The Daugavet property of $C^*$-algebras and von Neumann preduals.
Geometric characterizations

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February 2005 / Kent State University
The talk is based on these papers


- M. Martín, The alternative Daugavet property of $C^*$-algebras and $JB^*$-triples. *Preprint*
Outline

1. **Introduction**
   - Definitions and examples
   - Propaganda
   - Geometric characterizations

2. **A new sufficient condition**

3. **Applications**
   - von Neumann preduals
   - $C^*$-algebras

4. **The alternative Daugavet equation**
   - Definitions and basic results
   - Geometric characterizations
   - $C^*$-algebras and preduals

5. **Recommended readings**
The Daugavet equation

\( X \) Banach space, \( T \in L(X) \)

\[ \| Id + T \| = 1 + \| T \| \quad (DE) \]

Classical examples

1. Daugavet, 1963:
   Every compact operator on \( C[0,1] \) satisfies (DE).

2. Lozanoskii, 1966:
   Every compact operator on \( L_1[0,1] \) satisfies (DE).

3. Abramovich, Holub, and more, 80’s:
   \( X = C(K) \), \( K \) perfect compact space
   or \( X = L_1(\mu) \), \( \mu \) atomless measure
   \( \implies \) every weakly compact \( T \in L(X) \) satisfies (DE).
The Daugavet property

A Banach space $X$ is said to have the **Daugavet property** if every rank-one operator on $X$ satisfies (DE).

- Then, all weakly compact operators also satisfy (DE).
- Obviously, if $X^*$ has the Daugavet property, so does $X$.
  The converse is not true.


Prior versions of: *Chauveheid, 1982; Abramovich–Aliprantis–Burkinshaw, 1991*

Some examples...

1. $K$ perfect, $\mu$ atomeless, $X$ arbitrary Banach space
   $\implies C(K, X), L_1(\mu, X),$ and $L_\infty(\mu, X)$ have the Daugavet property.
   *(Kadets, 1996; Nazarenko, —; Shvidkoy, 2001)*

2. $K$ arbitrary. If $X$ has the Daugavet property, then so does $C(K, X)$.
   *(M.–Payá, 2000)*
More examples...

3. The $c_0$, $\ell_1$, and $\ell_\infty$ sums of Banach spaces with the Daugavet property have the Daugavet property.

4. $A(\mathbb{D})$ and $H^\infty$ have the Daugavet property.

   \[(\text{Wojtaszczyk, 1992})\]

5. $R \subset L_1[0, 1] =: L_1$ reflexive, then $L_1/R$ has the Daugavet property.

   \[(\text{Kadets–Shvidkoy–Sirotkin–Werner, 2000})\]

6. A $C^*$-algebra has the Daugavet property if and only if it is non-atomic.

7. The predual of a von Neumann algebra has the Daugavet property if and only if the algebra is non-atomic.

   \[(\text{Oikhberg, 2002})\]
Introduction

A new sufficient condition

Applications

The alternative Daugavet equation

Recommended readings

Definitions and examples

Propaganda

Geometric characterizations

Some propaganda…

Let $X$ be a Banach space with the Daugavet property. Then

- $X$ does not have the Radon-Nikodým property.
  
  *(Wojtaszczyk, 1992)*

- Every slice of $B_X$ and every $w^*$-slice of $B_{X^*}$ have diameter 2.
  
  *(Kadets–Shvidkoy–Sirotkin–Werner, 2000)*

- Actually, every weakly-open subset of $B_X$ has diameter 2.
  
  *(Shvidkoy, 2000)*

- $X$ contains a copy of $\ell_1$. $X^*$ contains a copy or $L_1[0,1]$.

- $X$ has no unconditional basis.
  
  *(Kadets, 1996)*

- Actually, $X$ does not embed into a space with unconditional basis.
  
  *(Kadets–Shvidkoy–Sirotkin–Werner, 2000)*
Geometric characterizations

**Theorem [KSSW]**

- **$X$** has the Daugavet property.
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that
  \[ \text{Re } x^*(y) > 1 - \varepsilon \quad \text{and} \quad \|x - y\| \geq 2 - \varepsilon. \]
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y^* \in S_{X^*}$ such that
  \[ \text{Re } y^*(x) > 1 - \varepsilon \quad \text{and} \quad \|x^* - y^*\| \geq 2 - \varepsilon. \]
- For every $x \in S_X$ and every $\varepsilon > 0$, we have
  \[ B_X = \overline{co}(\{y \in B_X : \|x - y\| \geq 2 - \varepsilon\}). \]
A new sufficient condition
A new sufficient condition

**Theorem**

Let $X$ be a Banach space such that

$$X^* = Y \oplus_1 Z$$

with $Y$ and $Z$ norming subspaces. Then, $X$ has the Daugavet property.

A closed subspace $W \subseteq X^*$ is **norming** if

$$\|x\| = \sup \{|w^*(x)| : w^* \in W, \|w^*\| = 1\}$$

or, equivalently, if $B_W$ is $w^*$-dense in $B_{X^*}$. 

Miguel Martín  The Daugavet property
Proof of the theorem

We have...

\[ X^* = Y \oplus_1 Z, \]
\[ B_Y, B_Z \text{ w}^*\text{-dense in } B_{X^*}. \]

We need...

fixed \( x_0 \in S_X, x_0^* \in S_{X^*}, \varepsilon > 0 \), find \( y^* \in S_{X^*} \) such that
\[ \|x_0^* + y^*\| > 2 - \varepsilon \quad \text{and} \quad \text{Re } y^*(x_0) > 1 - \varepsilon. \]

- Write \( x_0^* = y_0^* + z_0^* \) with \( y_0^* \in Y, z_0^* \in Z, \|x_0^*\| = \|y_0^*\| + \|z_0^*\| \), and write
  \[ U = \{ x^* \in B_{X^*} : \text{Re } x^*(x_0) > 1 - \varepsilon \}. \]

- Take \( z^* \in B_Z \cap U \) and a net \((y^*_\lambda)\) in \( B_Y \cap U \), such that \((y^*_\lambda) \stackrel{w^*}{\longrightarrow} z^* \).

- \((y^*_\lambda + y_0^*) \longrightarrow z^* + y_0^* \) and the norm is \( w^*\) -lower semi-continuous, therefore
  \[ \lim \inf \|y^*_\lambda + y_0^*\| \geq \|z^* + y_0^*\| = \|z^*\| + \|y_0^*\| > 1 + \|y_0^*\| - \varepsilon. \]

- Then, we may find \( \mu \) such that \( \|y^*_\mu + y_0^*\| \geq 1 + \|y_0^*\| - \varepsilon/2. \)

- Finally, observe that
  \[ \|x_0^* + y^*_\mu\| = \|(y_0^* + y^*_\mu) + z_0^*\| = \|y_0^* + y^*_\mu\| + \|z_0^*\| > 1 + \|y_0^*\| - \varepsilon + \|z_0^*\| = 2 - \varepsilon, \]
  and that \( \text{Re } y^*_\mu(x_0) > 1 - \varepsilon \) (since \( y^*_\mu \in U \)).
Some immediate consequences

**Corollary**

Let $X$ be an $L$-embedded space with $\text{ext} (B_X) = \emptyset$. Then, $X^*$ (and hence $X$) has the Daugavet property.

**Corollary**

If $Y$ is an $L$-embedded space which is a subspace of $L_1 \equiv L_1[0,1]$, then $(L_1/Y)^*$ has the Daugavet property.

**It was already known that...**

- If $Y \subset L_1$ is reflexive, then $L_1/Y$ has the Daugavet property.  
  *(Kadets–Shvidkoy–Sirotkin–Werner, 2000)*

- If $Y \subset L_1$ is $L$-embedded, then $L_1/Y$ does not have the RNP.  
  *(Godefroy–Li, 1990)*
Applications:
The Daugavet property of $C^*$-algebras and von Neumann preduals
A $C^*$-algebra $X$ is a von Neumann algebra if it is a dual space. In such a case, $X$ has a unique predual $X_\ast$. $X_\ast$ is always $L$-embedded. Therefore, if $\text{ext} (B_{X_\ast})$ is empty, then $X$ and $X_\ast$ have the Daugavet property.

Actually, much more can be proved:
Theorem

Let $X_*$ be the predual of the von Neumann algebra $X$. Then, TFAE:

- $X$ has the Daugavet property.
- $X_*$ has the Daugavet property.
- Every weakly open subset of $B_{X_*}$ has diameter 2.
- $B_{X_*}$ has no strongly exposed points.
- $B_{X_*}$ has no extreme points.
- $X$ is non-atomic (i.e. it has no atomic projections).

An atomic projection is an element $p \in X$ such that

$$p^2 = p^* = p \quad \text{and} \quad pXp = \mathbb{C}p.$$
Let $X$ be a $C^*$-algebra. Then, $X^{**}$ is a von Neumann algebra. Write $X^* = (X^{**})_* = A \oplus_1 N$, where

- $A$ is the atomic part,
- $N$ is the non-atomic part.

Every extreme point of $B_{X^*}$ is in $B_A$.

Therefore, $A$ is norming.

What’s about $N$?

**Theorem**

If $X$ is non-atomic, then $N$ is norming. Therefore, $X$ has the Daugavet property.

Actually, much more can be proved:
Theorem

Let $X$ be a $C^*$-algebra. Then, TFAE:

- $X$ has the Daugavet property.
- $X$ is non-atomic.
- The norm of $X$ is extremely rough, i.e.,
  \[
  \limsup_{\|h\| \to 0} \frac{\|x + h\| + \|x - h\| - 2}{\|h\|} = 2
  \]
  for every $x \in S_X$ (equivalently, every $w^*$-slice of $B_{X^*}$ has diameter 2).
- The norm of $X$ is not Fréchet-smooth at any point.
The alternative Daugavet equation
The alternative Daugavet equation

- **X** Banach space, **T** ∈ L(\(X\))

\[
\max_{|\omega|=1} \|Id + \omega T\| = 1 + \|T\| \quad (aDE)
\]

\[(Duncan–McGregor–Pryce–White, 1970; Holub, Abramovich... , 80's)\]

- Two possible properties
  - **X** is said to have the **alternative Daugavet property (ADP)** iff every rank-one (equivalently every compact) operator on **X** satisfies \((aDE)\).
    
    \[(Abramovich, 1991; M.–Oikhberg, 2004)\]
  - **X** is said to have **numerical index 1** iff EVERY operator on **X** satisfies \((aDE)\).
    
    \[(Lumer, 1968; Duncan–McGregor–Pryce–White, 1970)\]
Numerical index 1

- $C(K)$ and $L_1(\mu)$ have numerical index 1.

  *(Duncan–McGregor–Pryce–White, 1970)*

- $A(\mathbb{D})$ also has numerical index 1.

  *(Crabb–Duncan–McGregor, 1972)*

- In case $\dim(X) < \infty$, $X$ has numerical index 1 iff

  \[ |x^*(x)| = 1 \quad x^* \in \text{ext}(B_{X^*}), \ x \in \text{ext}(B_X). \]

  *(McGregor, 1971)*

- In case $\dim(X) = \infty$, if $X$ has numerical index 1 and it has the RNP, then $X \supseteq \ell_1$.

  *(López–M.–Payá, 1999)*

- A $C^*$-algebra has numerical index 1 iff it is commutative.

  *(Huruya, 1977)*
The alternative Daugavet property

- The ADP is weaker than the Daugavet property and the numerical index 1.
- $c_0 \oplus_{\infty} C([0,1],\ell_2)$ has the ADP, but neither the Daugavet property, nor numerical index 1.
- Every Banach space with the ADP can be renormed still having the ADP but lacking the Daugavet property.
Geometric characterizations

**Theorem**

- $X$ has the ADP.
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that
  \[ |x^*(y)| > 1 - \varepsilon \quad \text{and} \quad \|x - y\| \geq 2 - \varepsilon. \]
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y^* \in S_{X^*}$ such that
  \[ |y^*(x)| > 1 - \varepsilon \quad \text{and} \quad \|x^* - y^*\| \geq 2 - \varepsilon. \]
- For every $x \in S_X$ and every $\varepsilon > 0$, we have
  \[ B_X = \overline{\text{co}}(\bigcap \{ y \in B_X : \|x - y\| \geq 2 - \varepsilon \}). \]
Let $V_\ast$ be the predual of the von Neumann algebra $V$.

**The Daugavet property of $V_\ast$ is equivalent to:**

- $V$ has no atomic projections, or
- the unit ball of $V_\ast$ has no extreme points.

**$V_\ast$ has numerical index 1 iff:**

- $V$ is commutative, or
- $|v^*(v)| = 1$ for $v \in \text{ext}(B_V)$ and $v^* \in \text{ext}(B_{V_\ast})$.

**The alternative Daugavet property of $V_\ast$ is equivalent to:**

- the atomic projections of $V$ are central, or
- $|v(v_\ast)| = 1$ for $v \in \text{ext}(B_V)$ and $v_\ast \in \text{ext}(B_{V_\ast})$, or
- $V = C \oplus \infty N$, where $C$ is commutative and $N$ has no atomic projections.
Let $X$ be a $C^*$-algebra.

The Daugavet property of $X$ is equivalent to:
- $X$ does not have any atomic projection, or
- the unit ball of $X^*$ does not have any $w^*$-strongly exposed point.

$X$ has numerical index 1 iff:
- $X$ is commutative, or
- $|x^{**}(x^*)| = 1$ for $x^{**} \in \text{ext}(B_{X^{**}})$ and $x^* \in \text{ext}(B_{X^*}).$

The alternative Daugavet property of $X$ is equivalent to:
- the atomic projections of $X$ are central, or
- $|x^{**}(x^*)| = 1,$ for $x^{**} \in \text{ext}(B_{X^{**}}),$ and $x^* \in B_{X^*} \ w^*$-strongly exposed, or
- $\exists$ a commutative ideal $Y$ such that $X/Y$ has the Daugavet property.
Recommended readings...

Y. Abramovich, and C. Aliprantis,
*An invitation to operator theory.*

Y. Abramovich, and C. Aliprantis,
*Problems in operator theory.*

V. Kadets, R. Shvidkoy, G. Sirotkin, and D. Werner,
Banach spaces with the Daugavet property.

R. Shvidkoy,
Geometric aspects of the Daugavet property.

D. Werner,
Recent progress on the Daugavet property.