theta \in T} \| \text{Id} + \theta T \| = 1 + \| T \| \).
- \( X \text{ DPr} + T \text{ SCD-operator} \iff \| \text{Id} + T \| = 1 + \| T \| \).

**Remark**

Separability is not needed!

Main corollary

\( X \text{ ADP} + T \) does not fix copies of \( \ell_1 \) \( \iff \max_{\theta \in T} \| \text{Id} + \theta T \| = 1 + \| T \| \).
Open questions

On SCD-sets

- Find more sufficient conditions for a set to be SCD.
- For instance, if $X$ has 1-symmetric basis, is $B_X$ an SCD-set?
- Is SCD equivalent to the existence of a countable $\pi$-base for the weak topology?
## Open questions

### On SCD-sets
- Find more sufficient conditions for a set to be SCD.
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### On SCD-spaces
- $E$ with unconditional basis. Is $E$ SCD?
- $X, Y$ SCD. Are $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$ SCD?
Open questions

On SCD-sets

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On SCD-spaces

- $E$ with unconditional basis. Is $E$ SCD?
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On SCD-operators

- $T_1$, $T_2$ SCD-operators, is $T_1 + T_2$ an SCD-operator?
- $T : X \rightarrow Y$ hereditary SCD, is there $Z$ SCD-space such that $T$ factor through $Z$?
Remarks on two recent results

- Containment of $c_0$ or $\ell_1$
- On the numerical index of $L_p(\mu)$

A. Avilés, V. Kadets, M. Martín, J. Merí, and V. Shepelska.
Slicely countably determined Banach spaces.

V. Kadets, M. Martín, J. Merí, and R. Payá.
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M. Martín, J. Merí, and M. Popov.
On the numerical index of real $L_p(\mu)$-spaces.
*Preprint.*
Open question (Godefroy, private communication)

\[ X \text{ real}, \ dim(X) = \infty, \ n(X) = 1 \implies X \supset c_0 \text{ or } X \supset \ell_1 ? \]
Remarks on two recent results

Containment of $c_0$ or $\ell_1$

Open question (Godefroy, private communication)

$X$ real, $\dim(X) = \infty$, $n(X) = 1 \implies X \supset c_0$ or $X \supset \ell_1$?

★ Old approaches to this problem:
Remarks on two recent results Containment of $c_0$ or $\ell_1$

## Containment of $c_0$ or $\ell_1$

### Open question (Godefroy, private communication)

$$X \text{ real, } \dim(X) = \infty, \ n(X) = 1 \implies X \supset c_0 \text{ or } X \supset \ell_1?$$

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★ Old approaches to this problem:

- **López–M.–Payá, 1999:**

  $X$ real, **RNP**, $\dim(X) = \infty, \ n(X) = 1 \implies X \supset \ell_1.$

- **Kadets–M.–Merí–Payá, 2009:**

  $X$ real lush, $\dim(X) = \infty \implies X^* \supset \ell_1.$

- **Avilés–Kadets–M.–Merí–Shepelska, 2010:**

  $X$ real, $\dim(X) = \infty \implies X^* \supset \ell_1.$
Remarks on two recent results

Containment of $c_0$ or $\ell_1$

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Containment of $c_0$ or $\ell_1$

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Remarks on two recent results

Containment of $c_0$ or $\ell_1$

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- **(LMP 1999):** This gives $X^* \supset c_0$ or $X^* \supset \ell_1 \implies X^* \supset \ell_1$ ✓
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 فإذا مشكلة

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★ Equivalent reformulation of the problem:

**Equivalent open problem**

\[
X \text{ real separable, } X \not\supset \ell_1, \text{ exists } G \subseteq S_{X^*} \text{ norming with }
\]

\[
B_X = \overline{\text{aconv}} \left( \{ x \in B_X : x^*(x) = 1 \} \right) \quad (x^* \in G).
\]

Does $X \supset c_0$?
On the numerical index of $L_p(\mu)$. 1
On the numerical index of $L_p(\mu)$. I

The numerical radius for $L_p(\mu)$

For $T \in L(L_p(\mu))$, $1 < p < \infty$, one has

$$v(T) = \sup \left\{ \left| \int_{\Omega} x^#Tx \, d\mu \right| : x \in L_p(\mu), \|x\|_p = 1 \right\}.$$

where for $x \in L_p(\mu)$, $x^# = |x|^{p-1} \text{sign}(x) \in L_q(\mu)$ satisfies (unique)

$$\|x\|_p^p = \|x^#\|_q^q \quad \text{and} \quad \int_{\Omega} x x^# \, d\mu = \|x\|_p \|x^#\|_q = \|x\|_p^p.$$
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The absolute numerical radius

For $T \in L(L_p(\mu))$ we write

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On the numerical index of $L_p(\mu)$ (II)

Theorem
For $T \in L_p(\mu)$, $1 < p < \infty$, one has $v(T) \geq M_p^{4/|v|}(T)$, where $M_p = \max_{t \in [0,1]} |t^p - 1 - t|^{1/2} + t^p$.

Theorem
For $T \in L_p(\mu)$, $1 < p < \infty$, one has $2|v| \geq v(T) \geq n(L_{\mathcal{C}p}(\mu)) \|T\|$, where $T_{\mathcal{C}}$ complexification of $T$, $n(L_{\mathcal{C}p}(\mu))$ numerical index complex case.

Consequence
For $1 < p < \infty$, $n(L_p(\mu)) \geq M_p^{8e}$. If $p \neq 2$, then $n(L_p(\mu)) > 0$, so $v$ and $\|\cdot\|$ are equivalent in $L_p(\mu)$.
Remarks on two recent results

On the numerical index of $L_p(\mu)$ (II)

<table>
<thead>
<tr>
<th>Theorem</th>
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</table>

\[ v(T) \geq \frac{M_p}{4} |v|(T), \quad \text{where} \quad M_p = \max_{t \in [0,1]} \frac{|t^{p-1} - t|}{1 + t^p}. \]
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Extremely non-complex Banach spaces

- Motivation
- Extremely non-complex Banach spaces
- Surjective isometries

V. Kadets, M. Martín, and J. Merí.
Norm equalities for operators on Banach spaces.

P. Koszmider, M. Martín, and J. Merí.
Extremely non-complex \( C(K) \) spaces.

P. Koszmider, M. Martín, and J. Merí.
Isometries on extremely non-complex Banach spaces.
Isometries and duality. Reminder
Example (produced with numerical ranges)

There is a Banach space $X$ such that

- $\text{Iso}(X)$ has no exponential one-parameter semigroups.
- $\text{Iso}(X^*)$ contains infinitely many exponential one-parameter semigroups.
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★ In terms of linear dynamical systems:

- There is no $A \in L(X)$ such that

  $$x' = Ax \quad (x : \mathbb{R}_0^+ \longrightarrow X)$$

  is given by a semigroup of isometries.
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- But there are **unbounded** $A$'s on $X$ such that the solution of the linear dynamical system is a one-parameter $C_0$ semigroup of isometries.
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- But there are \textbf{unbounded} $A$'s on $X$ such that the solution of the linear dynamical system is a one-parameter $C_0$ semigroup of isometries.

We would like to find $\mathcal{X}$ such that

- $\text{Iso}(\mathcal{X})$ has no $C_0$ semigroup of isometries.
- $\text{Iso}(\mathcal{X}^*)$ has exponential semigroup of isometries
Numerical range of unbounded operators (1960’s)

Let $X$ be a Banach space, $T : D(T) \to X$ a linear operator, and define

$$V(T) = \{ x^*(Tx) : x^* \in X^*, x \in D(T), x^*(x) = \|x^*\| = \|x\| = 1 \}.$$
Numerical range of unbounded operators (1960’s)

$X$ Banach space, $T : D(T) \rightarrow X$ linear,

$$V(T) = \{ x^*(Tx) : x^* \in X^*, x \in D(T), x^*(x) = \|x^*\| = \|x\| = 1 \}.$$  

Teorema (Stone, 1932)

$H$ Hilbert space, $A$ densely defined operator. TFAE:

- $A$ generates an strongly continuous one-parameter semigroup of unitary operators (onto isometries).
- $A^* = -A$.
- $\text{Re}(Ax \mid x) = 0$ for every $x \in D(A)$. 

Numerical range of unbounded operators. II

Which Banach spaces have unbounded operators with numerical range zero?

Examples

In $C^0(\mathbb{R})$, $\Phi(t)(f)(s) = f(t+s)$ is a strongly continuous one-parameter semigroup of isometries (generated by the derivative).

In $C^0([0, 1]_{\|\Delta\|})$ there are also strongly continuous one-parameter semigroups of isometries.

Consequence

We have to completely change our approach to the problem.
Numerical range of unbounded operators. II

**Difficulty**

Which Banach spaces have unbounded operators with numerical range zero?
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Which Banach spaces have unbounded operators with numerical range zero?

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Complex structures

Definition

$X$ has complex structure if there is $T \in L(X)$ such that $T^2 = -\text{Id}$. 
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Some remarks

- This gives a structure of vector space over $\mathbb{C}$:

$$ (\alpha + i \beta) x = \alpha x + \beta T(x) \quad (\alpha + i \beta \in \mathbb{C}, \ x \in X) $$
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- Defining

\[
\|x\| = \max\{ \|e^{i\theta}x\| : \theta \in [0, 2\pi] \} \quad (x \in X)
\]

one gets that \((X, \| \cdot \|)\) is a complex Banach space.
Complex structures

**Definition**

*X* has **complex structure** if there is *T* ∈ *L*(X) such that *T*² = −Id.

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- If *T* is an isometry, then actually the given norm of *X* is complex.
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- This gives a structure of vector space over \( \mathbb{C} \):
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  one gets that \((X, \| \cdot \|)\) is a complex Banach space.

- If \( T \) is an isometry, then actually the given norm of \( X \) is complex.

- Conversely, if \( X \) is a complex Banach space, then
  \[
  T(x) = i x \quad (x \in X)
  \]
  satisfies \( T^2 = -\text{Id} \) and \( T \) is an isometry.
Some examples

1. If $\dim(X) < \infty$, $X$ has a complex structure iff $\dim(X)$ is even.

2. If $X \cong \mathbb{Z} \oplus \mathbb{Z}$ (in particular, $X \cong X^2$), then $X$ has a complex structure.

3. There are infinite-dimensional Banach spaces without complex structure:
   - Dieudonné, 1952: the James' space $J$ (since $J^{\ast\ast} \equiv J \oplus \mathbb{R}$).
   - Szarek, 1986: uniformly convex examples.
   - Ferenczi-Medina Galego, 2007: there are odd and even infinite-dimensional spaces $X$.
     - $X$ is even if it admits a complex structure but its hyperplanes do not.
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Definition

$X$ is extremely non-complex if $\operatorname{dist}(T^2, -\operatorname{Id})$ is the maximum possible, i.e.

$$
\|\operatorname{Id} + T^2\| = 1 + \|T^2\|,
$$

for $T \in L(X)$. 

Miguel Martín (University of Granada (Spain))

Numerical index theory

Bangalore, June 2009

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**Complex structures II**

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### The Daugavet equation

#### What Daugavet did in 1963

The norm equality

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holds for every *compact* \( T \in L(C[0,1]) \).
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\( X \) Banach space, \( T \in L(X) \), \( \| \text{Id} + T \| = 1 + \| T \| \) (DE).
## The Daugavet equation

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### The Daugavet equation

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### Classical examples

1. **Daugavet, 1963:**
   Every compact operator on \( C[0,1] \) satisfies (DE).

2. **Lozanovskyi, 1966:**
   Every compact operator on \( L_1[0,1] \) satisfies (DE).

3. **Abramovich, Holub, and more, 80's:**
   \( X = C(K) \), \( K \) perfect compact space
   or \( X = L_1(\mu) \), \( \mu \) atomless measure
   \( \implies \) every weakly compact \( T \in L(X) \) satisfies (DE).
The Daugavet property (Kadets–Shvidkoy–Sirotkin–Werner, 1997)

A Banach space $X$ is said to have the Daugavet property iff every rank-one operator on $X$ satisfies (DE).
The Daugavet property

The Daugavet property (Kadets–Shvidkoy–Sirotkin–Werner, 1997)

A Banach space $X$ is said to have the **Daugavet property** iff every rank-one operator on $X$ satisfies (DE).

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Geometric characterization: $X$ has the Daugavet property iff for each $x \in S_X$

$$\overline{co} \left( B_X \setminus (x + (2 - \varepsilon)B_X) \right) = B_X.$$ 

The Daugavet property II

The following spaces have the Daugavet property:

Wojtaszczyk, 1992: The disk algebra and $\mathcal{H}_\infty$.

Werner, 1997: "Nonatomic" function algebras.

Oikhberg, 2005: Non-atomic $C^*$-algebras and preduals of non-atomic von Neumann algebras.


Ivankhno, Kadets, Werner, 2007: $\text{Lip}(K)$ when $K \subseteq \mathbb{R}^n$ is compact and convex.
More examples

The following spaces have the Daugavet property.

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  The disk algebra and $H^\infty$.

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- **Oikhberg, 2005:**
  Non-atomic $C^*$-algebras and preduals of non-atomic von Neumann algebras.

- **Becerra–M., 2005:**
  Non-atomic $JB^*$-triples and their preduals.

- **Becerra–M., 2006:**
  Preduals of $L_1(\mu)$ without Fréchet-smooth points.

- **Ivankhno, Kadets, Werner, 2007:**
  $\text{Lip}(K)$ when $K \subseteq \mathbb{R}^n$ is compact and convex.
Daugavet–type inequalities

For every $1 < p < \infty$, $p \neq 2$, there exists $\psi_p : (0, \infty) \to (0, \infty)$ such that

$$\|\text{Id} + T\| \geq 1 + \psi_p(\|T\|)$$

for every compact operator $T$ on $L^p[0, 1]$.

If $p = 2$, then there is a non-null compact $T$ on $L^2[0, 1]$ such that

$$\|\text{Id} + T\| = 1.$$
Daugavet–type inequalities

Some examples

- **Benyamini–Lin, 1985:**
  For every $1 < p < \infty$, $p \neq 2$, there exists $\psi_p : (0, \infty) \to (0, \infty)$ such that
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    \[\|\text{Id} + T\| = 1.\]

- **Boyko–Kadets, 2004:**
  If $\psi_p$ is the best possible function above, then
  \[
  \lim_{p \to 1^+} \psi_p(t) = t \quad (t > 0).
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- **Oikhberg, 2005:**  
  If $K(\ell_2) \subseteq X \subseteq L(\ell_2)$, then
  \[ \| \text{Id} + T \| \geq 1 + \frac{1}{8\sqrt{2}} \|T\| \]
  for every compact $T$ on $X$.  

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Numerical index theory  
Bangalore, June 2009
Motivation

Norm equalities for operators

Motivating question

Are there other norm equalities which could define interesting properties of Banach spaces?

Concretely

We looked for non-trivial norm equalities of the forms

\[ \|Id + T\| = f(\|T\|) \]

or

\[ \|g(T)\| = f(\|T\|) \]

or

\[ \|Id + g(T)\| = f(g(T)) \]

\((g\text{ analytic, } f\text{ arbitrary})\) satisfied by all rank-one operators on a Banach space.

Solution

We proved that there are few possibilities.
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Concretely

We looked for non-trivial norm equalities of the forms

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Solution

We proved that there are few possibilities...
Equalities of the form $\|\text{Id} + T\| = f(\|T\|)$
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**Proposition**

$X$ real or complex, $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ arbitrary, $a, b \in \mathbb{K}$. If the norm equality

$$\|a \text{Id} + b T\| = f(\|T\|)$$

holds for every rank-one operator $T \in L(X)$, then

$$f(t) = |a| + |b| t \quad (t \in \mathbb{R}_0^+) .$$

If $a \neq 0$, $b \neq 0$, then $X$ has the Daugavet property.
Proposition

\[ X \text{ real or complex, } f : \mathbb{R}_0^+ \longrightarrow \mathbb{R} \text{ arbitrary, } a, b \in K. \text{ If the norm equality} \]
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holds for every rank-one operator \( T \in L(X) \), then

\[ f(t) = |a| + |b| t \quad (t \in \mathbb{R}_0^+). \]

If \( a \neq 0, b \neq 0 \), then \( X \) has the Daugavet property.

Then, we have to look for Daugavet-type equalities in which \( \text{Id} + T \) is replaced by something different.
Proof

We have...

\[ \|a \text{Id} + b T\| = f(\|T\|) \quad \forall T \in L(X) \text{ rank-one} \]
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We want... \[ f(t) = |a| + |b| t \quad (t \in \mathbb{R}^+_0). \]

- Trivial if \( a \cdot b = 0 \). Suppose \( a \neq 0 \) and \( b \neq 0 \) and write \( \omega_0 = \frac{b}{|b|} \frac{a}{|a|} \in \mathbb{T} \).
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- Fix \( x_0 \in S_X, \ x_0^* \in S_{X^*} \) with \( x_0^*(x_0) = \omega_0 \) and consider
  \[ T_t = t \ x_0^* \otimes x_0 \in L(X) \quad (t \in \mathbb{R}_0^+) \].
Proof

We have...  
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- Fix \( x_0 \in S_X, x_0^* \in S_{X^*} \) with \( x_0^*(x_0) = \omega_0 \) and consider
  \[ T_t = t x_0^* \otimes x_0 \in L(X) \quad (t \in \mathbb{R}^+_0). \]
- Since \( \| T_t \| = t \), we have
  \[ f(t) = \| a \text{Id} + b T_t \| \quad (t \in \mathbb{R}^+_0). \]
Proof

We have...

\[ \|a \text{Id} + b T\| = f(\|T\|) \quad \forall T \in L(X) \quad \text{rank-one} \]

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\[ T_t = t x_0^* \otimes x_0 \in L(X) \quad (t \in \mathbb{R}^+_0). \]

- Since \( \|T_t\| = t \), we have

\[ f(t) = \|a \text{Id} + b T_t\| \quad (t \in \mathbb{R}^+_0). \]

- It follows that

\[ |a| + |b| t \geq f(t) = \|a \text{Id} + b T_t\| \geq \|[a \text{Id} + b T_t](x_0)\| \]

\[ = \|a x_0 + b \omega_0 t x_0\| = |a + b \omega_0 t| \|x_0\| = |a + b \bar{b} \frac{a}{|b|} \frac{t}{|a|}| = |a| + |b| t. \]
Proof

We have...

\[ \|a \text{Id} + b \, T\| = f(\|T\|) \quad \forall T \in L(X) \text{ rank-one} \]

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- Fix \( x_0 \in S_X, \ x_0^* \in S_X^* \) with \( x_0^*(x_0) = \omega_0 \) and consider

\[ T_t = tx_0^* \otimes x_0 \in L(X) \quad (t \in \mathbb{R}_0^+) \]

- Since \( \|T_t\| = t \), we have

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\[ = \|a \, x_0 + b \, \omega_0 \, t \, x_0\| = |a + b \, \omega_0 \, t| \, \|x_0\| = \left| a + \frac{\bar{b}}{|b|} \frac{a}{|a|} \, t \right| = |a| + \frac{\bar{b}}{|b|} \frac{a}{|a|} \, t \]

- Finally, for rank-one \( T \in L(X) \), write \( S = \frac{a}{b} \, T \) and observe

\[ |a| (1 + \|T\|) = |a| + |b| \, \|S\| = \|a \text{Id} + b \, S\| = |a| \, \|\text{Id} + T\| \]. \( \checkmark \)
Equalities of the form \( \|g(T)\| = f(\|T\|) \)
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**Theorem**

Let \( X \) be real or complex with \( \dim(X) \geq 2 \). Suppose that the norm equality

\[
\|g(T)\| = f(\|T\|)
\]

holds for every rank-one operator \( T \in L(X) \), where

- \( g : \mathbb{K} \to \mathbb{K} \) is analytic,
- \( f : \mathbb{R}_0^+ \to \mathbb{R} \) is arbitrary.

Then, there are \( a, b \in \mathbb{K} \) such that

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g(\zeta) = a + b \zeta \quad (\zeta \in \mathbb{K}).
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Then, there are $a, b \in \mathbb{K}$ such that

$$g(\zeta) = a + b \zeta \quad (\zeta \in \mathbb{K}).$$

**Corollary**

Only three norm equalities of the form

$$\|g(T)\| = f(\|T\|)$$

are possible:

- $b = 0$: $\|a \text{Id}\| = |a|$, (trivial cases)
- $a = 0$: $\|b T\| = |b| \|T\|$, (trivial cases)
- $a \neq 0, b \neq 0$: $\|a \text{Id} + b T\| = |a| + |b| \|T\|$, (Daugavet property)
Proof (complex case)

We have...

\[ \| g(T) \| = f(\| T \|) \quad \forall T \in L(X) \text{ rank-one} \]

We want...

\[ g \text{ is affine} \]

\[ \Rightarrow \]

\[ \| a_0 \text{Id} + \tilde{g}(\lambda) T_1 \| = \| g(\lambda T_1) \| = f(|\lambda|) = \| a_0 \text{Id} + a_1 \lambda T_0 \|. \]
Proof (complex case)

We have...
\[ \|g(T)\| = f(\|T\|) \quad \forall T \in L(X) \text{ rank-one} \]

- Write \( g(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k \) and \( \tilde{g} = g - a_0 \).

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- Write \( g(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k \) and \( \tilde{g} = g - a_0 \).
- Take \( x_0, x_1 \in S_X \) and \( x_0^*, x_1^* \in S_{X^*} \) such that \( x_0^*(x_0) = 0 \) and \( x_1^*(x_1) = 1 \),
  and define the operators \( T_0 = x_0^* \otimes x_0 \) and \( T_1 = x_1^* \otimes x_1 \).
We have. . .
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- Write \(g(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k\) and \(\tilde{g} = g - a_0\).
- Take \(x_0, x_1 \in S_X\) and \(x_0^*, x_1^* \in S_{X^*}\) such that
  \[x_0^*(x_0) = 0 \quad \text{and} \quad x_1^*(x_1) = 1,\]
  and define the operators \(T_0 = x_0^* \otimes x_0\) and \(T_1 = x_1^* \otimes x_1\).
- Then \(g(\lambda T_0) = a_0 \text{Id} + a_1 \lambda T_0\) and \(g(\lambda T_1) = a_0 \text{Id} + \tilde{g}(\lambda) T_1\) \((\lambda \in \mathbb{C})\).
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We have...

\[ \|g(T)\| = f(\|T\|) \quad \forall T \in L(X) \text{ rank-one} \]

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- Therefore, for \( \lambda \in \mathbb{C} \) we have
  \[ \|a_0 \text{Id} + \tilde{g}(\lambda) T_1\| = \|g(\lambda T_1)\| = f(|\lambda|) = \|g(\lambda T_0)\| = \|a_0 \text{Id} + a_1 \lambda T_0\|. \]
Proof (complex case)

We have
\[ \|g(T)\| = f(\|T\|) \text{ for all } T \in L(X) \text{ rank-one} \]

We want
\[ g \text{ is affine} \]

- Write \( g(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k \) and \( \tilde{g} = g - a_0 \).
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- Then \( g(\lambda T_0) = a_0 \text{Id} + a_1 \lambda T_0 \) and \( g(\lambda T_1) = a_0 \text{Id} + \tilde{g}(\lambda) T_1 \) \((\lambda \in \mathbb{C})\).
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- We use the triangle inequality to get
  \[ |\tilde{g}(\lambda)| \leq 2|a_0| + |a_1||\lambda| \quad (\lambda \in \mathbb{C}), \]
Proof (complex case)

We have...

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- Then \( g(\lambda T_0) = a_0 \text{Id} + a_1 \lambda T_0 \) and \( g(\lambda T_1) = a_0 \text{Id} + \tilde{g}(\lambda) T_1 \) (\( \lambda \in \mathbb{C} \)).

- Therefore, for \( \lambda \in \mathbb{C} \) we have
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- We use the triangle inequality to get
  \[ |\tilde{g}(\lambda)| \leq 2|a_0| + |a_1||\lambda| \quad (\lambda \in \mathbb{C}), \]

- and so \( \tilde{g} \) is a degree-one polynomial by Cauchy inequalities. \( \checkmark \)
Equalities of the form $\|\text{Id} + g(T)\| = f(\|g(T)\|)$
Equalities of the form \( \|\text{Id} + g(T)\| = f(\|g(T)\|) \)

**Remark**

If \( X \) has the Daugavet property and \( g \) is analytic, then

\[
\|\text{Id} + g(T)\| = |1 + g(0)| - |g(0)| + \|g(T)\|
\]

for every rank-one \( T \in L(X) \).
Equalities of the form $\|\text{Id} + g(T)\| = f(\|g(T)\|)$

Remark

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- but it is to see that for every $g$ another simpler equation can be found.
Equalities of the form \[ \|\text{Id} + g(T)\| = f(\|g(T)\|) \]

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If \( X \) has the Daugavet property and \( g \) is analytic, then

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- Our aim here is not to show that \( g \) has a suitable form,
- but it is to see that for every \( g \) another simpler equation can be found.
- From now on, we have to separate the complex and the real case.
Equalities of the form $\|\text{Id} + g(T)\| = f(\|g(T)\|)$

- **Complex case:**

  $\|\text{Id} + g(0)\| = |1 + g(0)| - |g(0)| + \|g(0)\|\|\text{Id} + T\|$ for every rank-one $T \in L(X)$. Where $g: \mathbb{C} \to \mathbb{C}$ analytic non-constant, $f: \mathbb{R}^+ \to \mathbb{R}$ continuous.
Equalities of the form $\|\text{Id} + g(T)\| = f(\|g(T)\|)$

- **Complex case:**

**Proposition**

Let $X$ be a complex Banach space with $\dim(X) \geq 2$. Suppose that

$$\|\text{Id} + g(T)\| = f(\|g(T)\|)$$

for every rank-one $T \in L(X)$, where

- $g : \mathbb{C} \to \mathbb{C}$ analytic non-constant,
- $f : \mathbb{R}^+_0 \to \mathbb{R}$ continuous.

Then

$$\|(1 + g(0))\text{Id} + T\|$$

$$= |1 + g(0)| - |g(0)| + \|g(0)\text{Id} + T\|$$

for every rank-one $T \in L(X)$. 
Equalities of the form $\|\text{Id} + g(T)\| = f(\|g(T)\|)$

- **Complex case:**

**Proposition**

$X$ complex, $\dim(X) \geq 2$. Suppose that

$$\|\text{Id} + g(T)\| = f(\|g(T)\|)$$

for every rank-one $T \in L(X)$, where

- $g : \mathbb{C} \rightarrow \mathbb{C}$ analytic non-constant,
- $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ continuous.

Then

$$\|(1 + g(0))\text{Id} + T\|$$

$$= |1 + g(0)| - |g(0)| + \|g(0)\text{Id} + T\|$$

for every rank-one $T \in L(X)$.

We obtain two different cases:

- $|1 + g(0)| - |g(0)| \neq 0$ or
- $|1 + g(0)| - |g(0)| = 0$. 


Equalities of the form $||\text{Id} + g(T)|| = f(||g(T)||)$. Complex case

**Theorem**

If $\text{Re} g(0) \neq -1/2$ and

$$||\text{Id} + g(T)|| = f(||g(T)||)$$

for every rank-one $T$, then $X$ has the Daugavet property.
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**Theorem**

If $\text{Re} \, g(0) = -1/2$ and

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for every rank-one $T$, then exists $\theta_0 \in \mathbb{R}$ s.t.

$$\|\text{Id} + e^{i\theta_0} \, T\| = \|\text{Id} + T\|$$

for every rank-one $T \in L(X)$. 

Miguel Martín (University of Granada (Spain)) Numerical index theory Bangalore, June 2009 118 / 136
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$$\|\text{Id} + e^{i\theta_0} T\| = \|\text{Id} + T\|$$

for every rank-one $T \in L(X)$.

**Example**

If $X = C[0, 1] \oplus_2 C[0, 1]$, then

- $\|\text{Id} + e^{i\theta} T\| = \|\text{Id} + T\|$ for every $\theta \in \mathbb{R}$, rank-one $T \in L(X)$.
- $X$ does **not** have the Daugavet property.
Equalities of the form $\|\Id + g(T)\| = f(\|g(T)\|)$. Real case

- **Real case:**
Equalities of the form $\|\text{Id} + g(T)\| = f(\|g(T)\|)$. Real case

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  Remarks
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- The proofs are not valid (we use Picard’s Theorem).
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- But we do not know what is the situation when $g$ is not onto, even in the easiest examples:
  - $\|\text{Id} + T^2\| = 1 + \|T^2\|$,
  - $\|\text{Id} - T^2\| = 1 + \|T^2\|$.
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**Example**

If $X = C[0, 1] \oplus_2 C[0, 1]$, then
- $\|\text{Id} - T\| = \|\text{Id} + T\|$ for every rank-one $T \in L(X)$.
- $X$ does not have the Daugavet property.

$g(0) = -1/2$: 
Is there any real Banach space $X$ (with $\dim(X) > 1$) such that

$$\|\operatorname{Id} + T^2\| = 1 + \|T^2\|$$

for every operator $T \in L(X)$?

In other words, are there extremely non-complex spaces other than $\mathbb{R}$?
The first attempts

1. If $\dim(X) < \infty$, $X$ has complex structure iff $\dim(X)$ is even.

2. Dieudonné, 1952: the James' space $J(J^{**} \equiv J \oplus \mathbb{R})$.


5. Ferenczi-Medina-Galego, 2007: there are odd and even infinite-dimensional spaces $X$.

$X$ is even if admits a complex structure but its hyperplanes does not. $X$ is odd if its hyperplanes are even (and so $X$ does not admit a complex structure).

(Un)fortunately. . .

This did not work and we moved to $C(K)$ spaces.
The first attempts

The first idea

We may try to check whether the known spaces without complex structure are actually extremely non-complex.
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Some examples
1. If $\dim(X) < \infty$, $X$ has complex structure iff $\dim(X)$ is even.
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1. If $\dim(X) < \infty$, $X$ has complex structure iff $\dim(X)$ is even.
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This did not work and we moved to $C(K)$ spaces.
The first example: weak multiplications

**Weak multiplication**

Let $K$ be a compact space. $T \in L(C(K))$ is a weak multiplication if

$$T = g \text{Id} + S$$

where $g \in C(K)$ and $S$ is weakly compact.
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Theorem

$K$ perfect, $T = g \text{Id} + S \in L(C(K))$ weak multiplication

$$\implies \|\text{Id} + T^2\| = 1 + \|T^2\|$$
Proof of the theorem
### Proof of the theorem

We have \( X = C(K) \), \( K \) perfect, \( T = g\text{Id} + S \)

- \( \max \|\text{Id} \pm T\| = 1 + \|T\| \) (true for every \( K \) and every \( T \))
- \( \|\text{Id} + S\| = 1 + \|S\| \) (if \( S \in W(X) \), \( K \) perfect)

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- **Step 1:** We assume $\|g^2\| \leq 1$ and $\min g^2(K) > 0$. 

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  $\text{for } g \in C(K) \text{ and } S \text{ weakly compact}$.
- **Step 1:** We assume $\|g^2\| \leq 1$ and $\min g^2(K) > 0$.

**Proof**

- It is enough to show that
  $$\|\text{Id} - (g^2\text{Id} + S)\| < 1 + \|g^2\text{Id} + S\|.$$
Proof of the theorem

We have \( X = C(K), \ K \) perfect, \( T = g \text{Id} + S \)

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  \]
- \( \| \text{Id} - (g^2 \text{Id} + S) \| \leq \| (1 - g^2) \text{Id} \| + \| S \| = 1 - \min g^2(K) + \| S \|. \)
Proof of the theorem

We have $X = C(K)$, $K$ perfect, $T = g\mathrm{Id} + S$

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$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|

**Proof**

- It is enough to show that
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  \]

- $\|\mathrm{Id} - (g^2 \mathrm{Id} + S)\| \leq \|(1 - g^2)\mathrm{Id}\| + \|S\| = 1 - \min g^2(K) + \|S\|.$

- $\|g^2 \mathrm{Id} + S\| = \|\mathrm{Id} + S + (g^2 \mathrm{Id} - \mathrm{Id})\| \geq \|\mathrm{Id} + S\| - \|g^2 \mathrm{Id} - \mathrm{Id}\|
  = 1 + \|S\| - (1 - \min g^2(K)) = \|S\| + \min g^2(K).$
Proof of the theorem

We have $X = C(K)$, $K$ perfect, $T = g\text{Id} + S$

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Proof

Just think that the set of operators satisfying (DE) is closed.
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- **Step 3:** Finally, for every $g$ the above gives

$$\left\| \text{Id} + \frac{1}{\|g^2\|} \left( g^2 \text{Id} + S \right) \right\| = 1 + \frac{1}{\|g^2\|} \|g^2 \text{Id} + S\|$$

which gives us the result. ✓
Proof of the theorem

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Proof

If $\|u + v\| = \|u\| + \|v\| \implies \|\alpha u + \beta v\| = \alpha \|u\| + \beta \|v\|$ for $\alpha, \beta \in \mathbb{R}_0^+$. 
## Weak multiplication

Let $K$ be a compact space. $T \in L(C(K))$ is a **weak multiplication** if

$$T = g \text{Id} + S$$

where $g \in C(K)$ and $S$ is weakly compact.

## Theorem

$K$ perfect, $T = g \text{Id} + S \in L(C(K))$ weak multiplication

$$\implies \| \text{Id} + T^2 \| = 1 + \| T^2 \|$$
The first example: weak multiplications. II

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Example (Koszmider, 2004; Plebanek, 2004)

There are perfect compact spaces $K$ such that all operators on $C(K)$ are weak multiplications.
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There are perfect compact spaces $K$ such that all operators on $C(K)$ are weak multiplications.

Consequence

Therefore, there are extremely non-complex $C(K)$ spaces.
More examples: weak multipliers

**Weak multiplier**

Let $K$ be a compact space. $T \in L(C(K))$ is a weak multiplier if

$$T^* = g \text{Id} + S$$

where $g$ is a Borel function and $S$ is weakly compact.
More examples: weak multipliers

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If $K$ is perfect and all operators on $C(K)$ are weak multipliers, then $C(K)$ is extremely non-complex.
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**Example (Koszmider, 2004)**

There are infinitely many different perfect compact spaces $K$ such that all operators on $C(K)$ are weak multipliers.
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Example (Koszmider, 2004)

There are infinitely many different perfect compact spaces $K$ such that all operators on $C(K)$ are weak multipliers.

Corollary

There are infinitely many non-isomorphic extremely non-complex Banach spaces.
Further examples

Proposition
There is a compact infinite totally disconnected and perfect space $K$ such that all operators on $C(K)$ are weak multipliers.

Consequence
There is a family $(K_i)_{i \in I}$ of pairwise disjoint perfect and totally disconnected compact spaces such that every operator on $C(K_i)$ is a weak multiplier, for $i \neq j$, every $T \in L(C(K_i), C(K_j))$ is weakly compact.

Theorem
There are some compactifications $\tilde{K}$ of the above family $(K_i)_{i \in I}$ such that the corresponding $C(\tilde{K})$'s are extremely non-complex.
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Further examples II

There are perfect compact spaces $K_1, K_2$ such that:

- $C(K_1)$ and $C(K_2)$ are extremely non-complex.
- $C(K_1)$ contains a complemented copy of $C(\Delta)$.
- $C(K_2)$ contains a $1$-complemented isometric copy of $\ell_\infty$.

Observation: $C(K_1)$ and $C(K_2)$ have operators which are not weak multipliers. They are not indecomposable spaces.
Further examples II

Main consequence

There are perfect compact spaces $K_1$, $K_2$ such that:

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Observation

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Related open questions

Question 1
Find topological characterization of the compact Hausdorff spaces $K$ such that the spaces $C(K)$ are extremely non-complex.

Question 2
Find topological consequences on $K$ when $C(K)$ is extremely non-complex. For instance:
If $C(K)$ is extremely non-complex and $\psi: K \to K$ is continuous, are there an open subset $U$ of $K$ such that $\psi|_U = \text{id}$ and $\psi(K \setminus U)$ is finite?

We will show later that $\phi: K \to K$ homeomorphism $\Rightarrow \phi = \text{id}$.
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- We will show latter than $\phi : K \to K$ homeomorphism $\implies \phi = \text{id}$. 
**Extremely non-complex Banach spaces**

**Definition**

$X$ is **extremely non-complex** if $\text{dist}(T^2, -\text{Id})$ is the maximum possible, i.e.

$$\|\text{Id} + T^2\| = 1 + \|T^2\| \quad (T \in L(X))$$
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**Examples**

There are several extremely non-complex $C(K)$ spaces:

- If $T = g\text{Id} + S$ for every $T \in L(C(K))$ ($K$ Koszmider).
- If $T^* = g\text{Id} + S$ for every $T \in L(C(K))$ ($K$ weak Koszmider).
- One $C(K)$ containing a complemented copy of $C(\Delta)$.
- One $C(K)$ containing an isometric (1-complemented) copy of $\ell_\infty$. 
Isometries on extremely non-complex spaces. I

Theorem

$X$ extremely non-complex.

- $T \in \text{Iso}(X) \implies T^2 = \text{Id}$.
- $T_1, T_2 \in \text{Iso}(X) \implies T_1 T_2 = T_2 T_1$.
- $T_1, T_2 \in \text{Iso}(X) \implies \|T_1 - T_2\| \in \{0, 2\}$.
- $\Phi : \mathbb{R}_0^+ \longrightarrow \text{Iso}(X)$ one-parameter semigroup $\implies \Phi(\mathbb{R}_0^+) = \{\text{Id}\}$. 

Theorem

Let $X$ be extremely non-complex.

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- If $T_1, T_2 \in \text{Iso}(X)$, then $\|T_1 - T_2\| \in \{0, 2\}$.
- If $\Phi : \mathbb{R}^+_0 \to \text{Iso}(X)$ is a one-parameter semigroup, then $\Phi(\mathbb{R}^+_0) = \{\text{Id}\}$.

Proof.
Isometries on extremely non-complex spaces. 1

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Proof.

- Take $S = \frac{1}{\sqrt{2}} (T - T^{-1}) \implies S^2 = \frac{1}{2} T^2 - \text{Id} + \frac{1}{2} T^{-2}$. 
**Isometries on extremely non-complex spaces. I**

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Isometries on extremely non-complex spaces. I

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- Then \( \text{Id} = \frac{1}{2} T^2 + \frac{1}{2} T^{-2}. \)
Isometries on extremely non-complex spaces. I

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- Take \( S = \frac{1}{\sqrt{2}} (T - T^{-1}) \implies S^2 = \frac{1}{2} T^2 - \text{Id} + \frac{1}{2} T^{-2}. \)
- \( 1 + \|S^2\| = \|\text{Id} + S^2\| = \left\| \frac{1}{2} T^2 + \frac{1}{2} T^{-2} \right\| \leq 1 \implies S^2 = 0. \)
- Then \( \text{Id} = \frac{1}{2} T^2 + \frac{1}{2} T^{-2}. \)
- Since \( \text{Id} \) is an extreme point of \( B_L(X) \implies T^2 = T^{-2} = \text{Id}. \)  

✓
Theorem

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Proof.

\[
\text{Id} = (T_1 T_2)(T_1 T_2) \\
\implies T_1 T_2 = T_1 (T_1 T_2 T_1 T_2) T_2 = (T_1 T_1) T_2 T_1 (T_2 T_2) = T_2 T_1.
\]
Isometries on extremely non-complex spaces. 1

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Proof.

- \( (\text{Id} - T)^2 = 2(\text{Id} - T) \implies 2\|\text{Id} - T\| = \|(\text{Id} - T)^2\| \leq \|\text{Id} - T\|^2 \).
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- So \( \|\text{Id} - T\| \in \{0, 2\}. \)
- \( \|T_1 - T_2\| = \|T_1(\text{Id} - T_1 T_2)\| = \|\text{Id} - T_1 T_2\| \in \{0, 2\}. \) \( \checkmark \)
Isometries on extremely non-complex spaces. I

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- $\Phi : \mathbb{R}_0^+ \longrightarrow \text{Iso}(X)$ one-parameter semigroup $\implies \Phi(\mathbb{R}_0^+) = \{\text{Id}\}$.

Proof.

$\Phi(t) = \Phi(t/2 + t/2) = \Phi(t/2)^2 = \text{Id}$. ✓
Isometries on extremely non-complex spaces. I

**Theorem**

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Isometries on extremely non-complex spaces. I

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- $\text{Iso}(X)$ is a Boolean group for the composition operation.
Isometries on extremely non-complex spaces. I

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- $\text{Iso}(X)$ is a Boolean group for the composition operation.
- $\text{Iso}(X)$ identifies with the set $\text{Unc}(X)$ of unconditional projections on $X$:  
  
  $P \in \text{Unc}(X) \iff P^2 = P, \ 2P - \text{Id} \in \text{Iso}(X)$
  
  $\iff P = \frac{1}{2}(\text{Id} - T), \ T \in \text{Iso}(X), \ T^2 = \text{Id}.$
Extremely non-complex Surjective isometries

Isometries on extremely non-complex spaces. I

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  $$\iff P = \frac{1}{2}(\text{Id} - T), \ T \in \text{Iso}(X), \ T^2 = \text{Id}.$$

- $\text{Iso}(X) \equiv \text{Unc}(X)$ is a Boolean algebra
  
  $$\iff P_1 P_2 \in \text{Unc}(X) \text{ when } P_1, P_2 \in \text{Unc}(X)$$
  
  $$\iff \left\| \frac{1}{2} \left( \text{Id} + T_1 + T_2 - T_1 T_2 \right) \right\| = 1 \text{ for every } T_1, T_2 \in \text{Iso}(X).$$
Extremely non-complex $C_E(K\|L)$ spaces.
Extremely non-complex \( C_E(K\|L) \) spaces.

**Theorem**

\( K \) perfect weak Koszmider, \( L \) closed nowhere dense, \( E \subset C(L) \)
\[ \implies C_E(K\|L) \text{ is extremely non-complex.} \]
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**Proposition**

$K$ perfect $\implies \exists L \subset K$ closed nowhere dense with $C[0,1] \subset C(L)$. 
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### Observation: exists a non \( C(K) \) extremely non-complex space

\( C_{\ell_2}(K\|L) \) is not isomorphic to a \( C(K') \) space since \( \ell_2 \xrightarrow{\text{comp}} C_{\ell_2}(K\|L)^* \).
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**Observation: exists a non $C(K)$ extremely non-complex space**

$C_{\ell^2}(K\|L)$ is not isomorphic to a $C(K')$ space since $\ell^2 \overset{\text{comp}}{\rightarrow} C_{\ell^2}(K\|L)^*$.

**Important consequence: Example**

Take $K$ perfect weak Koszmider, $L \subset K$ closed nowhere dense with $E = \ell^2 \subset C[0,1] \subset C(L)$:

- $C_{\ell^2}(K\|L)$ has no non-trivial one-parameter semigroup of isometries.
- $C_{\ell^2}(K\|L)^* = \ell^2 \oplus C_0(K\|L)^*$, so $\text{Iso}(C_{\ell^2}(K\|L)^*) \supset \text{Iso}(\ell^2)$. 

But we are able to give a better result...
Theorem

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But we are able to give a better result...
Isometries on extremely non-complex $C_E(K\|L)$ spaces

Theorem

Let $T \in \text{Iso}(C_E(K\|L)) = \exists \theta : K \setminus L \rightarrow \{-1, 1\}$ continuous such that $T(f)(x) = \theta(x)f(x)$ for $x \in K \setminus L$, $f \in C_E(K\|L)$.
Isometries on extremely non-complex $C_E(K\|L)$ spaces

**Theorem**

$C_E(K\|L)$ extremely non-complex, $T \in \text{Iso}(C_E(K\|L))$

$\implies$ exists $\theta : K \setminus L \to \{-1, 1\}$ continuous such that

$$[T(f)](x) = \theta(x)f(x) \quad (x \in K \setminus L, f \in C_E(K\|L))$$
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**Sketch of the proof.**

- $D_0 = \{x \in K \setminus L : \exists y \in K \setminus L, \theta_0 \in \{-1, 1\} \text{ with } T^*(\delta_x) = \theta_0\delta_y\}$ dense in $K$.  

- $\phi(x) = x$ for all $x \in D_0$.  

- $\theta$ is continuous.  

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- Consider $\phi : D_0 \longrightarrow D_0$ and $\theta : D_0 \longrightarrow \{-1, 1\}$ with

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- Consider $\phi : D_0 \to D_0$ and $\theta : D_0 \to \{-1, 1\}$ with

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- $\phi^2 = \text{id}, \ \theta(x)\theta(\phi(x)) = 1$, $\phi$ homeomorphism.
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- $D_0 = \{ x \in K \setminus L : \exists y \in K \setminus L, \theta_0 \in \{-1, 1\} \text{ with } T^*(\delta_x) = \theta_0 \delta_y \}$ dense in $K$.
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  \[ T^*(\delta_x) = \theta(x) \delta_{\phi(x)} \]

- $\phi^2 = \text{id}$, $\theta(x) \theta(\phi(x)) = 1$, $\phi$ homeomorphism.
- $\phi(x) = x$ for all $x \in D_0$.
- $D_0 = K \setminus L$. 
**Theorem**

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  $$T^*(\delta_x) = \theta(x)\delta_{\phi(x)}$$

- $\phi^2 = \text{id}$, $\theta(x)\theta(\phi(x)) = 1$, $\phi$ homeomorphism.
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- $\theta$ is continuous. \(\surd\)
Isometries on extremely non-complex $C_E(K\|L)$ spaces

**Theorem**

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**Consequences: cases $E = C(L)$ and $E = 0$**

- $C(K)$ extremely non-complex, $\varphi: K \to K$ homeomorphism $\implies \varphi = \text{id}$
- $C_0(K \setminus L) \equiv C_0(K\|L)$ extremely non-complex, $\varphi: K \setminus L \to K \setminus L$ homeomorphism $\implies \varphi = \text{id}$
- In both cases, the group of surjective isometries identifies with a Boolean algebra of clopen sets.
Isometries on extremely non-complex $C_E(K\|L)$ spaces

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$$[T(f)](x) = \theta(x)f(x) \quad (x \in K \setminus L, f \in C_E(K\|L))$$

**Consequences: general case**

- If for every $x \in L$, there is $f \in E$ with $f(x) \neq 0$

  $\implies$ $\theta$ extends to the whole $K$ and

  $$[T(f)](x) = \theta(x)f(x) \quad (x \in K, f \in C_E(K\|L))$$

  for every surjective isometry $T$. 
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**Consequences: general case**

- If for every $x \in L$, there is $f \in E$ with $f(x) \neq 0$
  $\implies$ $\theta$ extends to the whole $K$ and

$$[T(f)](x) = \theta(x)f(x) \quad (x \in K, \ f \in C_E(K\|L))$$

  for every surjective isometry $T$.

- If this happens, then $0 \notin \text{ext} \left(B_E^*\right)^{w^*}$ (V. Kadets).
Isometries on extremely non-complex $C_E(K\|L)$ spaces

**Theorem**

$C_E(K\|L)$ extremely non-complex, $T \in \text{Iso}(C_E(K\|L))$

$\implies$ exists $\theta : K \setminus L \to \{-1, 1\}$ continuous such that

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- But for $E = \ell_2$, $0 \in \overline{\text{ext} \left( B_{E^*} \right)^{w^*}}$. 
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$C_E(K\|L)$ extremely non-complex, $T \in \text{Iso}(C_E(K\|L))$

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$(x \in K \setminus L, f \in C_E(K\|L))$

Consequence: connected case

If $K$ and $K \setminus L$ are connected, then

$$\text{Iso}(C_E(K\|L)) = \{-\text{Id}, +\text{Id}\}$$
The main example
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Koszmider, 2004

∃ \mathcal{K} weak Koszmider space such that \mathcal{K} \setminus F is connected if |F| < \infty.
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\[ \exists \mathcal{K} \text{ weak Koszmider space such that } \mathcal{K} \setminus F \text{ is connected if } |F| < \infty. \]

Observation on the above construction

There is \( \mathcal{L} \subset \mathcal{K} \) closed nowhere dense with

- \( \mathcal{K} \setminus \mathcal{L} \) connected
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The best example

Consider \( X = \text{C}_\ell_2(\mathcal{K} \| \mathcal{L}) \). Then:

\[
\text{Iso}(X) = \{-\text{Id}, +\text{Id}\} \quad \text{and} \quad \text{Iso}(X^*) \supset \text{Iso}(\ell_2)
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Proof.
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- \( \mathcal{K} \) weak Koszmider, \( \mathcal{L} \) nowhere dense, \( \ell_2 \subset C(\mathcal{L}) \)
  \( \implies \) \( X \) well-defined and extremely non-complex.
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  \[ \implies X \text{ well-defined and extremely non-complex.} \]
- \( \mathcal{K} \setminus \mathcal{L} \) connected \[ \implies \text{Iso}(X) = \{-\text{Id}, +\text{Id}\}. \]
The main example

Koszmider, 2004

\[ \exists K \text{ weak Koszmider space such that } K \setminus F \text{ is connected if } |F| < \infty. \]

Observation on the above construction

There is \( L \subset K \) closed nowhere dense with
- \( K \setminus L \) connected
- \( C[0,1] \subseteq C(L) \)

The best example

Consider \( X = C_{\ell^2}(K\|L) \). Then:

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Proof.

- \( K \) weak Koszmider, \( L \) nowhere dense, \( \ell^2 \subset C(L) \)
  \[ \implies X \text{ well-defined and extremely non-complex.} \]
- \( K \setminus L \) connected \( \implies \text{Iso}(X) = \{-\text{Id}, +\text{Id}\}. \)
- \( X^* = \ell^2 \oplus C_0(K\|L)^* \), so \( \text{Iso}(\ell^2) \subset \text{Iso}(X^*) \). \( \checkmark \)
Open questions on extremely non-complex Banach spaces

1. Does $X$ have the Daugavet property?
2. Stronger: Does $Y$ have the Daugavet property if $\|\text{Id} + T\| = 1 + \|T\|$ for every rank-one $T \in \mathcal{L}(Y)$?
3. Is it true that $\text{n}(X) = 1$?
4. We actually know that $\text{n}(X) \geq C > 0$.
5. Is $\text{Iso}(X) \equiv \text{Unc}(X)$ a Boolean algebra?
6. If $Y \leq X$ is 1-codimensional, is $Y$ extremely non complex?
7. Is it possible that $X \cong Z \oplus Z \oplus Z$?
Open questions on extremely non-complex Banach spaces

Questions

- Does $X$ have the Daugavet property?

If $Y \leq X$ is 1-codimensional, is $Y$ extremely non complex?
Open questions on extremely non-complex Banach spaces

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$X$ extremely non complex

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**X** extremely non complex

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- Is it true that \( n(\mathcal{X}) = 1 \)?
  - We actually know that \( n(\mathcal{X}) \geq C > 0 \).
- Is \( \text{Iso}(\mathcal{X}) \equiv \text{Unc}(\mathcal{X}) \) a Boolean algebra?
- If \( \mathcal{Y} \leq \mathcal{X} \) is 1-codimensional, is \( \mathcal{Y} \) extremely non complex?
- Is it possible that \( \mathcal{X} \cong \mathcal{Z} \oplus \mathcal{Z} \oplus \mathcal{Z} \)?
Schedule of the talk

1. Basic notation
2. Numerical range of operators
3. Two results on surjective isometries
4. Numerical index of Banach spaces
5. The alternative Daugavet property
6. Lush spaces
7. Slicely countably determined spaces
8. Remarks on two recent results
9. Extremely non-complex Banach spaces