Existence of hermitian operators on finite-dimensional Banach spaces: geometrical consequences

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• **Hermitian operators**

$H$ complex Hilbert space, $T \in L(H)$

$T$ is **Hermitian**  ⇐⇒  $T = T^*$

  ⇐⇒  $(Tx \mid x) \in \mathbb{R} \quad \forall x \in H$

  ⇐⇒  $\| \exp(i\rho T) \| = 1 \quad \forall \rho \in \mathbb{R}$

$iT$ is Hermitian  ⇐⇒  $T^* = -T$

  ⇐⇒  $\text{Re} \ (Tx \mid x) = 0 \quad \forall x \in H$

  ⇐⇒  $\| \exp(\rho T) \| = 1 \quad \forall \rho \in \mathbb{R}$
\( i T \) is Hermitian \( \iff \) \( \text{Re} (Tx \mid x) = 0 \quad \forall x \in H \)
\( \iff \) \( \| \exp(\rho T) \| = 1 \quad \forall \rho \in \mathbb{R} \)

\[
\text{Re} (Tx \mid x) = 0 \quad \forall x \in H \\
\updownarrow
\text{Re} x^*(Tx) = 0 \quad \text{when} \ x \in S_H, \ x^* \in S_{H^*}, \ x^*(x) = 1 \\
\updownarrow
x^*(Tx) = 0 \quad \text{when} \ x \in S_X, \ x^* \in S_{X^*}, \ x^*(x) = 1
\]
(calling \( X = H_{\mathbb{R}} \))
**Numerical Range**

$X$ Banach space, $T \in L(X)$

- **Numerical range:**

  $$V(T) = \{ x^*(Tx) : x \in S_X, \ x^* \in S_{X^*}, \ x^*(x) = 1 \}$$

- **Numerical radius:**

  $$\nu(T) = \sup\{ |\lambda| : \lambda \in V(T) \}$$
THEOREM (BOHNNENBLUST-KARLIN, 1955)

□ X real Banach space, $T \in L(X)$

$$\nu(T) = 0 \iff \| \exp(\rho T) \| = 1 \ \forall \rho \in \mathbb{R}$$

($iT$ is hermitian in $L(X)_\mathbb{C}$)
Numerical index:

\[ n(X) = \inf \{ v(T) : \| T \| = 1 \} = \max \{ k \geq 0 : k \| T \| \leq v(T) \ \forall T \} \]

+ $0 \leq n(X) \leq 1$ if $X$ is real
+ $\frac{1}{e} \leq n(X) \leq 1$ if $X$ is complex

\[ \exists T \neq 0 \text{ with } v(T) = 0 \Rightarrow n(X) = 0 \]

Of course, if $\dim(X) < \infty$, the equivalence holds.
Some known examples:

- $H$ Hilbert space, $\dim(H) > 1 \implies n(H) = 0$
- $X$ complex Banach space $\implies n(X_{\mathbb{R}}) = 0$
  
  $\left( Tx = ix \quad \forall x \in X, \quad v(T) = 0 \right)$

- $n(Z) = 0$, $Y$ arbitrary, $X = Y \oplus Z$ (absolute sum)
  
  $\implies n(X) = 0$
**Proposition**

Let $X$ be a real Banach space, $Y, Z \subseteq X$, $Z \neq 0$. If $Z$ has a complex structure, $X = Y \oplus Z$, and

$$\|y + e^{i\rho}z\| = \|y + z\| \quad (\rho \in \mathbb{R}, \ y \in Y, \ z \in Z),$$

then $n(X) = 0$.

Moreover, $T(y, z) = (0, iz)$ has numerical radius $0$. 
Example

Exists a real polyhedral Banach space $X$ such that $n(X) = 0$

□ $X$ does not contain $\mathbb{C}$ isometrically

□ $\nu(T) > 0$ for every $T \in L(X) \setminus \{0\}$
Finite dimension

**Theorem**

$X$ finite-dimensional real space. TFAE:

(i) $n(X) = 0$

(ii) Exist nonzero complex spaces $X_1, \ldots, X_n$, a real space $X_0$, and $q_1, \ldots, q_n \in \mathbb{N}$ such that $X = X_0 \oplus X_1 \oplus \cdots \oplus X_n$ and

$$
\|x_0 + e^{iq_1 \rho} x_1 + \cdots + e^{iq_n \rho} x_n\| = \|x_0 + \cdots + x_n\|
$$

for every $\rho \in \mathbb{R}$ and every $x_j \in X_j$ ($j = 0, 1, \ldots, n$).
Sketch of the proof

□ Fix $T \in L(X) \setminus \{0\}$ such that $v(T) = 0$

□ A Theorem by Auerbach: there exists a Hilbert space $H$ with $\dim(H) = \dim(X)$ such that every surjective isometry in $L(X)$ remains isometry in $L(H)$

□ Apply the above to $\exp(\rho T)$ for every $\rho \in \mathbb{R}$

□ Use Kronecker’s Approximation Theorem to change the roots of the characteristic polynomial of $T$ by rational numbers
**Corollary**

$X$ real Banach space with $n(X) = 0$

☐ If $\dim(X) = 2$, then $X \equiv \mathbb{C}$

☐ If $\dim(X) = 3$, then $X \equiv \mathbb{R} \oplus \mathbb{C}$ (absolute sum)
**Corollary**

Let $X$ be a real Banach space, consider the subspace of $L(X)$

$$Z(X) = \{ T \in L(X) : \nu(T) = 0 \}$$

\[\begin{align*}
\text{dim}(X) = n & \implies \text{dim}(Z(X)) \leq \frac{n(n-1)}{2} \\
\text{dim}(Z(X)) = \frac{n(n-1)}{2} & \iff X \text{ is a Hilbert space}
\end{align*}\]
Example

Exists a real Banach space $X$ such that $\dim(X) = 4$, $n(X) = 0$, and the numbers of complex spaces in the theorem cannot be reduced to one.

$$X = (\mathbb{R}^4, \| \cdot \|),$$

where

$$\| (a, b, c, d) \| = \frac{1}{4} \int_0^{2\pi} \left| \mathbf{Re} \left( e^{2it}(a + ib) + e^{it}(c + id) \right) \right| \, dt,$$

which satisfies

$$\| e^{2i\rho}(a, b, 0, 0) + e^{i\rho}(0, 0, c, d) \| = \| (a, b, c, d) \|$$

for every $\rho \in \mathbb{R}$ and every $a, b, c, d \in \mathbb{R}$. 
In this case, \( \dim(Z(X)) = 1 \) and \( Z(X) \) is generated by

\[
T \equiv \begin{pmatrix}
0 & 2 & 0 & 0 \\
-2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\]