STABLE EMBEDDED MINIMAL SURFACES BOUNDED BY A STRAIGHT LINE

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Abstract. We prove that if $M \subset \mathbb{R}^3$ is a properly embedded oriented stable minimal surface whose boundary is a straight line and the area of M in extrinsic balls grows quadratically in the radius, then M is a halfplane or half of the classical Enneper minimal surface. This solves a conjecture posed by B. White in [20].

1. Introduction.

The simplest version of the *Plateau problem* asks for surfaces that minimize area among all competitors with boundary given by a prescribed Jordan curve $\Gamma \subset \mathbb{R}^3$. Douglas [4] and Rado [17] gave a satisfactory answer to this question by proving that if Γ is a rectifiable Jordan curve, then there exists a conformal harmonic map (hence a possibly branched minimal immersion) from the open unit disk $\mathbb{D}$ into $\mathbb{R}^3$ that extends continuously to the closed disk, whose restriction to $\partial\mathbb{D}$ parametrizes monotonically Γ, and whose area is the smallest possible among all disks with the same boundary. We will refer to such a disk M as a *Douglas-Rado solution* of the Plateau problem for Γ.

Surprisingly, the natural question of deciding if a Douglas-Rado solution M must be actually an immersion up to the boundary (assuming that Γ is regular) remains nowadays open. The number of branch points of M is necessarily finite (Heinz and Hildebrandt [11]). Osserman [15], Gulliver [8] and Alt [1, 2] used a cut-and-paste argument to prove that M is free of *interior* branch points. Gulliver and Lesley [9] excluded boundary branch points when Γ is real-analytic. It is also remarkable that these boundary regularity questions remain open when dealing with solutions of the Plateau problem with *prescribed* topology: any area-minimizing integral current with boundary Γ is a smooth regular surface up to the boundary (Fleming [7], Hardt and Simon [10]).

White [20] contrived a blow-up argument on the scale of curvature for ruling out boundary branch points of the Douglas-Rado solution, that would work provided that the halfplane were the only complete area minimizing surface in $\mathbb{R}^3$ with either finite total curvature or quadratic area growth, and a straight line as boundary (see Section 2 for details on this argument). But as White showed in [20], this uniqueness does not hold since half of the Enneper minimal surface (see Figure 1 left) also satisfies the same properties. He then conjectured that

> *Half of the Enneper surface is the unique properly embedded nonflat orientable area minimizing surface in $\mathbb{R}^3$ with straight line boundary and quadratic area growth.*

Embeddedness cannot be removed from the above conjecture, as shows the counterexample given by the Weierstrass data $g(z) = \frac{z^2-1}{z}$, $dh = iz(z^2-1)dz$ defined on the upper halfplane, see Figure 1 right (here g and dh denote respectively the complex Gauss map and height differential). White's

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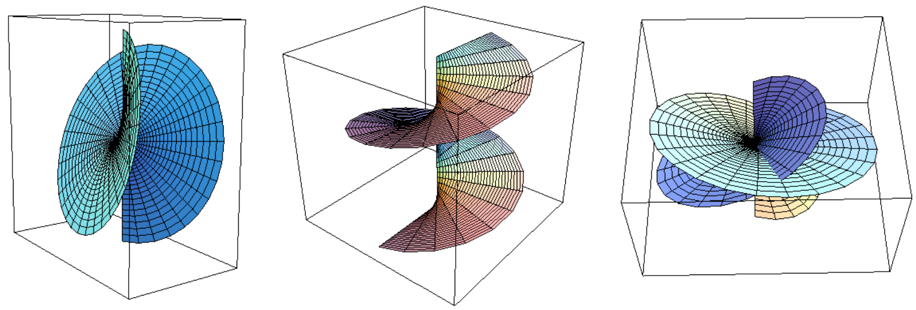

FIGURE 1. Half of Enneper (left), helicoid (center) and an immersed example (right).

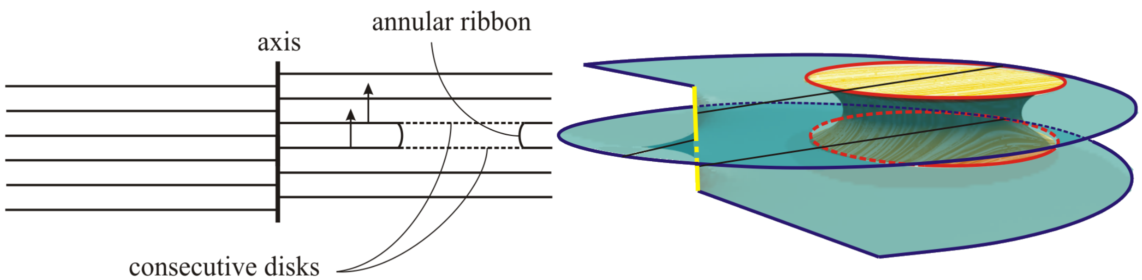

FIGURE 2. Half of the helicoid does not minimize area mod 2.

Conjecture neither holds true if we drop the quadratic area growth hypothesis, as demonstrates half of the helicoid (its boundary line being the axis of the helicoid), see Figure 1 center.

In a previous work [19], White had proven a partial version of the above conjecture, assuming additionally that the surface lies in one of the four regions in which two nonparallel planes Π_1, Π_2 divide the space, with boundary line being $\Pi_1 \cap \Pi_2$. In this paper we will give a general positive answer to White's conjecture, as a consequence of the following result.

Theorem 1.1. *If $M \subset \mathbb{R}^3$ is a properly embedded orientable stable minimal surface whose boundary is a straight line and M has quadratic ara growth, then M is a half-plane or half of the Enneper surface.*

So far, we have used the term "area minimizing" for orientable surfaces M by comparison with *orientable* competitors. This occurs, for instance, when M minimizes area as an integral current. When a possibly nonorientable immersed surface M *minimizes area mod 2,* then any compact subdomain in M minimizes area among all surfaces with its same boundary (independently of the orientability character). Any orientable surface which minimizes area mod 2 also minimizes area as integral current. Thus an interesting related question to White's conjecture is the following.

What surfaces with straight line boundary in $\mathbb{R}^3$ minimize area mod 2?

Neither half of the helicoid nor half of Enneper surface minimize area mod 2, as one can deduce from considering the nonorientable piecewise smooth surface obtained by substitution in each of these two surfaces of two appropriate consecutive almost flat disks by an annular ribbon (see Figure 2 for the helicoidal case; the construction is similar in the case of half of Enneper). The following result is the answer to our last question in the orientable setting.

Corollary 1.2. *If $M \subset \mathbb{R}^3$ is a properly embedded orientable surface which minimizes area mod 2 and whose boundary is a straight line, then M is a half-plane.*

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2. Preliminaries.

We start this section by sketching White's blow-up argument [20] for ruling out boundary branch points, that we mentioned in the introduction. Consider a smooth Jordan curve $\Gamma \subset \mathbb{R}^3$ and a Douglas-Rado solution M of the Plateau problem for Γ, with a boundary branch point $p \in \Gamma$. By pushing Γ along M towards the interior locally around a point in Γ different from p, one can assume that M is the only Douglas-Rado solution of the Plateau problem for Γ (Morgan [13]). Let $\{\Gamma_n\}_n$ be a sequence of real analytic Jordan curves converging to Γ, and M_n a Douglas-Rado solution of the Plateau problem for Γ_n, $n \in \mathbb{N}$. The sequence of Gaussian curvatures of the M_n cannot be uniformly bounded (otherwise after passing to a subsequence the M_n will converge to a Douglas-Rado solution of the Plateau problem for Γ, which can only be M; but the analyticity of Γ_n implies that M_n is free of branch points, hence the same holds for M, a contradiction). Rescaling the M_n by curvature, one produces a limit nonflat stable minimal disk $M_\infty \subset \mathbb{R}^3$ which is complete up to its boundary. Since M_∞ is not a plane, it must have nonvoid boundary ∂M_∞ (Do Carmo and Peng [3], Fischer-Colbrie and Schoen [6]), which arises from blowing-up portions of the Γ_n, hence ∂M_∞ must be a straight line. As the area of M_n is bounded independently of n, an application of the monotonicity formula for minimal surfaces insures that M_∞ has quadratic area growth (in fact, M_∞ has finite total curvature by the Gauss-Bonnet formula). If the unique such surface M_∞ were a halfplane (which is not true, and one of the goals of this paper is to show that the only possible counterexample is half of Enneper), we would have a contradiction to the existence of the branch point $p \in \Gamma$. Hopefully, the knowledge of that M_∞ must be half of Enneper could be useful for excluding boundary branch points for M.

Next we fix some notation to be used throughout the paper. Given a proper orientable minimal surface $M \subset \mathbb{R}^3$ with straight line boundary, we will assume $\partial M = \{(0,0,x_3) \mid x_3 \in \mathbb{R}\}$ after a rotation. By reflecting M across its boundary, we define a smooth proper minimal surface without boundary

$$M' = M \cup \mathrm{Rot}_{180^\circ}(M).$$

Note that if M is assumed to be embedded or stable, then M' needs not to inherit the same property. The Gauss map of M will be denoted by N with third component $N_3 = \langle N, (0,0,1)\rangle$, and the Gaussian curvature of the surface by K.

3. A general condition for M to be a disk.

Let $\pi \colon \mathbb{R}^3 \to \mathbb{R}^2 \times \{0\}$ be the orthogonal projection from space to the (x_1, x_2)-plane, and $\mathbf{p} \colon \mathcal{P} = \mathbb{R}^+ \times \mathbb{R} \to \{x_3 = 0\} - \{\vec{0}\}$ the universal covering of the punctured plane in polar coordinates (r, θ), so that $x_1 + ix_2 = re^{i\theta} = \mathbf{p}(r, \theta)$. Given an open subset $\Omega \subset \mathcal{P}$, a surface $\Sigma \subset \mathbb{R}^3 - \pi^{-1}(0)$ is said to be a *multigraph* over $\mathbf{p}(\Omega)$ if it is the graph of a function u defined on Ω. Clearly, multigraphs need not to be embedded, unless the so called *separation function* $(r, \theta) \in \Omega \mapsto u(r, \theta + 2\pi) - u(r, \theta)$ has no zeros. A multigraph is said to be *minimal* if its mean curvature vanishes identically, i.e. u satisfies the minimal surface equation. It is not difficult to check that half of the Enneper surface is actually a graph over an open sector of amplitude 2π, i.e. can be written as the graph of a function defined on $(0,\infty) \times (0, 2\pi) \subset \mathcal{P}$ (recall that we have normalized our surfaces to have boundary the x_3-axis), and half of the helicoid is an ∞-valued multigraph over $\mathcal{P}$. Clearly, the Gauss map along a multigraph is never horizontal. In the case of a straight line boundary, this geometric condition is enough to conclude the graphical property, as explained in Proposition 3.3 below. To prove that proposition, we shall need some preliminar results.

Let $U \subset \mathbb{R}^2$ be an open set of the plane. A point $x \in \partial U$ is called a *point of concavity of* U if there exists a circle C (called a *circle of inner contact at* x) through x and a disk $D(x,\varepsilon)$ whose intersection with the exterior of C lies in U. Given $x, y \in \mathbb{R}^2$, we will denote by $[x, y]$ the closed segment whose extrema are x and y, and use the notation $[x, y[,]x, y],]x, y[$ for the corresponding segments where one of two of their extrema have been removed.

Lemma 3.1. *Let $U \subset \mathbb{R}^2$ be an open set. Given $x_0 \in U$, we let $A(x_0) = \{x \in \partial U \mid [x_0, x[\subset U\}$.*

1. *If U is convex, then $A(x_0) = \partial U$.*
2. *If U is starshaped with respect to x_0 and $x \in \partial U$ is a point of concavity of U, then $x \in A(x_0)$.*
3. *U is convex if and only if no points in ∂U are points of concavity of U.*

Proof. Assume U is convex, and take a point $x \in \partial U - A(x_0)$. Thus, we can find a point $y \in [x_0, x[- U$. Since U is convex, the same holds for its closure $\overline{U}$, and thus $y \in \partial U$ (because $x_0, x \in \overline{U}$). Applying the Hahn-Banach theorem to the closed convex set $\overline{U}$, there exists a *supporting line* $l \subset \mathbb{R}^2$ passing through y, i.e. a straight line such that $y \in l$ and $\overline{U}$ is contained in one of the two closed halfplanes in which l divides $\mathbb{R}^2$. If l were transversal to $[x_0, x]$ then l would leave x_0, x in opposite open halfplanes, which contradicts that l is a supporting line. Thus $[x_0, x] \subset l$, which contradicts that x_0 belongs to the open set U. This proves item (1).

Now assume that U is starshaped with respect to x_0 and $x \in \partial U - A(x_0)$ is a point of concavity of U. As before, there exists $y \in [x_0, x[- U$. Since U is starshaped with respect to x_0, the same holds for $\overline{U}$, and thus, $[x_0, x] \subset \overline{U}$, from where we have $[y, x] \subset \partial U$. As x is a point of concavity of U, there exists a circle of inner contact C at x. Since U is starshaped with respect to x_0, U cannot contain points z such that $x \in [x_0, z)$. Thus, C intersects $[x_0, x]$ transversally at x and points of $[y, x)$ sufficiently close to x belong to U. This contradicts that $[y, x] \subset \partial U$, which proves item (2). Finally, item (3) is Lemma 10.4 in Osserman [16]. □

The next result is based on Finn [5], see also Lemma 10.3 in [16]. Consider a vertical line $l \subset \mathbb{R}^3$. Given a point $y \in l$ and a positive number t, let $\mathcal{C}(y, t)$ be the vertical catenoid with axis l and waist circle centered at y with radius t. For $y \in l$ fixed, let $\Theta(y)$ be the shallowest vertical double cone with axis l and vertex at y such that $\cup_{t>0}\mathcal{C}(y, t)$ is contained in the nonsimply connected component of $\mathbb{R}^3 - \Theta(y)$.

Lemma 3.2. *Let $0 < R_1 < R_2$. Suppose M is a properly immersed minimal surface in $\mathbb{R}^3$ contained in the closed region between the cylinders C_{R_1}, C_{R_2} of axis l and respective radii R_1, R_2. Then:*

1. *If there exist points $y_1, y_2 \in l$ with $x_3(y_1) < x_3(y_2)$, such that $\partial M - C_{R_1}$ lies strictly above $\Theta(y_1)$ and below $\Theta(y_2)$, then M is contained in the horizontal slab $\{x_3(y_1) \leq x_3 \leq x_3(y_2)\}$.*
2. *$\partial M - C_{R_1}$ is nonempty.*

Proof. To see the first item, choose $t_0 > R_2$. Then M lies in the simply connected component of $\mathbb{R}^3 - \mathcal{C}(y_i, t_1)$, $i = 1, 2$. Now shrink the radius t of the waist circle of $\mathcal{C}(y_i, t)$ in the range $t \in (R_1, t_0]$. Since $\partial M - C_{R_1}$ lies strictly above $\Theta(y_1)$ and below $\Theta(y_2)$, then ∂M is disjoint from $\mathcal{C}(y_i, t)$ for any $t \in (R_1, t_0]$. By the maximum principle, no first point of contact can occur between M and $\mathcal{C}(y_i, t)$ for each $t \in (R_1, t_0]$. Since M lies outside C_{R_1} and the waist circles of $\mathcal{C}(y_1, R_1), \mathcal{C}(y_2, R_1)$ are contained in C_{R_1}, we deduce that M lies between the heights of such waist circles. This proves item (1). Item (2) follows by contradiction, assuming $\partial M - C_{R_1} = \emptyset$ and applying item (1) to two pairs of points $\{y_1, y_2\}, \{y_1', y_2'\} \subset l$ such that $x_3(y_1) < x_3(y_2) < x_3(y_1') < x_3(y_2')$. □

Proposition 3.3. *Let $M \subset \mathbb{R}^3$ be a properly embedded minimal surface with boundary the x_3-axis. If $N_3^{-1}(0) = \partial M$, then there exists a halfstrip $\Omega = (0, \infty) \times (\theta_1, \theta_2) \subset \mathcal{P}$ with $-\infty \leq \theta_1 < \theta_2 \leq \infty$*

such that $\pi(M-\partial M)=\mathbf{p}(\Omega)$ *and* $M-\partial M$ *is a minimal multigraph over* $\mathbf{p}(\Omega)$*. In particular,* M *is topologically a disk. Furthermore if* θ_i *is finite for some* $i=1,2$*, then the third coordinate function of* M *diverges to* ∞ *(or to* $-\infty$*) along any point of the halfline* $\mathbf{p}((0,\infty)\times\{\theta_i\})$*.*

Proof. Fix a point $p_0\in M-\partial M$ and an open halfplane $H\subset\{x_3=0\}$ with $\pi(p_0)\in H$ and $\vec{0}\in\partial H$. Let $\mathcal{V}$ be the collection of open subsets $V\subset H$ that satisfy:

- V is starshaped with respect to $\pi(p_0)$.
- There exists a solution $v:V\to\mathbb{R}$ of the minimal surface equation whose graph G_v passes through p_0 and is contained in M.

Then $\mathcal{V}\neq\emptyset$ since $N_3(p_0)\neq 0$. Let U be the union of all elements in $\mathcal{V}$. Note that U also belongs to $\mathcal{V}$, so we have a minimal graph $u:U\to\mathbb{R}$ such that $p_0\in G_u\subset M-\partial M$. Let $A(\pi(p_0))=\{x\in\partial U\mid [\pi(p_0),x[\subset U\}$.

Assertion 1. *For any point* $x\in A(\pi(p_0))-\partial H$ *and sequence* $\{x_n\}_n\subset U$ *with* $x_n\to x$*, it holds* $|u(x_n)|\to\infty$ *as* $n\to\infty$*.*
Proof of Assertion 1. Otherwise, after extracting a subsequence $u(x_n)$ converges to a number $a\in\mathbb{R}$. Since M is proper, the corresponding points $(x_n,u(x_n))\in M$ converge as $n\to\infty$ to a point $(x,a)\in M$, which cannot lie at ∂M because $x\neq\vec{0}$. As $N_3(x,a)\neq 0$, we can write M locally around (x,a) as the graph of a function defined on a planar disk $D_1=D(x,\delta_1)\subset H$ with center x and radius $\delta_1>0$. Since $x\in A(\pi(p_0))$, for any $t\in[0,1)$ there exists $\delta_t>0$ such that the disk $D_t=D((1-t)\pi(p_0)+tx,\delta_t)$ is contained in U. By compactness, $[\pi(p_0),x]$ is covered by a finite number of disks D_{t_i}, from where one can construct an open set U_x with $\pi(p_0),x\in U_x$ such that U_x is starshaped with respect to $\pi(p_0)$ and where u can be extended. Clearly $U\cup U_x$ contradicts the maximality of U and Assertion 1 is proved.

Assertion 2. *U is convex (in particular, $A(\pi(p_0))=\partial U$ by item (1) of Lemma 3.1).*
Proof of Assertion 2. By item (3) of Lemma 3.1, it suffices to check that ∂U does not contain points of concavity of U. Fix a point $x\in\partial U$. Since $U\subset H$, a simple consequence of the definition of point of concavity shows that x cannot be a point of concavity of U provided that $x\in\partial H$. Now assume $x\in\partial U-\partial H$ is a point of concavity of U. Since U is starshaped with respect to $\pi(p_0)$, item (2) in Lemma 3.1 implies that $x\in A(\pi(p_0))$. As x is a point of concavity of U, there exists a circle of inner contact C at x. Thus for some $\varepsilon>0$, the intersection Δ of $D(x,\varepsilon)$ with the exterior of C is an open set contained in U. Exchanging C by an inner contact circle at x of slightly bigger radius, we can assume that u is defined on $\overline{\Delta}-\{x\}$ and takes finite values along $\partial\Delta-\{x\}$. By item (1) of Lemma 3.2, the fact that $u|_{\partial\Delta-C}$ is bounded implies $u|_\Delta$ is also bounded, which is impossible by Assertion 1. This proves Assertion 2.

Assertion 3. *Each component of* $\partial U-\partial H$ *is a straight line segment (possibly of infinite length).*
Proof of Assertion 3. Since U is convex, each component of ∂U is a continuous curve. Pick two distinct points x,y in a component Γ of $\partial U-\partial H$ and let γ be subarc of Γ with extrema x,y (note that Γ is a continuous curve because $\partial U-\partial H$ is obtained by removing a collection of straight line segments from the continuous curves in ∂U). Since U is convex, γ and $[x,y]$ form the boundary of a subdomain $D\subset U$. By Assertions 1 and 2, u tends to infinity at each point of γ, which contradicts Theorem 10.5 in [16]. For the sake of completeness, we shall give a short argument to find a contradiction. Pick a point $P\in D$ and let C be the circle passing through x,y,P. Then there exists a subdomain $D'\subset D$ lying outside C, bounded by portions of C and γ. Furthermore, the graph G of $u|_{D'}$ has noncompact boundary lying on the cylinder $C\times\mathbb{R}$, and G is contained in the solid region between $C\times\mathbb{R}$ and another cylinder $C'\times R$ with the same axis as $C\times\mathbb{R}$ and bigger radius. This situation contradicts item (2) of Lemma 3.2, hence Assertion 3 is proved.

ASSERTION 4. *If Γ is a component of $\partial U - \partial H$, then Γ cannot be a whole straight line.*
Proof of Assertion 4. Reasoning by contradiction, suppose Γ is a complete straight line (which must be parallel to ∂H). Then U is either an infinite strip or a halfplane strictly contained in H, and G_u is a graph over U, asymptotic to the vertical plane $\Gamma \times \mathbb{R}$ and contained in one of the halfspaces, W, bounded by $\Gamma \times \mathbb{R}$. Applying the proof of the Strong Halfspace Theorem (Hoffman and Meeks [12]) to G_u inside W (i.e. by considering halfcatenoids with waist circle inside W and parallel to $\Gamma \times \mathbb{R}$), we will find a contradiction.

ASSERTION 5. *Suppose $Q \in \partial H$ is the extremum of a halfline $r \subset \partial U - \partial H$. Then, $Q = \vec{0}$.*
Proof of Assertion 5. Assume $Q \neq \vec{0}$ and consider the open halfplane $H' \subset \{x_3 = 0\}$ with $\vec{0} \in \partial H'$, $\partial H'$ parallel to r and such that H' contains to r. Pick a point $p_1 \in G_u \cap \pi^{-1}(H')$ and repeat the arguments above exchanging p_0 by p_1 and H by H'. In this way we find a new maximal open domain U', which is convex, and a solution of the minimal surface equation $u' : U' \to \mathbb{R}$ whose graph $G_{u'}$ passes through p_1 and is contained in M. Clearly u coincides with u' in $U \cap U'$ (which contains $\pi(p_1)$). By Assertion 1 applied to U and H, u diverges to infinity along any sequence converging to a point of the interior of r, hence r is contained in a component Γ' of the boundary of U'. By Assertion 3 applied to U' and H', Γ' must be a whole straight line, which contradicts Assertion 4 thereby proving Assertion 5.

From the above arguments, we deduce that U must be an open sector contained in H, with $\vec{0}$ as vertex (U could coincide with H). We finish the proof of the proposition with the following description of the behavior of u along ∂U:

ASSERTION 6. *Let $r \subset \partial U - \{\vec{0}\}$ be a boundary halfline. Then one of the following possibilities hold:*

(1) *$r \subset \partial U - \partial H$. In this case, either u diverges to ∞ along any point of r, or it diverges to $-\infty$.*
(2) *$r \subset \partial U \cap \partial H$ and either u diverges to ∞ along any point of r, or it diverges to $-\infty$.*
(3) *$r \subset \partial U \cap \partial H$ and u extends smoothly along l. In this case, we can continue the description of M as a multigraph by rotating the halfplane H.*

Proof of Assertion 6. Firstly suppose $r \subset \partial U - \partial H$. Take a point $x \in r$ and a sequence $\{x_n\}_n \subset U$ converging to x as $n \to \infty$. By Assertion 1, we can assume $u(x_n) \to \infty$, and then item 1 of this Assertion 6 will be proved if we show that for any point $y \in r$ and sequence $\{y_n\}_n \subset U$ with $y_n \to y$, it holds $u(y_n) \to \infty$. Otherwise, we find y_n and y where this condition fails to hold. Using again Assertion 1, we have $u(y_n) \to -\infty$ as $n \to \infty$. Since U is convex, the segment $[x_n, y_n]$ is contained in U. By continuity, $[x_n, y_n]$ converges to a segment in r (possibly to a single point) and for all $n \in \mathbb{N}$ there exists $z_n \in]x_n, y_n[$ such that $u(z_n) \to 0$. Both facts together contradict Assertion 1.

Now assume $r \subset \partial U \cap \partial H$. If u is unbounded when evaluated on a sequence $x_n \in U$ that converges to a point in r, then after changing the halfplane H we can apply the arguments in the last paragraph to r, concluding that r diverges to the same signed infinity along all points in r. This concludes the proof of Assertion 6 and of Proposition 3.3. □

4. QUADRATIC AREA GROWTH AND LIMIT CONES AT INFINITY.

Along this section, we will let $\mathbb{B}(R) = \{x \in \mathbb{R}^3 \mid \|x\| < R\}$ and $\mathbb{S}^2(R) = \partial \mathbb{B}(R)$, for each $R > 0$. If $\Sigma \subset \mathbb{R}^3$ is a proper surface without boundary (we allow Σ to have a set of singularities with 2-dimensional Hausdorff measure zero), we can consider the functions

$$A_\Sigma(R) = \text{Area}(\Sigma \cap \mathbb{B}(R)), \qquad l_\Sigma(R) = \text{length}(\Sigma \cap \mathbb{S}^2(R)).$$

The classical monotonicity formula for minimal surfaces (see e.g. [14]) says that $R^{-2}A_\Sigma(R)$ is a monotonically nondecreasing function of R, which is constant if and only if Σ is a cone $C(\Delta) = \{tp \mid t \geq 0, p \in \Delta\}$ over a geodesic integral varifold $\Delta \subset \mathbb{S}^2(1)$, which consists of a finite configuration of geodesic arcs with positive integer multiplicities, joined by their common end points. Σ is said to have *quadratic area growth* if $R^{-2}A_\Sigma(R)$ is bounded from above.

Any properly immersed minimal surface $\Sigma \subset \mathbb{R}^3$ with finite total curvature has the area growth of k planes (here k is the number of ends of Σ counted with multiplicity) and so, Σ has quadratic area growth. The converse holds if we assume Σ has finite topology. Although this converse is well-known, we include a proof for the sake of completeness. Since Σ is assumed to have finite topology, it has a finite number of ends all being topological annuli. Clearly it suffices to check that if E is an annular end representative of Σ, then E has finite total curvature. Fix a point $p_0 \in \Sigma$ and consider the function $f(R) = \int_{E(R)} |K_\Sigma|$ that measures the absolute total curvature of the subannulus $E(R) = \{p \in E \mid d_\Sigma(p, p_0) < R\}$, where R is a large positive number and d_Σ denotes intrinsic distance on Σ. Assume R is a regular value of $p \mapsto d_\Sigma(p, p_0)$. Using the Gauss-Bonnet formula and the first variation of length,

$$f(R) = -\int_{E(R)} K = \int_{\partial E(R)} \kappa_g = \frac{d}{dR}\left(\text{length}(\partial E(R)\right) = \frac{d^2}{dR^2}\left(\text{Area}(E(R))\right),$$

where κ_g stands for the geodesic curvature of $\partial E(R)$. As $K \leq 0$ because Σ is minimal, we deduce that f is nondecreasing and $R \mapsto \text{Area}(E(R))$ is concave. As the intrinsic distance d_Σ dominates the Euclidean distance, the extrinsic triangle inequality implies $E(R) \subset \mathbb{B}(R + |x_0|)$. This inclusion together with the fact that Σ has quadratic area growth, imply that $\text{Area}(E(R)) \leq cR^2$ for a certain $c > 0$. Hence, f is bounded from above, i.e. E has finite total curvature.

A basic result in geometric measure theory asserts that for a properly immersed minimal surface $\Sigma \subset \mathbb{R}^3$ with quadratic area growth and $\partial\Sigma = \emptyset$, given a sequence of positive numbers $t_n \to 0$, the sequence of homothetic shrinkings $t_n\Sigma$ contains a subsequence that converges on compact subsets of $\mathbb{R}^3$ to a minimal cone $C(\Delta)$ over a finite collection $\Delta \subset \mathbb{S}^2(1)$ of geodesic arcs with positive integer multiplicities, joined by their common end points (it is unknown if such a limit minimal cone at infinity $C(\Delta)$ is independent of the sequence $\{t_n\}_n$).

Coming back to our case of a proper minimal surface $M \subset \mathbb{R}^3$ with boundary the x_3-axis, we will say that M has *quadratic area growth* if $R^{-2}A_M(R)$ is bounded from above, or equivalently, if the properly immersed minimal surface without boundary $M' = M \cup \text{Rot}_{180^\circ}(M)$ has quadratic area growth. Since half of Enneper has finite total curvature, it has quadratic area growth (the supremum of $R^{-2}A_M(R)$ is 3π). On the other hand, half of the helicoid has cubical area growth.

Remark 4.1. *There is a monotonicity formula for properly immersed minimal surfaces* $\Sigma \subset \mathbb{R}^3$ *with boundary, that asserts that* $R \mapsto R^{-2}\left(A_\Sigma(R) + A_{C^*(\partial\Sigma)}(R)\right)$ *is not decreasing, where* $C^*(\partial\Sigma) = \{tp \mid t \geq 1, p \in \partial\Sigma\}$. *Clearly this monotonicity coincides with the usual one when* $\partial\Sigma$ *is a straight line passing through the origin, but in this case it is easier to consider the classical monotonicity for the doubled surface obtained by reflection of* Σ *across its boundary.*

In our case of stable minimal surfaces with straight line boundary and quadratic area growth, we can improve the control on the limit tangent cones at infinity.

Lemma 4.2. *Let* $M \subset \mathbb{R}^3$ *be a properly immersed stable minimal surface with boundary the* x_3*-axis and quadratic area growth. Given a sequence of positive numbers* $t_n \searrow 0$, *there exists a subsequence (denoted in the same way) and a finite collection* $\Delta \subset \mathbb{S}^2(1)$ *of great halfcircles joining the north and south poles, each with finite multiplicity, such that* t_nM *converges on compact subsets of* $\mathbb{R}^3$ *to the minimal cone* $C(\Delta)$ *over* Δ.

Proof. Since $M' = M \cup \text{Rot}_{180^\circ}(M)$ has quadratic area growth, after extracting a subsequence the $t_n M'$ converge to the cone over a finite collection $\Delta' \subset \mathbb{S}^2(1)$ of geodesic arcs with finite multiplicities, joined by their common end points. Since the x_3-axis lies in M', it follows that $(0,0,\pm 1) \in \Delta'$. By construction, Δ' is of the form $\Delta' = \Delta \cup \text{Rot}_{180^\circ}(\Delta)$, so it only remains to check that if Δ is not smooth at a point p, then $p = (0,0,\pm 1)$. Reasoning by contradiction, assume $p \in \Delta - \{(0,0,\pm 1)\}$. Thus we can find a positive number $\varepsilon > 0$ and a sequence of points $p_n \in M$ such that

$$\lim_{n\to\infty} t_n p_n = p \qquad \text{and} \qquad x_3^2(p_n) > \varepsilon(x_1^2(p_n) + x_2^2(p_n)) \quad \forall n \in \mathbb{N}. \tag{1}$$

Since M is stable, Schoen's curvature estimate [18] gives a number $c > 0$ such that for all n,

$$|K|(p_n) \leq \frac{c}{d_M(p_n, \partial M)^2} \leq \frac{c}{x_1(p_n)^2 + x_2(p_n)^2},$$

where d_M denotes the intrinsic distance on M. Since $K(x_1^2 + x_2^2)$ is scaling invariant, we deduce that $|K_{t_n M}|(x_1^2 + x_2^2)$ is bounded at $t_n p_n \in t_n M$, where $K_{t_n M}$ stands for the Gauss curvature function of the shrunk surface $t_n M$. Since (1) implies that $x_1^2(t_n p_n) + x_2^2(t_n p_n)$ is bounded away from zero, it follows that $\{|K_{t_n M}|(t_n p_n)\}_n$ is bounded. As consequence, the cone $\mathcal{C}(\Delta)$ is smooth at p, so Δ is also smooth at p, a contradiction. Now the Lemma is proved. $\square$

Remark 4.3. When M is half of Enneper, the associated limit tangent cone $C(\Delta)$ at infinity for M does not depend on the sequence $t_n \searrow 0$, and Δ consists of half a vertical great circle with multiplicity one, and the complementary halfcircle with multiplicity two.

5. Proof of the main results.

Proposition 5.1. *Let $M \subset \mathbb{R}^3$ be a surface as in Lemma 4.2. Then, $N_3^{-1}(0) = \partial M$.*

Proof. Let $L = \Delta - 2K$ be the Jacobi operator on M and $Q(v,v) = \int_M (|\nabla v|^2 + 2Kv)$ the index form associated to L, which is defined on compactly supported piecewise functions on M that vanish along ∂M. The (non rigorous) idea behind the proof is to use the stability inequality with $u = |N_3|$, thus $0 \leq Q(u,u) = 0$, where the last equality holds by splitting M into $N_3^{-1}(\mathbb{R}^+)$ and $N_3^{-1}(\mathbb{R}^-)$ and applying that N_3 is Jacobi. Thus u minimizes Q, hence elliptic regularity gives that u is smooth in $M - \partial M$, which implies $N_3^{-1}(0) = \partial M$. This argument is not a rigorous proof since u is not compactly supported, which is the reason to truncate it with cut-off functions.

Given $R > 0$, consider the extrinsic cut-off function $\widetilde{\varphi} : \mathbb{R}^3 \to \mathbb{R}$ given by

$$\widetilde{\varphi}(x) = \begin{cases} 1 & |x| \leq R, \\ 2 - \frac{|x|}{R} & R < |x| < 2R, \\ 0 & |x| \geq R. \end{cases}$$

The restriction $\varphi = \widetilde{\varphi}|_M$ is piecewise smooth and has compact support on M, thus φu is a piecewise smooth compactly supported function on M that vanishes along ∂M. In particular, $Q(\varphi u, \varphi u)$

makes sense. Integrating by parts several times we have

$$\begin{aligned} Q(\varphi u, \varphi u) &= \int_M \left(u^2|\nabla\varphi|^2 + \varphi^2|\nabla u|^2 + 2\varphi u\langle\nabla\varphi, \nabla u\rangle + 2K\varphi^2u^2\right) \\ &= \int_M \left(u^2|\nabla\varphi|^2 + \frac{1}{2}\varphi^2\Delta(u^2) - \varphi^2 u\Delta u + \frac{1}{2}\langle\nabla(\varphi^2), \nabla(u^2)\rangle + 2K\varphi^2u^2\right) \\ &= \int_M \left(u^2|\nabla\varphi|^2 + \frac{1}{2}\mathrm{div}(\varphi^2\nabla(u^2)) - \varphi^2 u\Delta u + 2K\varphi^2u^2\right) \\ &\overset{(A)}{=} \int_M \left(u^2|\nabla\varphi|^2 - \varphi^2 u(\Delta u - 2Ku)\right) \\ &\overset{(B)}{=} \int_M u^2|\nabla\varphi|^2, \end{aligned} \tag{2}$$

where in (A) we have used that $u = 0$ in ∂M, and (B) follows from $Lu = 0$ in $M - N_3^{-1}(0)$. In particular, $Q(\varphi u, \varphi u) \geq 0$ (although we knew this inequality by stability). On the other hand, $|\nabla\varphi| \leq |\overline{\nabla}\widetilde{\varphi}|$ on M (here $\overline{\nabla}$ denotes Euclidean gradient), thus

$$\int_M u^2|\nabla\varphi|^2 \leq \frac{1}{R^2}\int_{M\cap[\mathbb{B}(2R)-\mathbb{B}(R)]} u^2(x) = \int_{\left(\frac{1}{R}M\right)\cap[\mathbb{B}(2)-\mathbb{B}(1)]} N_3^2. \tag{3}$$

Taking $R = R_n \to \infty$ and using that M is stable with quadratic area growth, Lemma 4.2 allows us to extract a subsequence so that the $\frac{1}{R_n}M$ converge to a minimal cone $C(\Delta)$ for which N_3 vanishes almost everywhere. Hence, (2) and (3) imply

$$\lim_{n\to\infty} Q(\varphi_n u, \varphi_n u) = 0. \tag{4}$$

We will finish the proof by checking that (4) cannot hold provided that N_3 vanishes at some point $p \in M - \partial M$. Let $D \subset M - \partial M$ be a small disk centered at p, and v_D a minimum of the functional

$$Q_D(v, v) = \int_D (|\nabla v|^2 + 2Kv^2)\, dA$$

defined on $A = \{v \in H^1(D) \mid v|_{\partial D} = u|_{\partial D}\}$ (here $H^1(D)$ stands for the usual Sobolev space of L^2-functions whose differentials are square integrable). By elliptic regularity, any minimum of Q_D is smooth on D. Since u fails to be smooth at p, we deduce that $\varepsilon = Q_D(u, u) - Q_D(v_D, v_D)$ is strictly positive. Finally, let $\widetilde{v}$ the piecewise smooth function on M defined by

$$\widetilde{v} = \begin{cases} v_D & \text{in } D \\ u & \text{in } M - D. \end{cases}$$

Since $\varphi_n\widetilde{v}$ is piecewise smooth with compact support, the stability of M insures that

$$0 \leq Q(\varphi_n\widetilde{v}, \varphi_n\widetilde{v}) = Q_{M-D}(\varphi_n u, \varphi_n u) + Q_D(\varphi_n v_D, \varphi_n v_D),$$

where Q_{M-D} is defined analogously as Q_D by integrating on $M - D$. Since $D \subset \mathbb{B}(R_n)$ for n large, we have $\varphi_n = 1$ in D, and so

$$0 \leq Q(\varphi_n u, \varphi_n u) - Q_D(u, u) + Q_D(v_D, v_D) = Q(\varphi_n u, \varphi_n u) - \varepsilon,$$

which contradicts (4). This proves the Proposition. □

Proof of Theorem 1.1.
Let $M \subset \mathbb{R}^3$ be a properly embedded nonflat orientable stable minimal surface with quadratic area growth and the x_3-axis as boundary. By Propositions 5.1 and 3.3, $N_3^{-1}(0) = \partial M$ and M is topologically a disk. Thus the reflected surface $M' = M \cup \mathrm{Rot}_{180^\circ}(M)$ is a once punctured sphere.

Since M' has quadratic area growth and finite topology, it has finite total curvature. To finish the proof, it suffices to check that the total curvature of M' is -4π (see e.g. Theorem 9.4 in [16]).

Since M' has finite total curvature, its Gauss map extends as a meromorphic map to the sphere obtained after attaching to M' its end, which we label by q. For the remainder of this proof, we shall assume the limiting normal vector of M' at q is $(0,0,1)$ (so the straight line boundary of M is now horizontal, say at height zero). Note that the extended Gauss map N of M' is unbranched at q (otherwise the set $N^{-1}(N(\partial M))$ consists locally around q of a equiangular system with more than curve, which contradicts Proposition 5.1). This fact allows us to parametrize M' locally around q by its Gauss map, thus we can find a conformal coordinate w defined on $\{|w|<\varepsilon\}$ so that $w=0$ corresponds to q and the (stereographically projected) Gauss map of M' writes as $g(w)=\frac{1}{w}$. With respect to this local coordinate, the height differential of M' can be expressed as

$$dh = \left(\frac{a_{-k}}{w^k} + \frac{a_{-k+1}}{w^{k-1}} + \ldots + a_0 + \mathcal{O}(w)\right) dw, \qquad |w|<\varepsilon,$$

for certain coefficients $a_{-k},\ldots,a_0 \in \mathbb{C}$ with $a_{-k}\neq 0$ and a holomorphic function $\mathcal{O}(w)$ with $\mathcal{O}(0)=0$. A standard computation using the Weierstrass pair (g,dh) shows that the asymptotics of M' around q are $X=(x_1+ix_2,x_3)\in\mathbb{C}\times\mathbb{R}\equiv\mathbb{R}^3$ where

$$(x_1+ix_2,x_3)(w) \sim \left(\frac{a_{-k}}{2kw^k}, -\frac{1}{k-1}\Re\left(\frac{a_{-k}}{w^{k-1}}\right)\right) + \text{lower order terms}, \quad 0<|w|<\varepsilon.$$

Therefore for $\rho\in(0,\varepsilon)$ very small, the closed planar curve $\theta\in[0,2\pi]\mapsto(x_1+ix_2)(\rho e^{i\theta})$ winds k times around the origin, while the height coordinate $\theta\in[0,2\pi]\mapsto x_3(\rho e^{i\theta})$ vanishes $2(k-1)$ times (recall that for $\theta=0,\pi$, the point $X(\rho e^{i\theta})$ lies on the straight line boundary of M, which is at height zero). Using that the curve $\theta\in[0,\pi]\mapsto X(\rho e^{i\theta})$ is embedded, it is not hard to check that $k\leq 3$. On the other hand, since both the period and flux vectors at q vanish (because M' is one-ended), we deduce that $\text{Res}_0(\frac{1}{2}(\frac{1}{g}-g)dh)=\text{Res}_0(\frac{i}{2}(\frac{1}{g}+g)dh)=\text{Res}_0(dh)=0$, which implies $a_{-2}=a_{-1}=a_0=0$ and so, $k=3$. In particular, dh has a triple pole at the end q. Since $M'\cup\{q\}$ is a sphere and dh has no other poles than q, dh must have exactly one single zero at a point $p\in M'$ (so p lies on ∂M). By regularity of the induced metric, g has a pole or zero at p, and no other poles or zeros in M'. Hence, the degree of g is one, and the proof is finished. □

As finite total curvature implies quadratic area growth, we immediately have the following statement.

Corollary 5.2. *Let $M\subset\mathbb{R}^3$ be a complete embedded orientable minimal surface with straight line boundary and finite total curvature. Then, M is a half plane or half Enneper.*

Remark 5.3. Proposition 5.1 admits a simpler proof if we assume M has finite total curvature, since in this case N_3 extends to the compactification of M and one can apply a standard nodal argument together with the stability of M.

We finish the article by proving Corollary 1.2. Let $M\subset\mathbb{R}^3$ be a properly embedded orientable surface which minimizes area mod 2 among all surfaces with boundary the x_3-axis. We claim that M has quadratic area growth. To see this, fix $R>0$ such that M cuts transversely to $\mathbb{S}^2(R)$. Let $\Gamma\subset\mathbb{S}^2(R)$ be a great halfcircle joining the north and the south pole of $\mathbb{S}^2(R)$. Let D be the closed planar halfdisk bounded by Γ and the portion of the x_3-axis between heights $-R$ and R. Since M is proper and transverse to $\mathbb{S}^2(R)$, the set $[M\cap\mathbb{S}^2(R)]\cup\Gamma$ divides $\mathbb{S}^2(R)$ into a finite number of components. It is possible to choose some of these components, denoted by $A_1,\ldots,A_k$, such that $\widetilde{M}=D\cup[M-\mathbb{B}(R)]\cup A_1\cup\ldots\cup A_k$ is a piecewise smooth (possibly nonorientable) surface. Since $\partial\widetilde{M}$ equals the x_3-axis and $\widetilde{M}$ coincides with M outside $\mathbb{B}(R)$, the mod 2 minimizing property of

M implies

$$\text{Area}(M \cap \mathbb{B}(R)) \leq \text{Area}(\widetilde{M} \cap \mathbb{B}(R)) \leq \frac{\pi}{2}R^2 + 4\pi R^2.$$

Hence, M has quadratic area growth. Since M is orientable and stable with quadratic area growth, Theorem 1.1 implies M is a half-plane or half Enneper. But this last surface does not minimize area mod 2 (as explained in the introduction), and the proof of Corollary 1.2 is complete.

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