The talk is based on these papers

J. Becerra Guerrero and M. Martín,
The Daugavet Property of $C^*$-algebras, $JB^*$-triples, and of their isometric preduals.

M. Martín,
The alternative Daugavet property of $C^*$-algebras and $JB^*$-triples.
*Mathematische Nachrichten* (to appear)

M. Martín and T. Oikhberg,
An alternative Daugavet property.
Introduction and motivation
- Definitions and examples
- Propaganda
- Geometric characterizations
- From rank-one to other class of operators

A new sufficient condition

Application: $C^*$-algebras and von Neumann preduals
- von Neumann preduals
- $C^*$-algebras

The alternative Daugavet equation
- Definitions and basic results
- Geometric characterizations
- $C^*$-algebras and preduals

Recommended readings
In a Banach space $X$ with the **Radon-Nikodým property** the unit ball has many denting points.

- $x \in S_X$ is a denting point of $B_X$ if for every $\varepsilon > 0$ one has
  \[ x \notin \overline{co}\left( B_X \setminus (x + \varepsilon B_X) \right). \]

- $C[0, 1]$ and $L_1[0, 1]$ have an extremely opposite property: for every $x \in S_X$ and every $\varepsilon > 0$
  \[ \overline{co}\left( B_X \setminus (x + (2 - \varepsilon) B_X) \right) = B_X. \]

- This geometric property is equivalent to a property of operators on the space.
The Daugavet equation

$X$ Banach space, $T \in L(X)$

$$\|\text{Id} + T\| = 1 + \|T\| \quad \text{(DE)}$$

Classical examples

1. **Daugavet, 1963:**
   Every compact operator on $C[0, 1]$ satisfies (DE).

2. **Lozanoskii, 1966:**
   Every compact operator on $L_1[0, 1]$ satisfies (DE).

3. **Abramovich, Holub, and more, 80’s:**
   $X = C(K)$, $K$ perfect compact space
   or $X = L_1(\mu)$, $\mu$ atomless measure
   $\implies$ every weakly compact $T \in L(X)$ satisfies (DE).
The Daugavet property

- A Banach space $X$ is said to have the **Daugavet property** if every rank-one operator on $X$ satisfies (DE).
- If $X^*$ has the Daugavet property, so does $X$. The converse is not true: $C[0, 1]$ has it but $C[0, 1]^*$ not.


Prior versions of: Chauveheid, 1982; Abramovich–Aliprantis–Burkinshaw, 1991

Some examples...

1. $K$ perfect, $\mu$ atomeless, $E$ arbitrary Banach space
   $\implies C(K, E), L_1(\mu, E),$ and $L_\infty(\mu, E)$ have the Daugavet property.
   *(Kadets, 1996; Nazarenko, −; Shvidkoy, 2001)*

2. $A(\mathbb{D})$ and $H^\infty$ have the Daugavet property.
   *(Wojtaszczyk, 1992)*
3 A function algebra whose Choquet boundary is perfect has the Daugavet property.

\cite{Werner, 1997}

4 “Large” subspaces of \( C[0, 1] \) and \( L_1[0, 1] \) have the Daugavet property (in particular, this happens for finite-codimensional subspaces).

\cite{Kadets–Popov, 1997}

5 A \( C^* \)-algebra has the Daugavet property if and only if it is non-atomic.

6 The predual of a von Neumann algebra has the Daugavet property if and only if the algebra is non-atomic.

\cite{Oikhberg, 2002}
Some propaganda... 

Let $X$ be a Banach space with the Daugavet property. Then

- $X$ does not have the Radon-Nikodým property.
  
  \[(\text{Wojtaszczyk, 1992})\]

- Every slice of $B_X$ and every $w^*$-slice of $B_{X^*}$ have diameter 2.
  
  \[(\text{Kadets–Shvidkoy–Sirotkin–Werner, 2000})\]

- Actually, every weakly-open subset of $B_X$ has diameter 2.
  
  \[(\text{Shvidkoy, 2000})\]

- $X$ contains a copy of $\ell_1$. $X^*$ contains a copy or $L_1[0, 1]$.
  
  \[(\text{Kadets–Shvidkoy–Sirotkin–Werner, 2000})\]
Theorem [KSSW]

- **X** has the Daugavet property.

- For every \( x \in S_X \), \( x^* \in S_{X^*} \), and \( \varepsilon > 0 \), there exists \( y \in S_X \) such that
  \[
  \text{Re} \, x^*(y) > 1 - \varepsilon \quad \text{and} \quad \|x - y\| > 2 - \varepsilon.
  \]

- For every \( x \in S_X \), \( x^* \in S_{X^*} \), and \( \varepsilon > 0 \), there exists \( y^* \in S_{X^*} \) such that
  \[
  \text{Re} \, y^*(x) > 1 - \varepsilon \quad \text{and} \quad \|x^* - y^*\| > 2 - \varepsilon.
  \]

- For every \( x \in S_X \) and every \( \varepsilon > 0 \), we have
  \[
  B_X = \operatorname{co} \left( \{ y \in B_X : \|x - y\| > 2 - \varepsilon \} \right).
  \]
**Theorem**

Let $X$ be a Banach space with the Daugavet property.

- Every weakly compact operator on $X$ satisfies (DE).
  
  \[(\text{Kadets–Shvidkoy–Sirotkin–Werner, 2000})\]

- Actually, every operator on $X$ which does not fix a copy of $\ell_1$ satisfies (DE).
  
  \[(\text{Sirotkin, 2000})\]

**Consequences**

1. $X$ does not have unconditional basis.
   
   \[(\text{Kadets, 1996})\]

2. Moreover, $X$ does not embed into any space with unconditional basis.
   
   \[(\text{Kadets–Shvidkoy–Sirotkin–Werner, 2000})\]

3. Actually, $X$ does not embed into an unconditional sum of Banach spaces without a copy of $\ell_1$.
   
   \[(\text{Shvidkoy, 2000})\]
A new sufficient condition
Theorem

Let $X$ be a Banach space such that

$$X^* = Y \oplus_1 Z$$

with $Y$ and $Z$ norming subspaces. Then, $X$ has the Daugavet property.

A closed subspace $W \subseteq X^*$ is norming if

$$\|x\| = \sup \{ |w^*(x)| : w^* \in W, \|w^*\| = 1 \}$$

or, equivalently, if $B_W$ is $w^*$-dense in $B_{X^*}$. 
Proof of the theorem

We have...

- \( X^* = Y \oplus_1 Z \),
- \( B_Y, B_Z \) \( w^* \)-dense in \( B_{X^*} \).

We need...

- fixed \( x_0 \in S_X \), \( x_0^* \in S_{X^*} \), \( \varepsilon > 0 \), find \( y^* \in S_{X^*} \) such that
  \[
  \|x_0^* + y^*\| > 2 - \varepsilon \quad \text{and} \quad \Re y^*(x_0) > 1 - \varepsilon.
  \]

Write \( x_0^* = y_0^* + z_0^* \) with \( y_0^* \in Y \), \( z_0^* \in Z \), \( \|x_0^*\| = \|y_0^*\| + \|z_0^*\| \), and write

\[
U = \{x^* \in B_{X^*} : \Re x^*(x_0) > 1 - \varepsilon/2\}.
\]

Take \( z^* \in B_Z \cap U \) and a net \((y_\lambda^*)\) in \( B_Y \cap U \), such that \((y_\lambda^*) \overset{w^*}\longrightarrow z^*\).

\((y_\lambda^* + y_0^*) \longrightarrow z^* + y_0^* \) and the norm is \( w^* \)-lower semi-continuous, therefore

\[
\liminf \|y_\lambda^* + y_0^*\| \geq \|z^* + y_0^*\| = \|z^*\| + \|y_0^*\| > 1 + \|y_0^*\| - \varepsilon/2.
\]

Then, we may find \( \mu \) such that \( \|y_\mu^* + y_0^*\| \geq 1 + \|y_0^*\| - \varepsilon/2. \)

Finally, observe that

\[
\|x_0^* + y_\mu^*\| = \|(y_0^* + y_\mu^*) + z_0^*\| = \|y_0^* + y_\mu^*\| + \|z_0^*\| > 1 + \|y_0^*\| - \varepsilon + \|z_0^*\| = 2 - \varepsilon,
\]

and that \( \Re y_\mu^*(x_0) > 1 - \varepsilon \) \( \) (since \( y_\mu^* \in U \)).
Some immediate consequences

**Corollary**

Let $X$ be an $L$-embedded space with $\text{ext} (B_X) = \emptyset$. Then, $X^*$ (and hence $X$) has the Daugavet property.

**Corollary**

If $Y$ is an $L$-embedded space which is a subspace of $L_1 \equiv L_1[0,1]$, then $(L_1/Y)^*$ has the Daugavet property.

**It was already known that...**

- If $Y \subset L_1$ is reflexive, then $L_1/Y$ has the Daugavet property.
  
  *(Kadets–Shvidkoy–Sirotkin–Werner, 2000)*

- If $Y \subset L_1$ is $L$-embedded, then $L_1/Y$ does not have the RNP.
  
  *(Harmand–Werner–Werner, 1993)*
Application:

The Daugavet property of

$C^*$-algebras and von Neumann preduals
A $C^*$-algebra $X$ is a **von Neumann algebra** if it is a dual space.
In such a case, $X$ has a unique predual $X_*$.
$X_*$ is always $L$-embedded.
Therefore, if $\text{ext} \left( B_{X_*} \right)$ is empty, then $X$ and $X_*$ have the Daugavet property.
Example: $L_\infty[0, 1]$ and $L_1[0, 1]$.

Actually, much more can be proved:
Theorem

Let $X_*$ be the predual of the von Neumann algebra $X$. Then, TFAE:

- $X$ has the Daugavet property.
- $X_*$ has the Daugavet property.
- Every weakly open subset of $B_{X_*}$ has diameter 2.
- $B_{X_*}$ has no strongly exposed points.
- $B_{X_*}$ has no extreme points.
- $X$ is non-atomic (i.e. it has no atomic projections).

An atomic projection is an element $p \in X$ such that

\[ p^2 = p^* = p \quad \text{and} \quad p X p = \mathbb{C} p. \]
Let $X$ be a $C^*$-algebra. Then, $X^{**}$ is a von Neumann algebra.
Write $X^* = (X^{**})_* = A \oplus_1 N$, where
- $A$ is the atomic part,
- $N$ is the non-atomic part.

- Every extreme point of $B_{X^*}$ is in $B_A$.
- Therefore, $A$ is norming.
- What’s about $N$?

**Theorem**
If $X$ is non-atomic, then $N$ is norming. Therefore, $X$ has the Daugavet property.
Example: $C[0, 1]$
We have...

- $X$ non-atomic $C^*$-algebra, $X^* = A \oplus_1 N$.

We need...

- $N$ to be norming for $X$, i.e. $\|x\| = \sup\{|f(x)| : f \in B_N\} \quad (x \in X)$.

- Write $X^{**} = A \oplus_\infty N$ and $Y = A \cap X$.
- $Y$ is an ideal of $X$, so $Y$ has no atomic projections.
- Therefore, the norm of $Y$ has no point of Fréchet-smoothness.
- But $Y$ is an Asplund space, so $Y = 0$.
- Now, the mapping
  $$X \hookrightarrow X^{**} = A \oplus_\infty N \twoheadrightarrow N$$
  in injective. Since it is an homomorphism, it is an isometry.
- But $N^* \equiv N$, so $N$ is norming for $N$ and now, also for $X$. 
Theorem

Let $X$ be a $C^*$-algebra. Then, TFAE:

- $X$ has the Daugavet property.
- The norm of $X$ is extremely rough, i.e.
  \[
  \limsup_{\|h\| \to 0} \frac{\|x + h\| + \|x - h\| - 2}{\|h\|} = 2
  \]
  for every $x \in S_X$ (equivalently, every $w^*$-slice of $B_{X^*}$ has diameter 2).
- The norm of $X$ is not Fréchet-smooth at any point.
- $X$ is non-atomic.
The alternative Daugavet equation
The alternative Daugavet equation

Let $X$ be a Banach space, $T \in L(X)$.

\[
\max_{|\omega|=1} \|Id + \omega T\| = 1 + \|T\| \quad \text{(aDE)}
\]

(Duncan–McGregor–Pryce–White, 1970; Holub, Abramovich..., 80's)

Two equivalent formulations

- There exists $\omega \in \mathbb{T}$ such that $\omega T$ satisfies (DE).
- The numerical radius of $T$, $\nu(T)$, coincides with $\|T\|$, where

\[
\nu(T) := \sup\{|x^*(Tx)| : x^* \in S_{X^*}, x \in S_X, x^*(x) = 1\}.
\]
Two possible properties

Let $X$ be a Banach space.

- $X$ is said to have the **alternative Daugavet property (ADP)** iff every rank-one operator on $X$ satisfies (aDE).
  - Then, every weakly compact operator also satisfies (aDE).
  - If $X^*$ has the ADP, so does $X$. The converse is not true: $C([0,1],\ell_2)$.

  *(M.–Oikhberg, 2004; briefly appearance: Abramovich, 1991)*

- $X$ is said to have **numerical index 1** iff $v(T) = \|T\|$ for every operator on $X$. Equivalently, if every operator on $X$ satisfies (aDE).


Observation

No analogous property is possible for the Daugavet equation:

$$\|ld + (-ld)\| = 0 \neq 1 + \| -ld\|.$$
Numerical index 1

- $C(K)$ and $L_1(\mu)$ have numerical index 1.  
  \[(Duncan–McGregor–Pryce–White, 1970)\]

- All function algebras have numerical index 1.  
  \[(Werner, 1997)\]

- A $C^*$-algebra has numerical index 1 iff it is commutative.  
  \[(Huruya, 1977; Kaidi–Morales–Rodríguez-Palacios, 2000)\]

- In case $\dim(X) < \infty$, $X$ has numerical index 1 iff  
  \[|x^*(x)| = 1 \quad x^* \in \text{ext} (B_{X^*}), \; x \in \text{ext} (B_X).\]  
  \[(McGregor, 1971)\]

- In case $\dim(X) = \infty$, if $X$ has numerical index 1 and the RNP, then $X \supseteq \ell_1$.  
  \[(López–M.–Payá, 1999)\]
The alternative Daugavet property

- $c_0 \oplus_\infty C([0, 1], \ell_2)$ has the ADP, but neither the Daugavet property, nor numerical index 1.
- For RNP or Asplund spaces, the ADP implies numerical index 1.
- Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.
Geometric characterizations

Theorem

- $X$ has the ADP.
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that
  \[ |x^*(y)| > 1 - \varepsilon \quad \text{and} \quad ||x - y|| \geq 2 - \varepsilon. \]
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y^* \in S_{X^*}$ such that
  \[ |y^*(x)| > 1 - \varepsilon \quad \text{and} \quad ||x^* - y^*|| \geq 2 - \varepsilon. \]
- For every $x \in S_X$ and every $\varepsilon > 0$, we have
  \[ B_X = \overline{co}(T \{ y \in B_X : ||x - y|| \geq 2 - \varepsilon \}). \]
Let $V^*$ be the predual of the von Neumann algebra $V$.

**The Daugavet property of $V^*$ is equivalent to:**

- $V$ has no atomic projections, or
- the unit ball of $V^*$ has no extreme points.

$V^*$ has numerical index 1 iff:

- $V$ is commutative, or
- $|v^*(v)| = 1$ for $v \in \text{ext}(B_V)$ and $v^* \in \text{ext}(B_{V^*})$.

**The alternative Daugavet property of $V^*$ is equivalent to:**

- the atomic projections of $V$ are central, or
- $|v(v^*)| = 1$ for $v \in \text{ext}(B_V)$ and $v^* \in \text{ext}(B_{V^*})$, or
- $V = C \oplus_{\infty} N$, where $C$ is commutative and $N$ has no atomic projections.
Let $X$ be a $C^*$-algebra.

The Daugavet property of $X$ is equivalent to:
- $X$ does not have any atomic projection, or
- the unit ball of $X^*$ does not have any $w^*$-strongly exposed point.

$X$ has numerical index 1 iff:
- $X$ is commutative, or
- $|x^*(x^*)| = 1$ for $x^* \in \text{ext}(B_{X^*})$ and $x^* \in \text{ext}(B_{X^*}).$

The alternative Daugavet property of $X$ is equivalent to:
- the atomic projections of $X$ are central, or
- $|x^*(x^*)| = 1$, for $x^* \in \text{ext}(B_{X^*})$, and $x^* \in B_{X^*}$ $w^*$-strongly exposed, or
- $\exists$ a commutative ideal $Y$ such that $X/Y$ has the Daugavet property.
Recommended readings...


