On a $C(K)$ space with few operators

Miguel Martín
http://www.ugr.es/local/mmartins

Work in progress with Piotr Koszmider and Javier Merí

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Notation and objective

Basic notation

\( X \) real Banach space.
- \( S_X \) unit sphere, \( B_X \) unit ball,
- \( X^* \) dual space, \( L(X) \) bounded linear operators
- \( T^* \in L(X^*) \) adjoint operator of \( T \in L(X) \).

Main Objective

We show that there exists a real Banach space \( X \) such that

\[
\|\text{Id} + T^2\| = 1 + \|T^2\| \quad \text{(for every } T \in L(X)\text{)}
\]

For topologists...

Actually, we may take as \( X \) any \( C(K) \) space, \( K \) perfect compact space such that

\[
T^* = g \text{Id} + S \quad \text{(} g \text{ Borel function, } S \text{ weakly compact)}.
\]

for every \( T \in L(X) \)

(existence of such \( K \)'s proved by Koszmider in 2004).
Motivation
- The Daugavet property
- Daugavet–type inequalities
- Norm equalities for operators

The examples

Consequences

Open Problems
Motivation
The Daugavet equation

What Daugavet did in 1963

The norm equality

\[ \| \text{Id} + T \| = 1 + \| T \| \]

holds for every \textbf{compact} \( T \) on \( C[0, 1] \).

The Daugavet equation

\( X \) Banach space, \( T \in L(X) \), \( \| \text{Id} + T \| = 1 + \| T \| \) \quad (DE).

Classical examples

1. **Daugavet, 1963:**
   Every compact operator on \( C[0, 1] \) satisfies (DE).

2. **Lozanovskii, 1966:**
   Every compact operator on \( L_1[0, 1] \) satisfies (DE).

3. **Abramovich, Holub, and more, 80’s:**
   \( X = C(K) \), \( K \) perfect compact space
   or \( X = L_1(\mu) \), \( \mu \) atomless measure
   \( \implies \) every weakly compact \( T \in L(X) \) satisfies (DE).
The Daugavet property (Kadets–Shvidkoy–Sirotkin–Werner, 1997)

A Banach space $X$ is said to have the Daugavet property iff every rank-one operator on $X$ satisfies (DE).

Some results

Let $X$ be a Banach space with the Daugavet property. Then

- Every weakly compact operator on $X$ satisfies (DE).
- $X$ contains $\ell_1$.
- $X$ does not embed into a Banach spaces with unconditional basis.
- **Geometric characterization**: $X$ has the Daugavet property iff for each $x \in S_X$

\[
\overline{\text{co}} \left( B_X \setminus (x + (2 - \varepsilon)B_X) \right) = B_X.
\]

The Daugavet property II

For $C(K)$ spaces

A compact space, $C(K)$ has the Daugavet property if and only if $K$ is perfect.

A related result

For every compact space $K$ and every $T \in L(C(K))$,

$$\|\text{Id} + T\| = 1 + \|T\| \quad \text{or} \quad \|\text{Id} - T\| = 1 + \|T\|.$$ 

More examples

The following spaces have the Daugavet property.

- **Wojtaszczyk, 1992**: The disk algebra $\mathbb{A}$ and $H^\infty$.
- **Oikhberg, 2005**: Non-atomic $C^*$-algebras and preduals of non-atomic von Neumann algebras.
- **Ivankhno, Kadets, Werner, 2007**: $\text{Lip}(K)$ when $K \subseteq \mathbb{R}^n$ is compact and convex.
Daugavet–type inequalities

Some examples

- **Benyamini–Lin, 1985:**
  For every $1 < p < \infty$, $p \neq 2$, there exists $\psi_p : (0, \infty) \rightarrow (0, \infty)$ such that
  \[ \|\text{Id} + T\| \geq 1 + \psi_p(\|T\|) \]
  for every compact operator $T$ on $L_p[0, 1]$.
  If $p = 2$, then there is a non-null compact $T$ on $L_2[0, 1]$ such that
  \[ \|\text{Id} + T\| = 1. \]

- **Boyko–Kadets, 2004:**
  If $\psi_p$ is the best possible function above, then
  \[ \lim_{p \rightarrow 1^+} \psi_p(t) = t \quad (t > 0). \]

- **Oikhberg, 2005:**
  If $K(\ell_2) \subseteq X \subseteq L(\ell_2)$, then
  \[ \|\text{Id} + T\| \geq 1 + \frac{1}{8 \sqrt{2}} \|T\| \]
  for every compact $T$ on $X$.  

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Motivating question

Are there other norm equalities which could define interesting properties of Banach spaces?

Concretely

We looked for non-trivial norm equalities of the forms

\[ \|g(T)\| = f(\|T\|) \quad \text{or} \quad \|\text{Id} + g(T)\| = f(\|g(T)\|) \]

(\(g\) analytic, \(f\) arbitrary) in such a way that all rank-one operators on a Banach space \(X\) satisfy.

Solution

We proved that there are not to many possibilities…
**Theorem**

$X$ real or complex with $\dim(X) \geq 2$. Suppose that the norm equality

$$\|g(T)\| = f(\|T\|)$$

holds for every rank-one operator $T \in L(X)$, where

- $g : \mathbb{K} \rightarrow \mathbb{K}$ is analytic,
- $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ is arbitrary.

Then, there are $a, b \in \mathbb{K}$ such that

$$g(\zeta) = a + b \zeta \quad (\zeta \in \mathbb{K}).$$

**Corollary**

Only three norm equalities of the form

$$\|g(T)\| = f(\|T\|)$$

are possible:

- $b = 0$: $\|a \text{Id}\| = |a|$,
- $a = 0$: $\|b \ T\| = |b| \|T\|$,
  (trivial cases)
- $a \neq 0, b \neq 0$: $\|a \text{Id} + b \ T\| = |a| + |b| \|T\|$,
  (Daugavet property)
Theorem

X complex with \( \dim(X) \geq 2 \). Suppose that the norm equality

\[
\|\text{Id} + g(T)\| = f(\|g(T)\|)
\]

holds for every rank-one operator \( T \in L(X) \), where

- \( g : \mathbb{K} \rightarrow \mathbb{K} \) is analytic with \( g(0) = 0 \),
- \( f : \mathbb{R}^+_0 \rightarrow \mathbb{R} \) is continuous.

Then, \( X \) has the Daugavet property

Remarks

- We do not know if the result is true in the real case.
- It is true if \( g \) is onto.
- Even the simplest case, \( g(t) = t^2 \), is not known. The only known thing is that, in this case, \( f(t) = 1 + t \), leading to the equation

\[
\|\text{Id} + T^2\| = 1 + \|T^2\|
\]
The question

Godefroy, private communication
Is there any real Banach space $X$ (with $\dim(X) > 1$) such that

$$||\text{Id} + T^2|| = 1 + ||T^2||$$

for every operator $T \in L(X)$?

Definition
We will call $\star$ the property defined above.
The examples
### The examples

**Weak multiplier**

Let $K$ be a compact space. $T \in L(C(K))$ is a **weak multiplier** if

\[ T^* = g \text{Id} + S \]

where $g$ is a Borel function and $S$ is weakly compact.

**Our main result**

If $K$ is perfect and all operators on $C(K)$ are weak multipliers, then $C(K)$ has $\star$.

**Theorem (Koszmider, 2004; Plebanek, 2004)**

There exist perfect compact spaces $K$ such that all operators on $C(K)$ are weak multipliers. There are examples of two kinds:

- $K$ connected where every operator in $L(C(K))$ is of the form $g \text{Id} + S$ for $g \in C(K)$ and $S$ weakly compact.
- $K$ totally disconnected and perfect.

In particular, there are nonisomorphic $C(K)$ spaces with $\star$. 
Proving a simple case...

Hypothesis

Every $T \in L(C(K))$ is of the form $g\text{Id} + S$, with $g \in C(K)$, $S$ weakly compact.

- If $T = g\text{Id} + S$, then $T^2 = g^2 \text{Id} + S'$ with $S'$ weakly compact.
- We will prove that $\|\text{Id} + g^2 \text{Id} + S\| = 1 + \|g^2 \text{Id} + S\|$ for $g \in C(K)$ and $S$ weakly compact.

**Step 1:** We assume $\|g^2\| \leq 1$ and $\min g^2(K) > 0$.

**Step 2:** We can avoid the assumption that $\min g^2(K) > 0$.

**Step 3:** Finally, for every $g$ the above gives

$$\left\|\text{Id} + \frac{1}{\|g^2\|} \left( g^2 \text{Id} + S \right) \right\| = 1 + \frac{1}{\|g^2\|} \|g^2 \text{Id} + S\|$$

which gives us the result.
Consequences
Consequences I: first results

Proposition

If $X$ has $\star$, then

- $X$ does not have the RNP.
- $X$ does not have unconditional basis.

For $C(K)$ spaces

This said not too much about $C(K)$…
Consequences II: isometries

**Theorem**

If $X$ has $\star$, then every surjective isometry $J$ on $X$ satisfies $J^2 = \text{Id}$. 

**For $C(K)$ spaces**

If all operators on a $C(K)$ space are weak multipliers, then every homeomorphism $\varphi$ of $K$ satisfies $\varphi^2 = \text{id}$. 

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Consequences III: complex structure

Complex structure

A complex structure in a real Banach space $X$ consists in viewing $X$ as a complex space under the multiplication $(\alpha + i \beta) x = [\alpha \text{Id} + \beta J](x)$ where $J \in L(X)$ satisfies $J^2 = -\text{Id}$.

Remark

If $X$ has $\star$, then it does not have any complex structure.

Theorem

$X$ having $\star$, $Y$ finite-codimensional subspace of $X$. Then

- $Y$ does not have any complex structure.
- In particular, $Y$ is not isomorphic to $Z \oplus Z$ for any $Z$.

For $C(K)$ spaces

- There is a connected compact space $K$ such that $C(K)$ has $\star$,
- the only complemented subspaces of $C(K)$ having complex structure are the finite-dimensional ones,
- In particular, no complemented subspace of $C(K)$ is isomorphic to $Z \oplus Z$ for any $Z$.

- There is another perfect compact space $K'$ such that $C(K')$ has $\star$,
- $C(K')$ contains a complemented copy of $c_0$. 

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Question 1

Find topological characterization of the compact Hausdorff spaces $K$ such that $C(K)$ has $\star$.

Question 2

Find other Banach spaces having $\star$.

Question 3

Characterize isometrically and/or isomorphically the property $\star$. 