Slicely Countably Determined spaces

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SLICELY COUNTABLY DETERMINED BANACH SPACES.
APPLICATIONS TO THE DAUGAVET AND THE ALTERNATIVE
DAUGAVET EQUATIONS

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ABSTRACT. We introduce the class of slicely countable determined Banach spaces which contains in particular all spaces with the RNP and all spaces without copies of $\ell_1$. We present many examples and several properties of this class. We give some applications to Banach spaces with the Daugavet and the alternative Daugavet properties, lush spaces and Banach spaces with numerical index 1. In particular, we show that the dual of a real infinite-dimensional Banach with the alternative Daugavet property contains $\ell_1$ and that operators which do not fix copies of $\ell_1$ on a space with the alternative Daugavet property satisfy the alternative Daugavet equation.
Basic notation and main objective

Basic notation

\( X \) real or complex Banach space.
- \( S_X \) unit sphere, \( B_X \) closed unit ball, \( T \) modulus-one scalars.
- \( X^* \) dual space, \( L(X) \) bounded linear operators.
- \( \text{conv}(\cdot) \) convex hull, \( \overline{\text{conv}}(\cdot) \) closed convex hull
- A slice of \( A \subset X \) is a subset of the form
  \[
  S(A, x^*, \alpha) = \{ x \in A : \Re x^*(x) > \sup \Re x^*(A) - \alpha \} \quad (x^* \in X^*, \alpha > 0)
  \]

Objective

- We introduce an isomorphic property for (separable) Banach spaces called **Slicely Countable Determined (SCD)** such that
  - it is satisfied by RNP spaces,
  - it is satisfied by spaces not containing \( \ell_1 \).
- We present some stability results.
- We apply SCD to get results for the Daugavet property, the alternative Daugavet property and spaces with numerical index 1.
Outline

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2. Slicely Countably Determined sets and spaces
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   - SCD spaces

3. Applications
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Motivation:
The Daugavet property, the alternative Daugavet property and spaces with numerical index 1
Definition of the properties

1. **Kadets-Shvidkoy-Sirotkin-Werner, 1997:**
   $X$ has the Daugavet property (DPr) if
   \[ \| \text{Id} + T \| = 1 + \| T \| \quad \text{(DE)} \]
   for every rank-one $T \in L(X)$.
   - Then every $T$ not fixing copies of $\ell_1$ also satisfies (DE).

2. **Lumer, 1968:** $X$ has numerical index 1 if every operator on $X$ satisfies
   \[ \max_{\theta \in \mathbb{T}} \| \text{Id} + \theta T \| = 1 + \| T \| \quad \text{(aDE)} \]
   Equivalently,
   \[ \| T \| = \sup \{ |x^*(Tx)| : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1 \} \]
   for every $T \in L(X)$.

3. **M.-Oikhberg, 2004:** $X$ has the alternative Daugavet property (ADP) if
   every rank-one $T \in L(X)$ satisfies (aDE).
   - Then every weakly compact $T$ also satisfies (aDE).
Relations between these properties

- **Daugavet property** ↔ **Numerical index 1**

**Examples**
- $C([0,1], K(ℓ_2))$ has DPr, but has not numerical index 1
- $c_0$ has numerical index 1, but has not DPr
- $c_0 ⊕ ∞ C([0,1], K(ℓ_2))$ has ADP, neither DPr nor numerical index 1

**Remarks**
- For RNP or Asplund spaces, **ADP** $\Rightarrow$ numerical index 1.
- Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.
Let $V_*$ be the predual of the von Neumann algebra $V$.

**The Daugavet property of $V_*$ is equivalent to:**
- $V$ has no atomic projections, or
- the unit ball of $V_*$ has no extreme points.

**$V_*$ has numerical index 1 iff:**
- $V$ is commutative, or
- $|v^*(v)| = 1$ for $v \in \text{ext}(B_V)$ and $v^* \in \text{ext}(B_{V^*})$.

**The alternative Daugavet property of $V_*$ is equivalent to:**
- the atomic projections of $V$ are central, or
- $|v(v_*)| = 1$ for $v \in \text{ext}(B_V)$ and $v_* \in \text{ext}(B_{V_*})$, or
- $V = C \oplus \infty N$, where $C$ is commutative and $N$ has no atomic projections.
Let $X$ be a $C^*$-algebra.

The Daugavet property of $X$ is equivalent to:
- $X$ does not have any atomic projection, or
- the unit ball of $X^*$ does not have any $w^*$-strongly exposed point.

$X$ has numerical index 1 iff:
- $X$ is commutative, or
- $|x^{**}(x^*)| = 1$ for $x^{**} \in \text{ext}(B_{X^{**}})$ and $x^* \in \text{ext}(B_{X^*}).$

The alternative Daugavet property of $X$ is equivalent to:
- the atomic projections of $X$ are central, or
- $|x^{**}(x^*)| = 1$, for $x^{**} \in \text{ext}(B_{X^{**}})$, and $x^* \in B_{X^*} \ w^*$-strongly exposed, or
- $\exists$ a commutative ideal $Y$ such that $X/Y$ has the Daugavet property.
Slicely Countably Determined spaces
**SCD sets: Definitions and preliminary remarks**

\(X\) Banach space, \(A \subset X\) bounded and convex.

**Determining sequence**

\(\{V_n : n \in \mathbb{N}\}\) family of subsets of \(A\) is **determining** if 
\(A \subset \text{conv}(B)\) for every \(B \subseteq A\) such that \(B \cap V_n \neq \emptyset\) for all \(n \in \mathbb{N}\).

**Remark**

\(\{V_n\}\) is determining \iff every slice of \(A\) contains one of the \(V_n\)'s.

**SCD sets**

\(A\) is **Slicely Countably Determined (SCD)** if it admits a determining sequence of slices.

**Remarks**

- \(A\) is SCD iff \(\overline{A}\) is SCD.
- If \(A\) is SCD, then it is separable.
**SCD sets: Elementary examples I**

**Example**

*A* separable and \( A = \overline{\text{conv}}(\text{dent}(A)) \) \( \implies \) \( A \) is SCD.

**Proof.**

- Take \( \{a_n : n \in \mathbb{N}\} \) denting points with \( A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\}) \).
- For every \( n, m \in \mathbb{N} \), take a slice \( S_{n,m} \) containing \( a_n \) and of diameter \( 1/m \).
- If \( B \cap S_{n,m} \neq \emptyset \) \( \implies \) \( a_n \in B \).
- Therefore, \( A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\}) \subseteq \overline{\text{conv}}(B) = \overline{\text{conv}}(B) \).

**Example**

In particular, \( A \) RNP separable \( \implies \) \( A \) SCD.

**Corollary**

- If \( X \) is separable LUR \( \implies \) \( B_X \) is SCD.
- So, every separable space can be renormed such that \( B(X,\|\cdot\|) \) is SCD.
Motivation

SCD sets & spaces

Applications

Open questions

SCD sets: Elementary examples II

Example

If $X^*$ is separable $\implies A$ is SCD.

Proof.

- Take $\{x_n^* : n \in \mathbb{N}\}$ dense in $S_{X^*}$.
- For every $n, m \in \mathbb{N}$, consider $S_{n,m} = S(A, x_n^*, 1/m)$.
- It is easy to show that any slice of $A$ contains one of the $S_{n,m}$

Example

If $X$ has the DPr $\implies B_X$ is not SCD. So, $B_{C[0,1]}$, $B_{L_1[0,1]}$ are not SCD.

Proof.

- Fix $x_0 \in B_X$ and $\{S_n\}$ sequence of slices of $B_X$.
- By [KSSW] there is a sequence $(x_n) \subset B_X$ such that
  - $x_n \in S_n$ for every $n \in \mathbb{N}$,
  - $(x_n)_{n \geq 0}$ is equivalent to the basis of $\ell_1$,
  - so $x_0 \notin \overline{\text{lin}}\{x_n : n \in \mathbb{N}\}$
Convex combination of slices

\[ W = \sum_{k=1}^{m} \lambda_k S_k \subset A \] where \( \sum \lambda_k = 1 \), \( S_k \) slices.

Proposition

In the definition of SCD we can use convex combination of slices.

Small combinations of slices

\( A \) has \textbf{small combinations of slices} iff every slice of \( A \) contains convex combinations of slices of \( A \) with arbitrary small diameter.

Example

If \( A \) has small combinations of slices + separable \( \implies A \) is SCD.

Example

\( A \) strongly regular + separable \( \implies A \) is SCD.
SCD sets: Further examples II

**Bourgain’s lemma**
Every relative weak open subset of $A$ contains a convex combination of slices.

**Corollary**
In the definition of SCD we can use relative weak open subsets.

**$\pi$-bases**
A $\pi$-base of the weak topology of $A$ is a family $\{V_i : i \in I\}$ such that every weak open subset of $A$ contains one of the $V_i$’s.

**Proposition**
If $(A, \sigma(X,X^*))$ has a countable $\pi$-base $\implies A$ is SCD.
**Theorem**

A separable without $\ell_1$-sequences $\implies (A, \sigma(X, X^*))$ has a countable $\pi$-base.

**Proof.**

- We see $(A, \sigma(X, X^*)) \subset C(T)$ where $T = (B_{X^*}, \sigma(X^*, X))$.
- By Rosenthal $\ell_1$ theorem, $(A, \sigma(X, X^*))$ is a relatively compact subset of the space of first Baire class functions on $T$.
- By a result of Todorčević, $(A, \sigma(X, X^*))$ has a $\sigma$-disjoint $\pi$-base.
- A $\sigma$-disjoint family of open subsets in a separable space is countable.

**Example**

A separable without $\ell_1$-sequences $\implies A$ is SCD.
## SCD spaces: definition and examples

### SCD space

$X$ is **Slicely Countably Determined (SCD)** if so are its convex bounded subsets.

### Examples of SCD spaces

1. $X$ separable strongly regular. In particular, RNP, CPCP spaces.
2. $X$ separable $X \not\subseteq \ell_1$. In particular, if $X^*$ is separable.

### Examples of NOT SCD spaces

1. $X$ having the Daugavet property.
2. In particular, $C[0,1]$, $L_1[0,1]$
3. There is $X$ with the Schur property which is not SCD.

### Remark

- Every subspace of a SCD space is SCD.
- This is false for quotients.
SCD spaces: stability properties

Theorem

$Z \subset X$. If $Z$ and $X/Z$ are SCD $\implies X$ is SCD.

Corollary

$X$ separable NOT SCD
- If $\ell_1 \simeq Y \subset X \implies X/Y$ contains a copy of $\ell_1$.
- If $\ell_1 \simeq Y_1 \subset X \implies$ there is $\ell_1 \simeq Y_2 \subset X$ with $Y_1 \cap Y_2 = 0$.

Corollary

$X_1, \ldots, X_m$ SCD $\implies X_1 \oplus \cdots \oplus X_m$ SCD.
SCD spaces: stability properties II

**Theorem**

Let $X_1, X_2, \ldots$ be SCD spaces, and $E$ be a Banach space with unconditional basis.

- If $E \nsubseteq c_0 = \ell_1$ implies that $\bigoplus_{n \in \mathbb{N}} X_n \subseteq E$ is SCD.
- If $E \nsubseteq \ell_1$ implies that $\bigoplus_{n \in \mathbb{N}} X_n \subseteq E$ is SCD.

**Examples**

1. $c_0(\ell_1)$ and $\ell_1(c_0)$ are SCD.
2. $c_0 \otimes \varepsilon c_0$, $c_0 \otimes \pi c_0$, $c_0 \otimes \varepsilon \ell_1$, $c_0 \otimes \pi \ell_1$, $\ell_1 \otimes \varepsilon \ell_1$, and $\ell_1 \otimes \pi \ell_1$ are SCD.
3. $K(c_0)$ and $K(c_0, \ell_1)$ are SCD.
4. $\ell_2 \otimes \varepsilon \ell_2 \cong K(\ell_2)$ and $\ell_2 \oplus \pi \ell_2 \cong \mathcal{L}_1(\ell_2)$ are SCD.
Applications
ADP + SCD $\implies$ numerical index 1

Characterization of ADP

$X$ Banach space. TFAE:

- $X$ has ADP (i.e. $\max_{\theta \in \mathbb{T}} \|\text{Id} + \theta T\| = 1 + \|T\|$ for all $T$ rank-one).
- Given $x \in S_X$, a slice $S$ of $B_X$ and $\varepsilon > 0$, there is $y \in S$ with
  \[
  \max_{\theta \in \mathbb{T}} \|x + \theta y\| > 2 - \varepsilon.
  \]
- Given $x \in S_x$, a sequence $\{S_n\}$ of slices of $B_X$, and $\varepsilon > 0$, there is $y^* \in S_{X^*}$ such that $x \in S(B_X, y^*, \varepsilon)$ and
  \[
  \overline{\text{conv}}(\mathbb{T} S(B_X, y^*, \varepsilon)) \bigcap S_n \neq \emptyset \quad (n \in \mathbb{N}).
  \]

Theorem

$X$ ADP + $B_X$ SCD $\implies$ given $x \in S_X$ and $\varepsilon > 0$, there is $y^* \in S_{X^*}$ such that

\[
  x \in S(B_X, y^*, \varepsilon) \quad \text{and} \quad B_X = \overline{\text{conv}}(\mathbb{T} S(B_X, y^*, \varepsilon)).
\]

- This implies numerical index 1 (i.e. $\max_{\theta \in \mathbb{T}} \|\text{Id} + \theta T\| = 1 + \|T\|$ for all $T$).
Some consequences

Corollary
- ADP + strongly regular $\implies$ numerical index 1.
- ADP + $X \not\subseteq \ell_1$ $\implies$ numerical index 1.

Corollary
$X$ real + dim($X$) = $\infty$ + ADP $\implies$ $X^* \supseteq \ell_1$.

In particular,

Corollary
$X$ real + dim($X$) = $\infty$ + numerical index 1 $\implies$ $X^* \supseteq \ell_1$. 
Motivation

SCD sets & spaces

Applications

Open questions

SCD operators

**SCD operator**

\[ T \in L(X) \text { is an SCD-operator if } T(B_X) \text { is an SCD-set.} \]

**Examples**

\( T \) is an SCD-operator when \( T(B_X) \) is separable and

1. \( T(B_X) \) is RPN,
2. \( T(B_X) \) has no \( \ell_1 \) sequences,
3. \( T \) does not fix copies of \( \ell_1 \)

**Theorem**

- \( X \) ADP + \( T \) SCD-operator \( \implies \) \( \max_{\theta \in T} \|\text{Id} + \theta T\| = 1 + \|T\| \).
- \( X \) DPr + \( T \) SCD-operator \( \implies \|\text{Id} + T\| = 1 + \|T\| \).

**Main corollary**

\( X \) ADP + \( T \) does not fix copies of \( \ell_1 \) \( \implies \) \( \max_{\theta \in T} \|\text{Id} + \theta T\| = 1 + \|T\| \).
Open questions
On SCD-sets
- Find more sufficient conditions for a set to be SCD.
- For instance, if $X$ has 1-symmetric basis, is $B_X$ an SCD-set?

On SCD-spaces
- $E$ with unconditional basis. Is $E$ SCD?
- $X, Y$ SCD. Are $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$ SCD?

On SCD-operators
- $T_1, T_2$ SCD-operators, is $T_1 + T_2$ an SCD-operator?
- $T : X \to Y$ SCD, is there $Z$ SCD-space such that $T$ factor through $Z$?