Modules and Comodules
Trends in Mathematics, 185–201

Corings with Exact Rational Functors and Injective Objects

L. El Kaoutit and J. Gómez-Torrecillas

Professor Robert Wisbauer gewidmet

Abstract. We describe how some aspects of abstract localization on module categories have applications to the study of injective comodules over some special types of corings. We specialize the general results to the case of Doi-Koppinen modules, generalizing previous results in this setting.

Introduction

The Wisbauer category $\sigma[M]$ subgenerated by a module M [20] is a flexible and useful tool when applied to some at a first look unrelated situations. This has been the case of the categories of comodules over corings, which, under suitable conditions, become Wisbauer's categories [2, 5, 12]. On the other hand, as it was explained in [6], the categories of entwined modules and, henceforth, of Doi-Koppinen modules, are instances of categories of comodules over certain corings, which ultimately enlarges the field of influence of the methods from Module Theory developed in [20]. The present paper has been deliberately written from this point of view, although with a necessarily different style. To illustrate how abstract results on modules may successfully be applied to more concrete situations, we have chosen a topic from the theory of Doi-Koppinen modules with roots in the theory of graded rings and modules, namely, the transfer of the injectivity from relative modules (Doi-Koppinen, graded) to the underlying modules over the ground ring (comodule algebra, graded algebra). This was studied at the level of Doi-Koppinen modules in [10], giving versions in this framework of results on graded modules from [9]. The methods developed in [10] rest on the exactness of the rational functor for semiperfect coalgebras over fields [17], which allows the construction of a suitable adjoint pair between the category of Doi-Koppinen modules and the category of modules over the smash product [10, Theorem 3.5]. The pertinent observation here, from the point of view of corings, is that one of the functors in that adjoint pair is already a rational functor for the coring associated to the comodule algebra [2, Proposition 3.21]. Thus, a relevant ingredient in [10] is, under this interpretation, the exactness of the trace functor defined by a Wisbauer category of modules or, equivalently, the exactness of the preradical associated to a closed subcategory of a category of modules. Here, we make explicit the fact that the exactness of such a preradical is equivalent to the property of being, up to an equivalence of categories, the canonical functor of a localization (Theorem 1.1), and, henceforth, it has a right adjoint, which is explicitly described. This right adjoint will preserve injective envelopes, since it is a section functor (Proposition 1.3). We then deduce the general form of the transfer of injective objects stated in [10].

In the rest of this paper, we specialize the former general scheme to corings with exact rational functors and, even more, to Doi-Koppinen modules where the coacting coalgebra has an exact rational functor.

The results of this paper should not be considered as completely new. In fact, most of them could be gathered, with suitable adaptations (not always obvious), from other sources. Thus, our text resembles a mini-survey. However, we believe that the reader will not find elsewhere the statements made here, nor the applications to the transfer of injectivity, since they do not intend to be reproductions of previously published results. We hope we have presented a study of some aspects of the theory of corings and their comodules in a new light.

Notations and basic notions. Throughout this paper the word ring will refer to an associative unital algebra over a commutative ring K. The category of all left modules over a ring R will be denoted by ${}_R\mathsf{Mod}$, being Mod_R the notation for the category of all right R-modules. The notation $X \in \mathcal{A}$ for a category $\mathcal{A}$ means that X is an object of $\mathcal{A}$, and the identity morphism attached to any object X will be denoted by the same character X.

Recall from [19] that an A-coring is a three-tuple $(\mathfrak{C}, \Delta_{\mathfrak{C}}, \varepsilon_{\mathfrak{C}})$ consisting of an A-bimodule $\mathfrak{C}$ and two homomorphisms of A-bimodules (the comultiplication and the counity)

$$\mathfrak{C} \xrightarrow{\Delta_{\mathfrak{C}}} \mathfrak{C} \otimes_A \mathfrak{C}\,, \qquad \mathfrak{C} \xrightarrow{\varepsilon_{\mathfrak{C}}} A$$

such that $(\Delta_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ \Delta_{\mathfrak{C}} = (\mathfrak{C} \otimes_A \Delta_{\mathfrak{C}}) \circ \Delta_{\mathfrak{C}}$ and $(\varepsilon_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ \Delta_{\mathfrak{C}} = (\mathfrak{C} \otimes_A \varepsilon_{\mathfrak{C}}) \circ \Delta_{\mathfrak{C}} = \mathfrak{C}$.

A right $\mathfrak{C}$-comodule is a pair (M, ρ_M) consisting of a right A-module M and a right A-linear map $\rho_M : M \to M \otimes_A \mathfrak{C}$, called right $\mathfrak{C}$-coaction, such that $(M \otimes_A \Delta_{\mathfrak{C}}) \circ \rho_M = (\rho_M \otimes_A \mathfrak{C}) \circ \rho_M$ and $(M \otimes_A \varepsilon_{\mathfrak{C}}) \circ \rho_M = M$. A morphism of right $\mathfrak{C}$-comodules (or a right $\mathfrak{C}$-colinear map) is a right A-linear map $f : M \to M'$ satisfying $\rho_{M'} \circ f = (f \otimes_A \mathfrak{C}) \circ \rho_M$. The K-module of all right $\mathfrak{C}$-colinear maps between two right comodules $M_{\mathfrak{C}}$ and $M'_{\mathfrak{C}}$ is denoted by $\mathrm{Hom}^{\mathfrak{C}}(M, M')$. Right $\mathfrak{C}$-comodules and their morphisms form a K-linear category $\mathsf{Comod}_{\mathfrak{C}}$. Although not abelian in general, $\mathsf{Comod}_{\mathfrak{C}}$ is a Grothendieck category provided ${}_A\mathfrak{C}$ is a flat module, see [12, Section 1]. The category ${}_{\mathfrak{C}}\mathsf{Comod}$ of left $\mathfrak{C}$-comodules is symmetrically defined.

For more information on corings and comodules, the reader is referred to [5] and its bibliography.

1. Exactness of a preradical, localization, and injective objects

In this section we will derive from [14] some facts on quotient categories that will be useful in the sequel. Recall that a full subcategory $\mathcal{C}$ of a Grothendieck category $\mathcal{G}$ is said to be *closed* if any subobject and any quotient object of an object belonging to $\mathcal{C}$ is in $\mathcal{C}$, and any direct sum of objects of $\mathcal{C}$ is in $\mathcal{C}$. A closed subcategory $\mathcal{C}$ of $\mathcal{G}$ defines a preradical $\mathfrak{r} : \mathcal{G} \to \mathcal{G}$, which sends an object X of $\mathcal{G}$ to its largest subobject $\mathfrak{r}(X)$ belonging to $\mathcal{C}$. This preradical is left exact, since it is right adjoint to the inclusion functor $\mathcal{C} \subseteq \mathcal{G}$. By $\mathsf{Ker}(\mathfrak{r})$ we denote the full subcategory of $\mathcal{G}$ with objects defined by the condition $\mathfrak{r}(X) = 0$.

A full subcategory $\mathcal{L}$ of $\mathcal{G}$ is *dense* if for any short exact sequence in $\mathcal{G}$

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0 \,,$$

Y is in $\mathcal{L}$ if, and only if, both X and Z are in $\mathcal{L}$. From [13, 15.11] we know that a dense subcategory $\mathcal{L}$ is *localizing* in the sense of [14] if and only if it is stable under coproducts. Following [14, Chapter III], every localizing subcategory $\mathcal{L}$ of $\mathcal{G}$ defines a new Grothendieck category $\mathcal{G}/\mathcal{L}$ (the *quotient* category), and an exact functor $\mathbf{T} : \mathcal{G} \to \mathcal{G}/\mathcal{L}$ (the *canonical functor*) that admits a right adjoint $\mathbf{S} : \mathcal{G}/\mathcal{L} \to \mathcal{G}$. The counit $\phi_- : \mathbf{T} \circ \mathbf{S} \to 1_{\mathcal{G}/\mathcal{L}}$ of this andjunction is a natural isomorphism. The unit $\psi_- : 1_{\mathcal{G}} \to \mathbf{S} \circ \mathbf{T}$ satisfies the property that both the kernel and the cokernel of $\psi_X : X \to (\mathbf{S} \circ \mathbf{T})(X)$ belong to $\mathcal{L}$ for every object X of $\mathcal{G}$.

The exactness of a preradical $\mathfrak{r}$ can be expressed in terms of quotient categories, as the following proposition shows. The underlying ideas of its proof can be traced back to [17, Theorem 2.3].

Proposition 1.1. *Let $\mathcal{C}$ be a closed subcategory of a Grothendieck category $\mathcal{G}$ with associated preradical $\mathfrak{r} : \mathcal{G} \to \mathcal{C}$, and inclusion functor $\mathfrak{l} : \mathcal{C} \to \mathcal{G}$. The following statements are equivalent:*

(i) *$\mathfrak{r}$ is an exact functor;*
(ii) *$\mathcal{K} = \mathsf{Ker}(\mathfrak{r})$ is a localizing subcategory of $\mathcal{C}$ with canonical functor $\mathbf{T}$, and there exists an equivalence of categories $H : \mathcal{G}/\mathcal{K} \to \mathcal{C}$ such that $H \circ \mathbf{T} = \mathfrak{r}$.*

Proof. (i) $\Rightarrow$ (ii) Since $\mathfrak{r}$ is exact and preserves coproducts, we easily get that $\mathcal{K} = \mathsf{Ker}(\mathfrak{r})$ is a localizing subcategory. Consider the canonical adjunctions

$$\mathcal{C} \underset{\mathfrak{r}}{\overset{\mathfrak{l}}{\rightleftarrows}} \mathcal{G} \,, \qquad \mathcal{G} \underset{\mathbf{S}}{\overset{\mathbf{T}}{\rightleftarrows}} \mathcal{G}/\mathcal{K}$$

where $\mathbf{S}$ is right adjoint to $\mathbf{T}$, and $\mathfrak{r}$ is right adjoint to the inclusion functor $\mathfrak{l}$. Composing we get a new adjoint pair $\mathbf{T} \circ \mathfrak{l} : \mathcal{C} \leftrightarrows \mathcal{G}/\mathcal{K} : \mathfrak{r} \circ \mathbf{S}$, which we claim to provide an equivalence of categories. The unit of this new adjunction is given by

$$\mathrm{id}_{\mathcal{C}} = \mathfrak{r}\mathfrak{l} \xrightarrow{\quad \mathfrak{r}\,\psi_{\mathfrak{l}} \quad} \mathfrak{r}\,\mathbf{S}\,\mathbf{T}\,\mathfrak{l}$$

where ψ_- is the unit of the adjunction $\mathbf{T} \dashv \mathbf{S}$. For any object M of $\mathcal{C}$, there is an exact sequence

$$0 \longrightarrow X \longrightarrow \mathfrak{l}(M) \xrightarrow{\psi_{\mathfrak{l}(M)}} \mathbf{S}\mathbf{T}\mathfrak{l}(M) \longrightarrow Y \longrightarrow 0$$

with X and Y in $\mathcal{K}$. Apply the exact functor $\mathfrak{r}$ to obtain an isomorphism $\mathfrak{r}(\psi_{\mathfrak{l}(M)}) : M = \mathfrak{r}\mathfrak{l}(M) \cong \mathfrak{r}\mathbf{S}\mathbf{T}\mathfrak{l}(M)$. Therefore, $\mathfrak{r}\psi_{\mathfrak{l}(-)}$ is a natural isomorphism. The counit of the adjunction $\mathbf{T} \circ \mathfrak{l} \dashv \mathfrak{r} \circ \mathbf{S}$ is given by the following composition

$$\mathbf{T}\,\mathfrak{l}\,\mathfrak{r}\,\mathbf{S} \xrightarrow{\mathbf{T}\,\lambda_{\mathbf{S}}} \mathbf{T}\,\mathbf{S} \xrightarrow[\cong]{\phi_-} id_{\mathcal{G}/\mathcal{K}}$$

where λ_- is the counit of the adjunction $\mathfrak{l} \dashv \mathfrak{r}$, and ϕ_- is the counit of the adjunction $\mathbf{T} \dashv \mathbf{S}$. For any object N of $\mathcal{G}/\mathcal{K}$, $\lambda_{\mathbf{S}(N)}$ is a monomorphism with cokernel in $\mathcal{K}$ since $\mathfrak{r}$ is exact. Thus, [14, Lemme 2, p. 366] implies that $\mathbf{T}(\lambda_{\mathbf{S}(N)})$ is an isomorphism. Therefore, $\phi_N\, \mathbf{T}(\lambda_{\mathbf{S}(N)})$ is an isomorphism. Therefore, $\mathbf{T} \circ \mathfrak{l}$ is an equivalence of categories. On the other hand, by [14, Corollaire 3, p. 368], there exists a functor $H : \mathcal{G}/\mathcal{K} \to \mathcal{C}$ such that $H \circ \mathbf{T} = \mathfrak{r}$. By composing on the right with $\mathfrak{l}$ we get $H \circ \mathbf{T} \circ \mathfrak{l} = \mathfrak{r} \circ \mathfrak{l} = id_{\mathcal{C}}$. From this, and using that $\mathbf{T} \circ \mathfrak{l}$ is an equivalence, we get that H is an equivalence.(ii) $\Rightarrow$ (i) This is obvious, since $\mathbf{T}$ is always exact. □

In the rest of this section we consider $\mathcal{G} = \mathsf{Mod}_B$, the category of right modules over a ring B. We fix the following notation: $\mathcal{C}$ is a closed subcategory of Mod_B, with preradical $\mathfrak{r} : \mathsf{Mod}_B \to \mathcal{C}$, and inclusion functor $\mathfrak{l} : \mathcal{C} \to \mathsf{Mod}_B$. We will consider the twosided ideal $\mathfrak{a} = \mathfrak{r}(B_B)$, and $\mathcal{K} = \mathsf{Ker}(\mathfrak{r})$.

The following proposition collects a number of well-known consequences of assuming that $\mathfrak{r}$ is exact. A short proof is included.

Proposition 1.2. *If $\mathfrak{r}$ is exact then $\mathfrak{a}$ is an idempotent ideal of B such that ${}_B(B/\mathfrak{a})$ is flat, and $\mathfrak{r}(M) = M\mathfrak{a}$ for every right B-module M. In this way, $\mathcal{K} = \mathsf{Ker}(\mathfrak{r})$ becomes a localizing subcategory of Mod_B stable under direct products and injective envelopes.*

Proof. Since $\mathfrak{r}$ preserves epimorphisms it follows easily that $\mathfrak{r}(M) = M\mathfrak{a}$, for any right B-module M. In particular, we get that $\mathcal{K} = \{M \in \mathsf{Mod}_B \mid M\mathfrak{a} = 0\}$. This easily implies that $\mathcal{K}$ is a localizing subcategory stable under direct products and essential extensions. Finally, the flatness of ${}_B(B/\mathfrak{a})$ can be proved as follows. We know that $\mathcal{K}$ is isomorphic to $\mathsf{Mod}_{B/\mathfrak{a}}$. Let $\pi : B \to B/\mathfrak{a}$ be the canonical projection; the functor $- \otimes_B (B/\mathfrak{a}) : \mathsf{Mod}_B \to \mathsf{Mod}_{B/\mathfrak{a}}$ is left adjoint to the restriction of scalars functor $\pi_* : \mathsf{Mod}_{B/\mathfrak{a}} \to \mathsf{Mod}_B$. Up to the isomorphism $\mathcal{K} \cong \mathsf{Mod}_{B/\mathfrak{a}}$, π_* is nothing but the inclusion functor $j : \mathcal{K} \to \mathsf{Mod}_B$. Since $\mathcal{K}$ is stable under injective envelopes, the functor $- \otimes_B (B/\mathfrak{a})$ has to be exact, that is, ${}_B(B/\mathfrak{a})$ is a flat module. □

If $\mathfrak{a}$ is any idempotent ideal of B such that ${}_B(B/\mathfrak{a})$ is flat, then there is a canonical isomorphism of B-bimodules $\mathfrak{a} \cong \mathfrak{a} \otimes_B \mathfrak{a}$. This isomorphism makes $\mathfrak{a}$ a B-coring with counit given by the inclusion $\mathfrak{a} \subseteq B$. We say that $\mathfrak{a}$ is a *left* *idempotent B-coring* to refer to this situation. The forgetful functor $U : \mathsf{Comod}_{\mathfrak{a}} \to \mathsf{Mod}_B$ induces then an isomorphism of categories between $\mathsf{Comod}_{\mathfrak{a}}$ and the full subcategory of Mod_B whose objects are the modules M_B such that $M\mathfrak{a} = M$.

Corollary 1.1. *Assume that* $\mathfrak{r}$ *is an exact functor. Then*

(i) *The ideal* $\mathfrak{a} = \mathfrak{r}(B_B)$ *is a left idempotent* B*-coring whose category of all right comodules* $\mathsf{Comod}_{\mathfrak{a}}$ *is isomorphic to the quotient category* $\mathsf{Mod}_B/\mathcal{K}$*. In particular* $\mathfrak{a}$ *is a generator of* $\mathsf{Mod}_B/\mathcal{K}$*.*

(ii) *The functor* $F = \mathrm{Hom}_B(\mathfrak{a}, -) \circ \mathfrak{l} : \mathcal{C} \to \mathsf{Mod}_B$ *is right adjoint to* $\mathfrak{r}$*, where* $\mathfrak{l} : \mathcal{C} \to \mathsf{Mod}_B$ *is the inclusion functor. In particular if* E *is an injective object of* $\mathcal{C}$*, then* $F(E)_B$ *is an injective right module.*

Proof. (i) By Proposition 1.2, $\mathfrak{a}$ is an idempotent B-coring. Its category of right comodules clearly coincides with the torsion class $\mathcal{C}$, and the stated isomorphism of categories follows by Proposition 1.1.

(ii) Given any object (M, M') in $\mathsf{Mod}_B \times \mathcal{C}$, we get natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(\mathfrak{r}(M), M') \cong \mathrm{Hom}_B(M \otimes_B \mathfrak{a}, \mathfrak{l}(M')) \cong \mathrm{Hom}_B(M, \mathrm{Hom}_B(\mathfrak{a}, \mathfrak{l}(M'))),$$

since $\mathfrak{r}(M) = M\mathfrak{a} \cong M \otimes_B \mathfrak{a}$. This means that F is right adjoint to $\mathfrak{r}$. In particular, F preserves injectives since $\mathfrak{r}$ is exact. □

Given a module M in Mod_B, the *Wisbauer category* $\sigma[M]$ associated to M is the full subcategory of Mod_B whose objects are all M-subgenerated modules (see [20]). By definition, it is a closed subcategory and, in fact, it is easy to prove that every closed subcategory of Mod_B is of the form $\sigma[M]$. Therefore, the following theorem, that summarizes some of the previous results, complements [5, 42.16].

Theorem 1.1. *Let* $\mathcal{C}$ *be a closed subcategory of a category of modules* Mod_B *with associated preradical* $\mathfrak{r} : \mathsf{Mod}_B \to \mathcal{C}$*. Let* $\mathfrak{l} : \mathcal{C} \to \mathsf{Mod}_B$ *be the inclusion functor, and* $\mathfrak{a} = \mathfrak{r}(B)$*. The following statements are equivalent.*

(i) $\mathfrak{r} : \mathsf{Mod}_B \to \mathcal{C}$ *is an exact functor;*

(ii) $\mathcal{K} = \mathsf{Ker}(\mathfrak{r})$ *is a localizing subcategory of* Mod_B *with canonical functor* $\mathbf{T}$*, and there exists an equivalence* $H : \mathsf{Mod}_B/\mathcal{K} \to \mathcal{C}$ *such that* $\mathfrak{r} = H \circ \mathbf{T}$*;*

(iii) $F = \mathrm{Hom}_B(\mathfrak{a}_B, -) \circ \mathfrak{l} : \mathcal{C} \to \mathsf{Mod}_B$ *is right adjoint to* $\mathfrak{r}$*;*

(iv) $\mathfrak{a}$ *is an idempotent* B*-coring and the forgetful functor* $U : \mathsf{Comod}_{\mathfrak{a}} \to \mathsf{Mod}_B$ *induces an isomorphism of categories* $\mathsf{Comod}_{\mathfrak{a}} \cong \mathcal{C}$*;*

(v) $\mathfrak{a}^2 = \mathfrak{a}$ *and* $M\mathfrak{a} = M$ *for every* M *in* $\mathcal{C}$*.*

Proof. The equivalences (i) ⇔ (ii) and (i) ⇔ (iii) are immediate from Proposition 1.1 and Corollary 1.1.

(i) ⇒ (iv) is a consequence of Proposition 1.1 and Corollary 1.1(i).

(iv) ⇒ (v) Obvious.

(v) ⇒ (i) We have easily that $\mathfrak{r}(M) = M\mathfrak{a}$, for every right B-module M. From this we get immediately that $\mathfrak{r}$ is a right exact functor. □

Given a right B-module $M \in \mathcal{C}$, by $E_{\mathcal{C}}(M)$ we denote its injective hull in the Grothendieck category $\mathcal{C}$. According to Theorem 1.1, if $\mathfrak{r}$ is exact, then it becomes essentially the canonical functor associated to a localization with a section functor (the terminology is taken from [14]). As a section functor, $\mathrm{Hom}_B(\mathfrak{a}, -)$ will preserve injective envelopes, as stated in Proposition 1.3. We give a detailed proof of this fact, suitable for the forthcoming applications to more concrete situations.

Proposition 1.3. *Assume that* $\mathfrak{r} : \mathsf{Mod}_B \to \mathcal{C}$ *is exact, and let* $M \in \mathcal{C}$*. The map*

$$\zeta_M : M \to \mathrm{Hom}_B(\mathfrak{a}, E_{\mathcal{C}}(M)) \qquad (m \mapsto \zeta_M(m)(a) = ma,\ m \in M, a \in \mathfrak{a})$$

gives an injective envelope of M *in* Mod_B*. As a consequence,* M *is injective in* Mod_B *if and only if* M *is injective in* $\mathcal{C}$ *and* ζ_M *is an isomorphism.*

Proof. By Theorem 1.1, the functor $F = \mathrm{Hom}_B(\mathfrak{a}, -) \circ \mathfrak{l} : \mathcal{C} \to \mathsf{Mod}_B$ is right adjoint to the exact functor $\mathfrak{r}$. Therefore, $F(E_{\mathcal{C}}(M)) = \mathrm{Hom}_B(\mathfrak{a}, E_{\mathcal{C}}(M))$ is injective in Mod_B. On the other hand, ζ_M is obviously a right B-linear map. Let us show that it is injective. Let $m \in M$ such that $\zeta_M(m) = 0$, that is, $m\mathfrak{a} = 0$. By Theorem 1.1, we have $mB = m\mathfrak{a}$, which implies $m = 0$. Let us prove that ζ_M is essential. Pick a non zero element $f \in \mathrm{Hom}_B(\mathfrak{a}, E_{\mathcal{C}}(M))$, so there exists $0 \neq u \in \mathfrak{a}$ such that $0 \neq f(u) \in E_{\mathcal{C}}(M)$. Since M is essential in $E_{\mathcal{C}}(M)$, there exists a non zero element $b \in B$ such that $0 \neq f(u)b \in M$. Since $B/\mathfrak{a}$ is flat as a left B-module (Proposition 1.2), there exists $w \in \mathfrak{a}$ with $ub = ubw$ (see, e.g., [5, 42.5]). If we consider the map $g = fub$, then $g(x) = (fub)(x) = f(ubx)$, for all $x \in \mathfrak{a}$, that is $g = fub = \zeta_M(f(ub))$ is a non zero element of $\zeta_M(M)$, as $g(w) = f(ub) \neq 0$. □

Definition 1.1. *Assume that* $\mathfrak{a}$ *has a set of local units in the sense of* [1]*, that is,* $\mathfrak{a}$ *contains a set* E *of commuting idempotents such that for every* $x \in \mathfrak{a}$ *there exits* $e \in E$ *such that* $xe = ex = x$*. A right* B*-module* M *is said to be of* finite support *if there exits a finite subset* $F \subseteq E$ *such that* $e \in E$ *and* $m \sum_{e \in F} e = m$ *for every* $m \in M$*.*

A straightforward argument proves that if M_B is of finite support, then every $f \in \mathrm{Hom}_B(\mathfrak{a}, M)$ is of the form $f(x) = mx$ for some $m \in M$. Therefore, we deduce from Proposition 1.3:

Corollary 1.2. *Assume that* $\mathfrak{a}$ *has a set of local units, and let* $M \in \mathcal{C}$ *of finite support. Then* M *is injective in* Mod_B *if and only if* M *is injective in* $\mathcal{C}$*. As a consequence, given a homomorphism of rings* $A \to B$ *with* ${}_AB$ *flat, we deduce that if* M *is injective in* $\mathcal{C}$*, then* M *is injective in* Mod_A*.*

Remark 1.1. In Definition 1.1 and Corollary 1.2, it suffices to assume that $\mathfrak{a}$ contains a set of commuting idempotents E such that $\mathfrak{a} = \sum_{e \in E} eB$.

In what follows we specialize our results to the case where the subcategory $\mathcal{C}$ is isomorphic to the category of right comodules over a given A-coring $\mathfrak{C}$. This is the case when $\mathfrak{C}$ is member of a rational pairing $\mathcal{T} = (\mathfrak{C}, B, \langle -, - \rangle)$. Rational pairings for coalgebras over commutative rings were introduced in [15] and used in [3] to study the category of right comodules over the finite dual coalgebra associated to certain algebras over Noetherian commutative rings. This development was adapted for corings in [12], see also [2].

Recall from [12, Section 2] that a three-tuple $\mathfrak{T} = (\mathfrak{C}, B, \langle -, - \rangle)$ consisting of an A-coring $\mathfrak{C}$, an A-ring B (i.e., B is an algebra extension of A) and a balanced A-bilinear form $\langle -, - \rangle : \mathfrak{C} \times B \to A$, is said to be *a right rational pairing over* A provided

(1) $\beta_A : B \to {}^*\mathfrak{C}$ is a ring anti-homomorphism, where ${}^*\mathfrak{C}$ is the left dual convolution ring of $\mathfrak{C}$ defined in [19, Proposition 3.2], and
(2) α_M is an injective map, for each right A-module M,

where α_- and β_- are the following natural transformations

$$\beta_N : B \otimes_A N \longrightarrow \mathrm{Hom}\,({}_A\mathfrak{C}\,,\, {}_AN)\,, \qquad \alpha_M : M \otimes_A \mathfrak{C} \longrightarrow \mathrm{Hom}\,(B_A\,,\, M_A)$$
$$b \otimes_A n \longrightarrow [c \mapsto \langle c, b\rangle n] \qquad\qquad m \otimes_A c \longrightarrow [b \mapsto m\langle c, b\rangle]\,.$$

Given a right rational pairing $\mathfrak{T} = (\mathfrak{C}, B, \langle -, - \rangle)$ over A, we can define a functor called the *right rational functor* as follows. An element m of a right B-module M is called *rational* if there exists a set of *right rational parameters* $\{(c_i, m_i)\} \subseteq \mathfrak{C} \times M$ such that $mb = \sum_i m_i\langle c_i, b\rangle$, for all $b \in B$. The set of all rational elements in M is denoted by $\mathrm{Rat}^{\mathfrak{T}}(M)$. As it was explained in [12, Section 2], the proofs detailed in [15, Section 2] can be adapted in a straightforward way in order to get that $\mathrm{Rat}^{\mathfrak{T}}(M)$ is a B-submodule of M and the assignment $M \mapsto \mathrm{Rat}^{\mathfrak{T}}(M)$ is a well-defined functor $\mathrm{Rat}^{\mathfrak{T}} : \mathsf{Mod}_B \to \mathsf{Mod}_B$, which is in fact a left exact preradical. Therefore, the full subcategory $\mathrm{Rat}^{\mathfrak{T}}(\mathsf{Mod}_B)$ of Mod_B whose objects are those B-modules M such that $\mathrm{Rat}^{\mathfrak{T}}(M) = M$ is a closed subcategory. Furthermore, $\mathrm{Rat}^{\mathfrak{T}}(\mathsf{Mod}_B)$ is a Grothendieck category which is shown to be isomorphic to the category of right comodules $\mathsf{Comod}_{\mathfrak{C}}$ as [12, Theorem 2.6'] asserts (see also [2, Proposition 2.8]).

Example 1.1. Let $\mathfrak{C}$ be an A-coring such that ${}_A\mathfrak{C}$ is a locally projective left module (see [21, Theorem 2.1] and [2, Lemma 1.29]). Consider the endomorphism ring $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$ as a subring of the endomorphism ring $\mathrm{End}({}_A\mathfrak{C})$, that is, with multiplication opposite to the composition of maps. Since $\Delta_{\mathfrak{C}}$ is a left $\mathfrak{C}$-colinear and a right A-linear map, the canonical ring extension $A \to \mathrm{End}({}_A\mathfrak{C})$ factors throughout the extension $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C}) \hookrightarrow \mathrm{End}({}_A\mathfrak{C})$. Therefore, the three-tuple $\mathfrak{T} = (\mathfrak{C}, \mathrm{End}({}_{\mathfrak{C}}\mathfrak{C}), \langle -, - \rangle)$, where the balanced A-bilinear $\langle -, - \rangle$ map is defined by $\langle c, f\rangle = \varepsilon_{\mathfrak{C}}(f(c))$, for $(c, f) \in \mathfrak{C} \times \mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$ is a rational pairing since $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$ is already a ring anti-isomorphic to ${}^*\mathfrak{C}$ via the beta map associated to $\langle -, - \rangle$. We refer to $\mathfrak{T}$ as *the right canonical pairing* associated to $\mathfrak{C}$.

The following theorem complements [5, 20.8].

Theorem 1.2. *Let* $\mathfrak{T} = (\mathfrak{C}, B, \langle -, - \rangle)$ *be a right rational pairing with rational functor* $\mathrm{Rat}^{\mathfrak{T}} : \mathsf{Mod}_B \to \mathsf{Mod}_B$*, and put* $\mathfrak{a} = \mathrm{Rat}^{\mathfrak{T}}(B_B)$*,* $\mathcal{K} = \mathsf{Ker}(\mathrm{Rat}^{\mathfrak{T}})$*. The following statements are equivalent:*

(i) $\mathrm{Rat}^{\mathfrak{T}} : \mathsf{Mod}_B \to \mathsf{Mod}_B$ *is an exact functor;*
(ii) $\mathcal{K}$ *is a localizing subcategory of* Mod_B *with canonical functor* $\mathbf{T}$*, and there exists an equivalence* $H : \mathsf{Mod}_B/\mathcal{K} \to \mathrm{Rat}^{\mathfrak{T}}(\mathsf{Mod}_B)$ *such that* $\mathrm{Rat}^{\mathfrak{T}} = H \circ \mathbf{T}$*;*

(iii) $F = \mathrm{Hom}_B(\mathfrak{a}_B, -) \circ \mathfrak{l} : \mathrm{Rat}^{\mathfrak{T}}(\mathsf{Mod}_B) \to \mathsf{Mod}_B$ *is right adjoint to* $\mathrm{Rat}^{\mathfrak{T}}$*;*

(iv) $\mathfrak{a}$ *is an idempotent* B*-coring and the forgetful functor* $U : \mathsf{Comod}_{\mathfrak{a}} \to \mathsf{Mod}_B$ *induces an isomorphism of categories* $\mathsf{Comod}_{\mathfrak{a}} \cong \mathrm{Rat}^{\mathfrak{T}}(\mathsf{Mod}_B) \cong \mathsf{Comod}_{\mathfrak{C}}$*;*

(v) ${}_B\mathfrak{a}$ *is a pure submodule of* ${}_BB$*,* $\mathfrak{a}^2 = \mathfrak{a}$*, and* $\mathfrak{C}\mathfrak{a} = \mathfrak{C}$*.*

Proof. By Theorem 1.1 we only need to show that (v) $\Rightarrow$ (iv) since (iv) $\Rightarrow$ (v) is clear. We have that $\mathfrak{a}$ is an idempotent B-coring and $\mathfrak{C} \cong \mathfrak{C} \otimes_B \mathfrak{a}$ as right B-modules. Given any rational right B-module X with its canonical structure of right $\mathfrak{C}$-comodule, we obtain a B-linear isomorphism $X \cong X\square_{\mathfrak{C}}(\mathfrak{C} \otimes_B \mathfrak{a})$ (recall that the comultiplication is a right B-linear map), where the symbol $-\square_{\mathfrak{C}}-$ refers to the cotensor bifunctor over $\mathfrak{C}$. Using the left version of [16, Lemma 2.2], we get

$$X \cong X\square_{\mathfrak{C}}\mathfrak{C} \cong X\square_{\mathfrak{C}}(\mathfrak{C} \otimes_B \mathfrak{a}) \cong (X\square_{\mathfrak{C}}\mathfrak{C}) \otimes_B \mathfrak{a} \cong X \otimes_B \mathfrak{a}.$$

That is, X is in fact a right $\mathfrak{a}$-comodule. □

Remark 1.2. Right rational pairings are instances of right coring measuring in the sense of [4]. In this way, given an exact rational functor $\mathrm{Rat}^{\mathfrak{T}}$ the isomorphism of categories $\mathsf{Comod}_{\mathfrak{C}} \cong \mathsf{Comod}_{\mathfrak{a}}$ stated in Theorem 1.2 can be interpreted as an isomorphism of corings in an adequate category. Following to [4, Definition 2.1], a B-coring $\mathfrak{D}$ is called a *right extension* of an A-coring $\mathfrak{C}$ provided $\mathfrak{C}$ is a $(\mathfrak{C}, \mathfrak{D})$-bicomodule with the left regular coaction $\Delta_{\mathfrak{C}}$. Corings understood as pairs $(\mathfrak{C} : A)$ (i.e., $\mathfrak{C}$ is an A-coring) and morphisms understood as right coring extensions (i.e., a pairs consisting of an action and coaction) with their bullet composition form a category denoted by $\mathbf{CrgExt}^r_K$ (see [4] for more details). If we apply this to the setting of Theorem 1.2, then it can be easily checked that $(\mathfrak{C} : A)$ and $(\mathfrak{a} : B)$ become isomorphic objects in the category $\mathbf{CrgExt}^r_K$.

From Proposition 1.3 and Corollary 1.2, we obtain:

Proposition 1.4. *Let* $\mathfrak{T} = (\mathfrak{C}, B, \langle -, - \rangle)$ *be a right rational pairing with rational functor* $\mathrm{Rat}^{\mathfrak{T}} : \mathsf{Mod}_B \to \mathsf{Mod}_B$*, and put* $\mathfrak{a} = \mathrm{Rat}^{\mathfrak{T}}(B_B)$*. Assume that* $\mathrm{Rat}^{\mathfrak{T}}$ *is an exact functor. Let* M *be a right* $\mathfrak{C}$*-comodule, and* $E(M_{\mathfrak{C}})$ *its injective hull in* $\mathsf{Comod}_{\mathfrak{C}}$*.*

(a) *The map*

$$\zeta_M : M \to \mathrm{Hom}_B(\mathfrak{a}, E(M_{\mathfrak{C}})) \quad (m \mapsto \zeta_M(m)(a) = ma,\ m \in M, a \in \mathfrak{a})$$

gives an injective envelope of M *in* Mod_B*.*

(b) M *is injective in* Mod_B *if and only if* M *is injective in* $\mathsf{Comod}_{\mathfrak{C}}$ *and* ζ_M *is an isomorphism.*

(c) *If* ${}_AB$ *is flat,* ζ_M *is an isomorphism, and* M *is injective in* $\mathsf{Comod}_{\mathfrak{C}}$*, then* M *in injective in* Mod_A*.*

(d) *If* $\mathfrak{a}$ *has a set of local units and* M *is of finite support, then* M *is injective in* $\mathsf{Comod}_{\mathfrak{C}}$ *if and only if* M *in injective in* Mod_B*.*

(e) *Assume that* $\mathfrak{a}$ *has a set of local units,* M *is of finite support and* ${}_AB$ *is flat. If* M *is injective in* $\mathsf{Comod}_{\mathfrak{C}}$*, then* M *is injective in* Mod_A*.*

2. Rational functors for entwined and Doi-Koppinen modules

In this section, we shall study the exactness of the rational functors for the corings coming from entwining structures. When specialized to the entwining structures given by a comodule algebra, we will obtain a result from [2]. Most of results in [10] are deduced.

2.1. Entwining structures with rational functor

Recall from [7] that an entwining structure over K is a three-tuple $(A, C)_\psi$ consisting of a K-algebra A with multiplication μ and unity 1, a K-coalgebra C with comultiplication Δ and counity ε, and a K-module map $\psi : C \otimes_K A \to A \otimes_K C$ satisfying

$$\begin{aligned} \psi \circ (C \otimes_K \mu) &= (\mu \otimes_K C) \circ (A \otimes_K \psi) \circ (\psi \otimes_K A), \\ (A \otimes_K \Delta) \circ \psi &= (\psi \otimes_K C) \circ (C \otimes_K \psi) \circ (\Delta \otimes_K A), \\ \psi \circ (C \otimes_K 1) &= 1 \otimes_K C, \quad (A \otimes_K \varepsilon) \circ \psi = \varepsilon \otimes_K A. \end{aligned} \tag{2.1}$$

By [6, Proposition 2.2] the corresponding A-coring is $\mathfrak{C} = A \otimes_K C$ with the A-bimodule structure given by $a''(a' \otimes_K c)a = a''a'\psi(c \otimes_K a)$, $a, a', a'' \in A$, $c \in C$, the comultiplication $\Delta_{\mathfrak{C}} = A \otimes_K \Delta$, and the counit $\varepsilon_{\mathfrak{C}} = A \otimes_K \varepsilon$. Furthermore, the category of right $\mathfrak{C}$-comodules is isomorphic to the category of right entwined modules.

The map $(\phi, \nu) : (C, K) \to (\mathfrak{C}, A)$ defined by $\nu(1) = 1$ and $\phi(c) = 1 \otimes_K c$, is a homomorphism of corings in the sense of [16]. As in [16] the associated induction and ad-induction functors to this morphism are, respectively, given by $\mathcal{O} : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Comod}_C$ and $- \otimes_K A : \mathsf{Comod}_C \to \mathsf{Comod}_{\mathfrak{C}}$, where $\mathcal{O}$ is the cotensor functor $-\square_{\mathfrak{C}}(A \otimes_K C)$. When $\mathsf{Comod}_{\mathfrak{C}}$ is interpreted as the category of entwined modules, $\mathcal{O}$ is naturally isomorphic to the forgetful functor. Moreover, there is a natural isomorphism

$$\begin{array}{rcl} \mathrm{Hom}_{\mathfrak{C}}(M \otimes_K A, N) & \xrightarrow{\cong} & \mathrm{Hom}_C(M, \mathcal{O}(N)) \\ f & \longmapsto & [m \mapsto f(m \otimes_K 1)] \\ [m \otimes_K a \mapsto g(m)a] & \longleftarrow & g, \end{array}$$

for every pair of comodules $(M_C, N_{\mathfrak{C}})$. Thus the functor $\mathcal{O}$ is a right adjoint functor of $- \otimes_K A$. If C_K is a flat module, then $\mathcal{O}$ is exact, since $U_A : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Mod}_A$ is already an exact functor (see, [12, Proposition 1.2]).

We know from [6] that the left dual convolution ring ${}^*\mathfrak{C}$ is isomorphic as a K-module to $\mathrm{Hom}_K(C, A)$. Up to this isomorphism the convolution multiplication reads

$$f \cdot g = \mu \circ (A \otimes_K f) \circ \psi \circ (C \otimes_K g) \circ \Delta_C, \quad f, g \in \mathrm{Hom}_K(C, A). \tag{2.2}$$

The connection between this convolution ring and the usual coalgebra convolution ring C^* is given by the following homomorphism of rings

$$\Phi : C^* \longrightarrow {}^*\mathfrak{C}, \qquad (x \longmapsto A \otimes_K x). \tag{2.3}$$

Proposition 2.1. *Let $(A,C)_\psi$ be an entwining structure over K such that C_K is a locally projective module and consider its corresponding A-coring $\mathfrak{C} = A \otimes_K C$. Suppose that there is a right rational pairing $\mathcal{T} = (\mathfrak{C}, B, \langle -,- \rangle)$ and an anti-morphism of K-algebras $\varphi : C^* \to B$ which satisfy the following two conditions: (1) $\beta \circ \varphi = \Phi$, where $\beta : B \to {}^*\mathfrak{C}$ is the anti-homomorphism of K-algebras associated to $\mathcal{T}$ and Φ is the homomorphism of rings given in equation* (2.3)*; (2) for every pair of elements $(a,x) \in A \times C^*$, there exists a finite subset of pairs $\{(x_i, a_i)\}_i \subseteq C^* \times A$ such that $a\varphi(x) = \sum_i \varphi(x_i)a_i$. Then, by restricting scalars we have*

$$\mathrm{Rat}^{\mathcal{T}}(M_B) = \mathrm{Rat}^r_C({}_{C^*}M)$$

for every right B-module M, where $\mathrm{Rat}^r_C(-)$ is the canonical right rational functor associated to the K-coalgebra C.

Proof. Start with an arbitrary element $m \in \mathrm{Rat}^{\mathcal{T}}(M_B)$ with right rational system of parameters $\{(\sum_k a_{kj} \otimes_K c_{kj},\, m_j)\}_j \subset \mathfrak{C} \times M$. Then for every $x \in C^*$, we have

$$\begin{aligned} xm = m\varphi(x) &= \sum_{k,j} m_j \langle a_{kj} \otimes_K c_{kj}, \varphi(x) \rangle \\ &= \sum_{k,j} m_j \beta(\varphi(x))(a_{kj} \otimes_K c_{kj}) \\ &= \sum_{k,j} m_j \Phi(x)(a_{kj} \otimes_K c_{kj}), \quad \Phi = \beta \circ \varphi, \\ &= \sum_{k,j} m_j a_{kj} x(c_{kj}), \end{aligned}$$

thus $\{c_{kj}, m_j a_{kj}\} \subset C \times M$ is a right rational system of parameters for $m \in {}_{C^*}M$; that is $m \in \mathrm{Rat}^r_C({}_{C^*}M)$. Therefore, $\mathrm{Rat}^{\mathcal{T}}(M_B) \subseteq \mathrm{Rat}^r_C({}_{C^*}M)$. Conversely, start with a pair of elements $(a,x) \in A \times C^*$, and let $\{(x_i,a_i)\}_i \subset C^* \times A$ be the finite system given by hypothesis, that is $a\varphi(x) = \sum_i \varphi(x_i)a_i$. So, for every element $m \in \mathrm{Rat}^r_C({}_{C^*}M)$ with right C-coaction $\rho_{\mathrm{Rat}^r_C({}_{C^*}M)}(m) = \sum_{(m)} m_{(0)} \otimes_K m_{(1)}$, we have

$$\begin{aligned} x(ma) &= (ma)\varphi(x) = m(a\varphi(x)) \\ &= \sum m(\varphi(x_i)a_i), \;\; a\varphi(x) = \sum \varphi(x_i)a_i \\ &= \sum (x_i m)a_i \\ &= \sum (m_{(0)} x_i(m_{(1)}))a_i \\ &= \sum m_{(0)} \left(\Phi(x_i)(1 \otimes_K m_{(1)})a_i \right) \\ &= \sum m_{(0)} \left(\beta(\varphi(x_i))(1 \otimes_K m_{(1)})a_i \right) \\ &= \sum m_{(0)} \left(\beta(\varphi(x_i)a_i)(1 \otimes_K m_{(1)}) \right), \;\; \beta \text{ is right } A\text{-linear} \end{aligned}$$

$$= \sum m_{(0)} \langle 1 \otimes_K m_{(1)}, \varphi(x_i)a_i \rangle$$
$$= \sum m_{(0)} \langle 1 \otimes_K m_{(1)}, a\varphi(x) \rangle$$
$$= \sum m_{(0)} \langle a_\psi \otimes_K m^\psi_{(1)}, \varphi(x) \rangle, \ \psi(m_{(1)} \otimes_K a) = \sum a_\psi \otimes_K m^\psi_{(1)}$$
$$= \sum m_{(0)} a_\psi x(m^\psi_{(1)}).$$

We conclude that $ma \in \mathrm{Rat}^r_C({}_{C^*}M)$ with right C-coaction $\rho_{\mathrm{Rat}^r_C({}_{C^*}M)}(ma) = \sum m_{(0)}a_\psi \otimes_K m^\psi_{(1)}$. From which we conclude that $\mathrm{Rat}^r_C({}_{C^*}M)$ is an entwined module, and thus a right $\mathfrak{C}$-comodule or, equivalently, a right rational B-submodule of M_B. Therefore, $\mathrm{Rat}^r_C({}_{C^*}M) \subseteq \mathrm{Rat}^{\mathcal{T}}(M_B)$. □

2.2. The category of Doi-Koppinen modules

We apply the results of Subsection 2.1 to the category of Doi-Koppinen modules. This category is identified with the category of right rational modules over a well-known ring. Some results of this section were proved by different methods for particular Hopf algebras in [8, Theorem 2.3], for algebras and coalgebras over a field in [10, Proposition 2.7], and more recently for bialgebras in [2, Theorem 3.18, Proposition 3.21].

Let $\mathbf{H}$ be a Hopf K-algebra, (A, ρ_A) a right $\mathbf{H}$-comodule K-algebra, and (C, ϱ_C) a left $\mathbf{H}$-module K-coalgebra. That is $\rho_A : A \to A \otimes_K \mathbf{H}$ and $\varrho_C : \mathbf{H} \otimes_K C \to C$ are, respectively, a K-algebra and a K-coalgebra map. We will use Sweedler's notation, that is $\Delta_C(c) = \sum_{(c)} c_{(1)} \otimes_K c_{(2)}$, $\Delta_{\mathbf{H}}(h) = \sum_{(h)} h_{(1)} \otimes_K h_{(2)}$, and $\rho_A(a) = \sum_{(a)} a_{(1)} \otimes_K a_{(2)}$, for every $c \in C$, $h \in \mathbf{H}$ and $a \in A$.

Following [11, 18], a *Doi-Koppinen module* is a left A-module M with a structure of right C-comodule ρ_M such that, for every $a \in A$, $m \in M$,

$$\rho_M(am) = \sum a_{(0)} m_{(0)} \otimes_K a_{(1)} m_{(1)}.$$

A morphism between two Doi-Koppinen modules is a left A-linear and right C-colinear map. Doi-Koppinen modules and their morphisms form the category ${}_A\mathcal{M}(\mathbf{H})^C$.

Consider the following K-map (A^o means the opposite ring of A)

$$\psi : C \otimes_K A^o \longrightarrow A^o \otimes_K C, \qquad (c \otimes_K a^o \longmapsto \textstyle\sum_{(a)} a^o_{(0)} \otimes_K a_{(1)}c). \quad (2.4)$$

It is easily seen that the map ψ satisfies all identities of equation (2.1). That is $(A^o, C)_\psi$ is an entwining structure over K. So, consider the associated A^o-coring $\mathfrak{C} = A^o \otimes_K C$, the A^o-biactions are then given by

$$b^o(a^o \otimes_K c) = (ab)^o \otimes_K c, \text{ and } (b^o \otimes_K c)a^o = b^o \psi(c \otimes_K a^o) = \sum (a_{(0)}b)^o \otimes_K a_{(1)}c,$$

for every $a^o, b^o \in A^o$ and $c \in C$. The convolution multiplication of $\mathrm{Hom}_K(C, A^o)$ comes out from the general equation (2.2), as

$$f.g(c) = \sum_{(c)} \big(f(g(c_{(2)})_{(1)}c_{(1)})\, g(c_{(2)})_{(0)}\big)^o \in A^o, \qquad f, g \in \mathrm{Hom}_K(C, A^o) \text{ and } c \in C. \tag{2.5}$$

This multiplication coincides with the generalized smash product of A by C, denoted by $\sharp(C, A)$ in [18, (2.1)].

Define the smash product $A\sharp C^*$ whose underling K-module is the tensor product $A \otimes_K C^*$ and internal multiplication is given by

$$(a\sharp x).(b\sharp y) = \sum ab_{(0)}\sharp(xb_{(1)})y,$$

for $a \otimes_K x$, $b \otimes_K y \in A \otimes_K C^*$, and where the left $\mathbf{H}$-action on C^* is induced by the right $\mathbf{H}$-action on C. The unit of this multiplication is $1\sharp\varepsilon_C$. Moreover, it is clear that the maps

$$-\sharp\varepsilon_C : A \longrightarrow A\sharp C^*, \quad a \longmapsto a\sharp\varepsilon_C \qquad 1\sharp- : C^* \longrightarrow A\sharp C^*, \quad x \longmapsto 1\sharp x$$

are K-algebra maps, and an easy computation shows that

$$\alpha_A : A\sharp C^* \longrightarrow \mathrm{Hom}_K(C, A^o), \qquad (a\sharp x \longmapsto [c \mapsto a^o x(c)])$$

is also a K-algebra morphism where $\mathrm{Hom}_K(C, A^o)$ is endowed with the multiplication of equation (2.5).

Proposition 2.2. [2, Proposition 3.21] *Let $\mathbf{H}$ be a Hopf K-algebra, A a right $\mathbf{H}$-comodule K-algebra and C a left $\mathbf{H}$-module K-coalgebra. Consider $\mathfrak{C} = A^o \otimes_K C$ the A^o-coring associated to the entwining structure $(A^o, C)_\psi$ where ψ is defined by* (2.4)*, and let $B = (A\sharp C^*)^o$. Suppose that C_K is a locally projective module. Then $\mathcal{T} = (\mathfrak{C}, B, \langle -, - \rangle)$ is right rational pairing over A^o with the bilinear form $\langle -, - \rangle$ defined by*

$$\mathfrak{C} \times B \longrightarrow A^o, \qquad (a^o \otimes_K c, (b\sharp x)^o) \longmapsto \langle a^o \otimes_K c, (b\sharp x)^o \rangle = a^o b^o x(c)$$

$a, b \in A$, $c \in C$, $x \in C^*$. Moreover, $(A^o, C)_\psi$, $\mathcal{T}$ and $\varphi = (1\sharp-)^o : C^* \to B$ satisfy the conditions (1) *and* (2) *stated in Proposition* 2.1*, and using restriction of scalars, we obtain*

$$\mathrm{Rat}^{\mathcal{T}}(M_B) = \mathrm{Rat}^r_C({}_{C^*}M),$$

for every right B-module M.

Proof. First we show that $\langle -, - \rangle$ is bilinear and balanced. For $a, b, e \in A$, $x \in C^*$ and $c \in C$, we compute

$$\begin{aligned}
\langle a^o \otimes_K c, (b\sharp x)^o e^o \rangle &= \langle a^o \otimes_K c, ((e\sharp\varepsilon_C)(b\sharp x))^o \rangle \\
&= \sum \langle a^o \otimes_K c, (eb_{(0)}\sharp(\varepsilon_C b_{(1)})x)^o \rangle \\
&= \sum a^o\, b^o_{(0)}\, e^o\, ((\varepsilon_C\, b_{(1)})x)(c) \\
&= \sum a^o\, b^o_{(0)}\, e^o\, \varepsilon_{\mathbf{H}}(b_{(1)})x(c) \\
&= a^o\, b^o\, e^o\, x(c) \\
&= \langle a^o \otimes_K c, (b\sharp x)^o \rangle\, e^o,
\end{aligned}$$

which shows that $\langle -, - \rangle$ is right A^o-linear, and

$$\begin{aligned}
\langle (a^o \otimes_K c)e^o, (b\sharp x)^o \rangle &= \langle a^o \psi(c \otimes_K e^o), (b\sharp x)^o \rangle \\
&= \sum \langle a^o e^o_{(0)} \otimes_K e_{(1)}c, (b\sharp x)^o \rangle \\
&= \sum a^o e^o_{(0)} b^o x(e_{(1)}c) \\
&= \sum a^o e^o_{(0)} b^o\ (xe_{(1)})(c) \\
&= \langle a^o \otimes_K c, \sum (be_{(0)})\sharp xe_{(1)} \rangle \\
&= \langle a^o \otimes_K c, (b\sharp x)(e\sharp\varepsilon_C) \rangle \\
&= \langle a^o \otimes_K c, e^o\, (b\sharp x)^o \rangle,
\end{aligned}$$

which proves that $\langle -, - \rangle$ is A^o-balanced. The pairing $\langle -, - \rangle$ is clearly left A^o-linear. Consider now the right natural transformation associated to $\langle -, - \rangle$:

$$\begin{aligned}
\alpha_N :\ N \otimes_{A^o} \mathfrak{C} &\longrightarrow \mathrm{Hom}_{A^o}(B_{A^o}, N) \\
n \otimes_{A^o} a^o \otimes_K c &\longmapsto [(b\sharp x)^o \mapsto n\langle a^o \otimes_K c, (b\sharp x)^o \rangle\ =\ n(a^o b^o x(c))]\,.
\end{aligned}$$

We need to show that α_{-} is injective. So let $\sum_i n_i \otimes_{A^o} 1 \otimes_K c_i \in N \otimes_{A^o} \mathfrak{C}$ whose image by α_N is zero. Since C_K is locally projective, associated to the finite set $\{c_i\}_i$ there exists a finite set $\{(c_l, x_l)\} \subset C \times C^*$ such that $c_i = \sum_l c_l x_l(c_i)$. The condition

$$\alpha_N(\sum_i n_i \otimes_{A^o} 1 \otimes_K c_i)((1\sharp x_l)^o)\ =\ \sum_i n_i x_l(c_i) = 0, \text{ for all the } l's,$$

implies that

$$\sum_i n_i \otimes_{A^o} 1 \otimes_K c_i = \sum_{i,\,l} n_i \otimes_{A^o} 1 \otimes_K x_l(c_i)c_l = \sum_l \left(\sum_i n_i x_l(c_i) \right) \otimes_{A^o} 1 \otimes_K c_l = 0,$$

That is α_N is an injective map for every right A^o-module N. Therefore, $\mathcal{T}$ is a right rational system. Lastly, the map $\beta : B \to {}^*\mathfrak{C}$ sending $(b\sharp x)^o \mapsto [a^o \otimes_K c \mapsto \langle a^o \otimes_K c, (b\sharp x)^o \rangle]$ is an anti-homomorphism of K-algebras, and B is a K-algebra extension of A^o, thus $\mathcal{T}$ is actually a right rational pairing.

Let $a^o \in A^o$, $c \in C$ and $x \in C^*$, then

$$\beta(\varphi(x))(a^o \otimes_K c) = \langle a^o \otimes_K c, (1\sharp x)^o \rangle \;=\; a^o x(c) \;=\; \Phi(x)(a^o \otimes_K c)$$

which implies the condition (1) of Proposition 2.1. For the condition (2), it is easily seen that the set $\{a^o_{(0)}, xa_{(1)}\}$, where $\rho_A(a) = \sum a_{(0)} \otimes_K a_{(1)}$, satisfies this condition for the pair $(a^o, x) \in A^o \times C^*$. The last stated assertion is a consequence of Proposition 2.1, and this finishes the proof. □

Theorem 2.1. *Let* $\mathbf{H}$ *be a Hopf* K*-algebra,* A *a right* $\mathbf{H}$*-comodule* K*-algebra and* C *a left* $\mathbf{H}$*-module* K*-coalgebra. Consider* $\mathfrak{C} = A^o \otimes_K C$ *the* A^o*-coring associated to the entwining structure* $(A^o, C)_\psi$ *where* ψ *is defined by* (2.4)*, and set* $B = (A\sharp C^*)^o$*. Suppose that* C_K *is a locally projective module and consider the right rational pairing* $\mathcal{T} = (\mathfrak{C}, B, \langle -,- \rangle)$ *over* A^o *of Proposition* 2.2*, and put* $\mathfrak{a} = \mathrm{Rat}^{\mathcal{T}}(B_B)$*. If* A_K *is a flat module and* $\mathrm{Rat}^r_C(-)$ *is an exact functor, then*

(a) $\mathrm{Rat}^r_C({}_{C^*}C^*)$ *is a right* $\mathbf{H}$*-submodule of* C^* *and* $\mathfrak{a} = A \otimes_K \mathrm{Rat}^r_C({}_{C^*}C^*)$*.*

(b) *For each right* B*-module* M*, the map*

$$\mathrm{Hom}_B(\mathfrak{a}_B, M) \longrightarrow \mathrm{Hom}_{C^*}(\mathrm{Rat}^r_C({}_{C^*}C^*), M) \tag{2.6}$$

sending f *onto the morphism* $\widehat{f}$ *defined by* $\widehat{f}(c^*) = f(1 \otimes c^*)$ *for* $c^* \in C^*$ *is an isomorphism of* K*-modules.*

(c) *If, for every left* $A\sharp C^*$*-module* M*, we endow* $\mathrm{Hom}_{C^*}(\mathrm{Rat}^r_C({}_{C^*}C^*), M)$ *with the structure of a left* $A\sharp C^*$*-module transferred from that of*

$$\mathrm{Hom}_{A\sharp C^*}({}_{A\sharp C^*}\mathfrak{a}, M)$$

via the isomorphism (2.6)*, then we obtain a functor*

$$\mathrm{Hom}_{C^*}(\mathrm{Rat}^r_C({}_{C^*}C^*), -) : {}_A\mathcal{M}(\mathbf{H})^C \longrightarrow {}_{A\sharp C^*}\mathsf{Mod},$$

which is right adjoint to the functor (*see Proposition* 2.2)

$$\mathrm{Rat}^r_C : {}_{A\sharp C^*}\mathsf{Mod} \longrightarrow {}_A\mathcal{M}(\mathbf{H})^C.$$

Proof. (a) Let $y \in \mathrm{Rat}^r_C({}_{C^*}C^*)$ with rational system of parameters $\{(y_i, c_i)\}_i \subset C^* \times C$. For any $h \in \mathbf{H}$ and $x \in C^*$, we obtain as in [10, Lemma 3.1]:

$$\begin{aligned}(x(yh))(c) &= \sum_{(c)} x(c_{(1)})y(hc_{(2)}) = \sum_{(c),(h)} x(\varepsilon_{\mathbf{H}}(h_{(1)})c_{(1)})y(h_{(2)}c_{(2)}) \\ &= \sum_{(c),(h)} x(S(h_{(1)})h_{(2)}c_{(1)})y(h_{(3)}c_{(2)}) = \sum_{(h)} (xS(h_{(1)})y)(h_{(2)}c) \\ &= \sum_{(h),i} y_i(h_{(2)}c)(xS(h_{(1)}))(c_i) = \sum_{(h),i} y_i(h_{(2)}c)x(S(h_{(1)})c_i),\end{aligned}$$

for every $c \in C$, where S is the antipode of $\mathbf{H}$. That is,

$$x(yh) = \sum_{(h),\, i} (y_i h_{(2)})\, x(S(h_{(1)})c_i).$$

Hence, $\{(y_i h_{(2)}, S(h_{(1)})c_i)\} \subset C^* \times C$ is a rational system of parameters for yh. Thus $yh \in \mathrm{Rat}^r_C({}_{C^*}C^*)$, and $\mathrm{Rat}^r_C({}_{C^*}C^*)$ is a right $\mathbf{H}$-submodule of C^*. Since A_K is a flat module, an easy computation shows now that $A \otimes_K \mathrm{Rat}^r_C({}_{C^*}C^*)$ is a two-sided ideal of $A\sharp C^*$. Let $x \in C^*$, $a \otimes_K y \in A \otimes_K \mathrm{Rat}^r_C({}_{C^*}C^*)$, and $\{(y_i, c_i)\}_i \subset C^* \times C$ a rational system of parameters for y. Applying the smash product, we get

$$x(a \otimes_K y) = \sum_{(a)} a_{(0)} \otimes_K ((xa_{(1)})y) = \sum_{(a),\, i} (a_{(0)} \otimes_K y_i) x(a_{(1)}c_i);$$

this means that $\{(a_{(0)} \otimes_K y_i, a_{(1)}c_i)\}_{(a),\, i} \subset (A \otimes_K \mathrm{Rat}^r_C({}_{C^*}C^*)) \times C$ is a rational system of parameters for $a \otimes_K y \in {}_{C^*}\left(A \otimes_K \mathrm{Rat}^r_C({}_{C^*}C^*)\right)$. Proposition 2.2, implies now that $A \otimes_K \mathrm{Rat}^r_C({}_{C^*}C^*) \subseteq \mathfrak{a}$. Conversely, we know that $\mathfrak{a}$ is a right $\mathfrak{C}$-comodule, so the underlying K-module is a right C-comodule, and, since Rat^r_C is exact, $\mathfrak{a} = \mathrm{Rat}^r_C({}_{C^*}C^*)\mathfrak{a}$. From this equality, it is easy to see that $\mathfrak{a} \subseteq A \otimes_K \mathrm{Rat}^r_C({}_{C^*}C^*)$, and the desired equality is derived.

(b) We know that $B = (A\sharp C^*)^o$ and, by (a), we have $\mathfrak{a} = A \otimes_K \mathrm{Rat}^r({}_{C^*}C^*)$. Consider the homomorphism of algebras $C^* \to A\sharp C^*$, which gives, as usual, the induction functor $(A\sharp C^*) \otimes_{C^*} - : {}_{C^*}\mathsf{Mod} \to {}_{A\sharp C^*}\mathsf{Mod}$ which is left adjoint to the restriction of scalars functor ${}_{A\sharp C^*}\mathsf{Mod} \to {}_{C^*}\mathsf{Mod}$. The mapping $f \mapsto \widehat{f}$ is then defined as the composition

$$\begin{aligned}&\mathrm{Hom}_{A\sharp C^*}\left(A \otimes_K \mathrm{Rat}^r_C({}_{C^*}C^*)\,,\, M\right)\\ &\quad\cong \mathrm{Hom}_{A\sharp C^*}\left((A\sharp C^*) \otimes_{C^*} \mathrm{Rat}^r_C({}_{C^*}C^*)\,,\, M\right) \cong \mathrm{Hom}_{C^*}\left(\mathrm{Rat}^r_C({}_{C^*}C^*)\,,\, M\right),\end{aligned}$$

where the second is the adjointness isomorphism, and the first one comes from the obvious isomorphism $(A\sharp C^*) \otimes_{C^*} \mathrm{Rat}^r_C({}_{C^*}C^*) \cong A \otimes_K \mathrm{Rat}^r_C({}_{C^*}C^*)$.

(c) This is a consequence of (b) and Theorem 1.2. □

Keep, in the following corollary, the hypotheses of Theorem 2.1.

Corollary 2.1. *If M is an object of ${}_A\mathcal{M}(\mathbf{H})^C$, and $E({}_AM^C)$ denotes its injective hull in the category ${}_A\mathcal{M}(\mathbf{H})^C$, then*

(a) *The map*

$$\zeta_M : M \to \mathrm{Hom}_{C^*}\left(\mathrm{Rat}^r_C({}_{C^*}C^*)\,,\, E({}_AM^C)\right)$$
$$(m \mapsto \zeta_M(m)(c^*) = c^*m,\ m \in M))$$

gives an injective envelope of M in ${}_{A\sharp C^*}\mathsf{Mod}$.

(b) *M is injective in ${}_{A\sharp C^*}\mathsf{Mod}$ if and only if M is injective in ${}_A\mathcal{M}(\mathbf{H})^C$ and ζ_M is an isomorphism.*

(c) *Assume that the antipode of $\mathbf{H}$ is bijective, and that C^* is flat as a K-module. Let M be injective in ${}_A\mathcal{M}(\mathbf{H})^C$. If M has finite support as a right C-comodule, then M is injective as a left A-module.*

Proof. The two first statements follow from Proposition 1.4 and Theorem 2.1. For the last statement, observe that $B_{A^{op}}$ is a flat module. Now, the proof of [10, Lemma 2.6] runs here to prove that ${}_{A^o}B$ is flat. □

Remark 2.1. We have proved that, under suitable conditions,

$$\mathrm{Rat}^{\mathcal{T}}(M_B) = \mathrm{Rat}^r_C({}_{C^*}M) \;=\; M\mathfrak{a} \;=\; \mathrm{Rat}^r_C({}_{C^*}C^*)M, \tag{2.7}$$

for every right B-module M. Therefore, equation (2.7) establishes a radical functor: $t : {}_{A\sharp C^*}\mathsf{Mod} \to {}_{A}\mathcal{M}(\mathbf{H})^C$ which acts on objects by $M \to \mathrm{Rat}^r_C({}_{C^*}C^*)M$. This radical was used in [10, Lemma 2.9] for left and right semiperfect coalgebras over a commutative field. In this way, if we apply our results and [17, Proposition 2.2] to this setting, then most part of the results stated in [10] become consequences of the results stated in this paper. In particular, let us mention that, for a semiperfect coalgebra over a field, any comodule of finite support in the sense of [10] becomes of finite support in the sense of Definition 1.1. Finally, let us note that the results from [10] are only applicable to group-graded algebras over a field. This restriction has been dropped by our approach, and we fully cover the case of graded rings (take $K = \mathbb{Z}$), since the $\mathbb{Z}$-coalgebra $\mathbb{Z}G$, where the elements of the group G are all group-like, is easily shown to have an exact rational functor. Of course, the category of comodules over this coalgebra is not semiperfect.

References

[1] G.D. Abrams, *Morita equivalence for rings with local units*, Comm. Algebra **11** (1983), 801–837.

[2] J.Y. Abuhlail, *Rational modules for corings*, Comm. Algebra **31** (2003), no. 12, 5793–5840.

[3] J.Y. Abuhlail, J. Gómez-Torrecillas, and F.J. Lobillo, *Duality and rational modules in Hopf algebras over commutative rings*, J. Algebra **240** (2001), 165–184.

[4] T. Brzeziński, *A note on coring extensions*, Ann. Univ. Ferrara, Sez. VII, Sc. Mat. **LI**(III) (2005), 15–27.

[5] T. Brzeziński and R. Wisbauer, *Corings and comodules*, LMS, vol. 309, Cambridge University Press, 2003.

[6] T. Brzeziński, *The structure of corings. Induction functors, Maschke-type theorem, and Frobenius and Galois-type properties*, Alg. Rep. Theory **5** (2002), 389–410.

[7] T. Brzeziński and S. Majid, *Coalgebra bundles*, Comm. Math. Phys. **191** (1998), 467–492.

[8] C. Cai and H. Chen, *Coactions, smash product, and Hopf modules*, J. Algebra **167** (1994), 85–99.

[9] S. Dăscălescu, C. Năstăsescu, A. del Rio, and F. van Oystaeyen, *Gradings of finite support. Applications to injective objects*, J. Pure Appl. Algebra **107** (1996), 193–206.

[10] S. Dăscălescu, C. Năstăsescu, and B. Torrecillas, *Co-Frobenius Hopf algebras: Integrals, Doi-Koppinen modules and injective objects*, J. Algebra **220** (1999), 542–560.

[11] Y. Doi, *Unifying Hopf modules*, J. Algebra. **153** (1992), 373–385.

[12] L. El Kaoutit, J. Gómez-Torrecillas, and F.J. Lobillo, *Semisimple corings*, Algebra Colloq. **11** (2004), 427–442.

[13] C. Faith, *Algebra: rings, modules and categories, I*, Grundlehren Math. Wiss., 190, Springer, New York, 1973.

[14] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448.

[15] J. Gómez-Torrecillas, *Coalgebras and comodules over commutative rings*, Rev. Roumaine Math. Pures Appl. **43** (1998), no. 5-6, 591–603.

[16] J. Gómez-Torrecillas, *Separable functors in corings*, Int. J. Math. Math. Sci. **30** (2002), no. 4, 203–225.

[17] J. Gómez-Torrecillas and C. Năstăsescu, *Quasi-co-Fronenius coalgebras*, J. Algebra, **174** (1995), 909–923.

[18] M. Koppinen, *Variations on the smash product with applications to group-graded rings*, J. Pure Appl. Algebra **104** (1995), 61–80.

[19] M. Sweedler, *The predual theorem to the Jacobson-Bourbaki theorem*, Trans. Amer. Math. Soc. **213** (1975), 391–406.

[20] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading-Paris, 1991.

[21] B. Zimmermann-Huisgen, *Pure submodules of direct products of free modules*, Math. Ann. **224** (1976), 233–245.

L. El Kaoutit
Departamento de Álgebra
Facultad de Educación y Humanidades
Universidad de Granada
El Greco N^o 10
E-51002 Ceuta, España
e-mail: `kaoutit@ugr.es`

J. Gómez-Torrecillas
Departamento de Álgebra
Facultad de Ciencias
Universidad de Granada
E-18071 Granada, España
e-mail: `gomezj@ugr.es`