The Daugavet property
of C*-algebras and von Neumann preduals

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The Daugavet equation

$X$ Banach space, $T \in L(X)$

\[ \|Id + T\| = 1 + \|T\| \quad \text{(DE)} \]

- **Daugavet, 1963:**
  Every compact operator on $C[0,1]$ satisfies (DE).

- **Lozanskii, 1966:**
  Every compact operator on $L_1[0,1]$ satisfies (DE).

- **Abramovich, Holub and more, 80’s:**
  $X = C(K)$, $K$ perfect compact space
  or $X = L_1(\mu)$, $\mu$ atomless measure,
  $\implies$ every weakly compact $T \in L(X)$ satisfies (DE).
The Daugavet property

A Banach space $X$ is said to have the Daugavet property if every rank-one operator $T \in L(X)$ satisfies (DE).

* Then, all weakly compact operators also satisfy (DE).


* $X^*$ Daugavet property $\implies X$ Daugavet property

**Examples**

- $K$ perfect, $\mu$ atomeless, $X$ arbitrary Banach space $\implies C(K,X)$, $L_1(\mu,X)$, and $L_\infty(\mu,X)$ have the Daugavet property.

(Kadets, 1996; Nazarenko, –; M.–Villena, 2003)
The Daugavet property

- $K$ arbitrary compact space. If $X$ has the Daugavet property, then so does $C(K, X)$.

  (M.–Payá, 2000)

- $A(1)$ and $H^\infty$ have the Daugavet property.

  (Wojtaszczyk, 1992)

- A $C^*$-algebra has the Daugavet property if and only if it is non-atomic.

- The predual of a von Neumann algebra has the Daugavet property if and only if the algebra is non-atomic.

  (Oikhberg, 2002)
**The Daugavet property**

**Some known properties**

Let $X$ be a Banach space with the Daugavet property. Then

- $X$ contains a copy of $\ell_1$.
- $X$ does not embed into a space with unconditional basis.
  
  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)

- $X$ does not have the Radon-Nikodým property.
  
  (Wojtaszczyk, 1992)

- Every weakly-open subset of $B_X$ has diameter 2.
  
  (Shvidkoy, 2000)
**Proposition (KSSW, 2000)**

$X$ Banach space. TFAE:

(i) $X$ has the Daugavet property.

(ii) For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that

\[
\Re x^*(y) > 1 - \varepsilon \quad \text{and} \quad \|x + y\| \geq 2 - \varepsilon.
\]

(iii) For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y^* \in S_{X^*}$ such that

\[
\Re y^*(x) > 1 - \varepsilon \quad \text{and} \quad \|x^* + y^*\| \geq 2 - \varepsilon.
\]

(iv) For every $x \in S_X$ and every $\varepsilon > 0$, we have

\[
B_X = \overline{\co}(\{y \in B_X : \|x - y\| \geq 2 - \varepsilon\}).
\]
New sufficient conditions

**Theorem**
Let $X$ be a Banach space such that

$$X^* = Y \oplus_1 Z$$

with $Y$ and $Z$ 1-norming subspaces. Then, $X$ has the Daugavet property.

**Corollary**

- $X$ $L$-embedded without extreme points. Then, $X^*$ (and hence $X$) has the Daugavet property.

- $Y \subseteq L_1[0,1]$, $Y$ $L$-embedded. Then $(L_1[0,1]/Y)^*$ has the Daugavet property.
Von Neumann preduals

Let \( X_\ast \) be the predual of the von Neumann algebra \( X \).

- \( X_\ast \) is \( L \)-embedded.

- Therefore, if \( \text{ex}(B_{X_\ast}) \) is empty, then \( X \) and \( X_\ast \) have the Daugavet property.

Actually, more can be proved:
THEOREM

$X_*$ the predual of the von Neumann algebra $X$. TFAE:

(i) $X$ has the Daugavet property.
(ii) $X_*$ has the Daugavet property.
(iii) Every relative weak open subset of $B_{X_*}$ has diameter 2.
(iv) $B_{X_*}$ has no strongly exposed points.
(v) $B_{X_*}$ has no extreme points.
(vi) $X$ is non-atomic, i.e., there is no $p \in X$ such that

\[ p^2 = p^* = p \quad \text{and} \quad p \, X \, p = \mathbb{C} \, p. \]
Let $X$ be a von Neumann algebra.

- $X$ decomposes as $A \oplus_{\infty} N$, where $A$ is purely atomic and $N$ has no atoms.

- Then, $X^*$ decomposes as $A \oplus_1 N$, where $A$ is generated by its extreme points and $N$ has no extreme points.

**Corollary**

In the natural decomposition $X^* = A \oplus_1 N$, we have

- $N$ has the Daugavet property and

- $A$ has the RNP.
Let $X$ be a $C^*$-algebra. Then, $X^{**}$ is a von Neumann algebra and, as before,

$$X^* = (X^{**})_* = A \oplus_1 N$$

- $A$ is generated by the extreme points of $X^*$
- $B_N$ has no extreme points

**Corollary**

- The dual of a $C^*$-algebra does not have the Daugavet property.
- A $C^*$-algebra $X = Z^{**}$ does not have the Daugavet property.
Let $X$ be a $C^*$-algebra. Write $X^* = A \oplus_1 N$.

- $A$ is 1-norming for $X$ (Krein-Milman Theorem)
- What’s about $N$?

**Proposition**

If $X$ is non-atomic, then $N$ is 1-norming for $X$. Therefore, $X$ has the Daugavet property.

Actually, more can be proved:
The Daugavet property \( C^*-\)algebras

**Theorem**

Let \( X \) be a \( C^*-\)algebra. TFAE:

(i) \( X \) has the Daugavet property.

(ii) \( X \) is non-atomic.

(iii) The norm of \( X \) is extremely rough, i.e.,

\[
\limsup_{\|h\| \to 0} \frac{\|x + h\| + \|x - h\| - 2}{\|h\|} = 2
\]

for every \( x \in S_X \).

(iv) The norm of \( X \) is not Fréchet-smooth at any point.
The Daugavet property

Remark

• If $X$ is an arbitrary infinite-dimensional $C^*$-algebra, then every relative weak-open subset of $B_X$ has diameter 2.

• If $X$ is an arbitrary infinite-dimensional von Neumann algebra, then the norm of $X_*$ is extremely rough.

(Becerra–López–Rodríguez-Palacios, 2003)
The uniform Daugavet property

A Banach space $X$ is said to have the **Uniform Daugavet property (UDP)** if, for every $\varepsilon > 0$,

$$ \inf\{n \in \mathbb{N} : \text{conv}_n(l^+(x, \varepsilon)) \supset S_X \quad \forall x \in S_X\} < \infty $$

where $\text{conv}_n$ denotes the set of convex combinations of $n$-point collections and

$$ l^+(x, \varepsilon) = \{y \in (1 + \varepsilon)B_X : \|x + y\| > 2 - \varepsilon\}.$$ 


- $X$ has the UDP iff $X_U$ has the Daugavet property for every free ultrafilter $U$ of $\mathbb{N}$. 

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The uniform Daugavet property

Examples

- If $K$ is perfect, $C(K)$ has the UDP.

- $L_1[0,1]$ has the UDP.


- There exists $X$ having the Daugavet property but not the UDP.

  (Kadets–Werner, 2004)

Theorem

The UDP and the Daugavet property are equivalent for $C^*$-algebras and for von Neumann preduals.
The uniform Daugavet property

Sketch of the proof

- For $C^*$-algebras:
  - The ultrapower of a $C^*$-algebra is a $C^*$-algebra.
  - The roughness of the norm passes to ultrapower.

- For von Neumann preduals:
  - We do not know if the ultrapower of a von Neumann predual is again a von Neumann predual.
  - But, it is the predual of a $JBW^*$-triple.
  - The geometrical characterization is valid for preduals of $JBW^*$-triples.
  - If all the slices of $B_X$ have diameter 2, then the unit ball of $X_U$ has no strongly exposed points.
The alternative Daugavet property

$X$ Banach space, $T \in L(X)$

$$\max_{\omega \in T} \|Id + \omega T\| = 1 + \|T\| \quad \text{(aDE)}$$

- $X$ is said to have the alternative Daugavet property if every rank-one operator $T \in L(X)$ satisfies (aDE).
  - Then, all weakly compact operators also satisfy (aDE).

  (M.–Oikhberg, 2004)

- If all the operators $T \in L(X)$ satisfy (aDE), $X$ is said to have numerical index 1.

  (Lumer, 1968)
The alternative Daugavet property

\[ \star \text{For a } C^*\text{-algebra } X:\]

- The Daugavet property is equivalent to:
  - \( X \) does not have any atomic projection, or
  - \( \mathcal{B}_{X^*} \) has no \( w^*\)-strongly exposed points.

- The numerical index 1 is equivalent to:
  - \( X \) is commutative, or
  - \(|x^{**}(x^*)| = 1\) for \( x^{**} \in \text{ex} (\mathcal{B}_{X^{**}}) \) and \( x^* \in \text{ex} (\mathcal{B}_{X^*}) \).

(Huruya, 1977)
The alternative Daugavet property is equivalent to:

- the atomic projections of $X$ are central, or
- \[ |x^{**}(x^*)| = 1 \] for every $x^{**} \in \text{ex}(B_{X^{**}})$ and every $w^*$-strongly exposed point $x^*$ of $B_{X^*}$, or
- There is a commutative ideal $Y$ of $X$ such that $X/Y$ has the Daugavet property.

(M.–Oikhberg, 2004)
For the predual $V_*$ of a von Neumann algebra $V$:

- The Daugavet property of $V_*$ is equivalent to:
  - $V$ has the Daugavet property, or
  - $V_*$ has no extreme points.

- The numerical index 1 of $V_*$ is equivalent to:
  - $V$ has numerical index 1, or
  - $\|v^*(v)\| = 1$ for $v^* \in \text{ex}(B_{V^*})$ and $v \in \text{ex}(B_V)$.

- The alternative Daugavet property of $V_*$ is equivalent to:
  - $V$ has the alternative Daugavet property, or
  - $\|v(v_*)\| = 1$ for $v \in \text{ex}(B_V)$ and $v_* \in \text{ex}(B_{V_*})$, or
  - $V = C \oplus_\infty N$, where $C$ has numerical index 1 and $N$ has the Daugavet property.