The geometry of minimal surfaces of finite genus II; nonexistence of one limit end examples.

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Abstract

We demonstrate that a properly embedded minimal surface in $\mathbb{R}^3$ with finite genus cannot have one limit end.

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1 Introduction.

This manuscript is the second in a series of papers whose goal is to describe the topology, geometry, asymptotic behavior and conformal structure of properly embedded minimal surfaces M in $\mathbb{R}^3$ with finite genus [23, 24, 25, 26].

By the recent results of Collin [9] and Meeks-Rosenberg [27], the asymptotic behavior of an M with finite topology can be characterized as follows: each annular end of M is asymptotic to an end of a plane, catenoid or helicoid, with the helicoid-type end occurring only in the case M has one end and is not a plane. These results together with theorems in [18, 27, 32] imply that the catenoid is the only properly embedded minimal surface of

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finite topology with either two ends or with genus zero and the plane and the helicoid are the only simply connected properly embedded minimal surfaces in $\mathbb{R}^3$. The main goal of this paper is to prove that a properly embedded minimal surface M in $\mathbb{R}^3$ of finite genus cannot have one limit end. This just means that M cannot be homeomorphic to a compact surface $\overline{M}$ punctured in closed countable set $\mathcal{E} = \{p_\infty, p_1, \ldots, p_n, \ldots\}$ of points with exactly one limit point p_∞ in $\overline{M}$. This result and theorems in [23] imply the following descriptive theorem.

Theorem 1 *If M is a connected properly embedded minimal surface in $\mathbb{R}^3$ with finite genus, then one of the following holds:*

1. *M is a plane;*
2. *M has one end and is asymptotic to the end of a helicoid;*
3. *M has a finite number of ends greater than one, has finite total curvature and each end is asymptotic to a plane or to the end of a catenoid;*
4. *M has two limit ends.*

Furthermore, M has bounded Gaussian curvature and is conformally diffeomorphic to a compact Riemann surface punctured in a countable closed subset which has exactly two limit points if the subset is infinite. In particular, M is recurrent for Brownian motion.

The main theorem in [10] states that a limit end of a properly embedded minimal surface M with horizontal limit tangent plane must be a top or bottom end and hence M can have at most two limit ends (see Section 2 for definitions). If M is a properly embedded minimal surface in $\mathbb{R}^3$ with finite topology, then Collin's theorem [9] and the main theorem in [27] imply that parts 1, 2 or 3 of Theorem 1 hold. Furthermore, when M has finite topology the theorems in [9, 27] show that M is conformally diffeomorphic to a compact Riemann surface $\overline{M}$ punctured in a finite number of points and M can be defined analytically in terms of meromorphic data on $\overline{M}$. If M has finite genus and two limit ends, then results in our previous paper [23] show that M has bounded curvature and that M is conformally a compact Riemann surface punctured in a countable closed subset with two limit points. Thus, Theorem 1 will follow once we prove that a properly embedded minimal surface in $\mathbb{R}^3$ cannot have finite genus and one limit end; this result is Theorem 5 in Section 5.

We conjecture that the plane, the catenoid, the helicoid and a one-parameter family $\mathcal{R}_t,\ t \in \mathbb{R}^+$, of periodic examples with two limit ends defined by Riemann [30] are the only properly embedded minimal surfaces in $\mathbb{R}^3$ of genus zero. Note that these classical surfaces are foliated by circles and lines in parallel planes. We further conjecture that if M is a properly embedded minimal surface of finite genus, has horizontal tangent plane at infinity and an infinite number of ends, then the top and bottom limit ends of M are in a natural sense asymptotic to the limit ends of one of the Riemann examples as the third coordinate diverges to infinity (see Section 2 for definitions).

The proof of Theorem 5 in Section 5 depends on several important recent advances to the classical theory of minimal surfaces. For example, we have already mentioned some of these results that appear in [9, 10, 23, 27]. Our proof relies on a series of deep papers [3, 4, 5, 6, 7, 8] by Colding and Minicozzi on the structure of embedded minimal planar domains in $\mathbb{R}^3$. Also see [2] and [22] for detailed surveys of the relevant theorems. Specifically we use the results of Colding and Minicozzi to prove that for any properly embedded minimal surface M in $\mathbb{R}^3$ with finite genus and one limit end, any sequence of homothetic scalings $M_n = \lambda_n M,\ \lambda_n \to 0$, has a subsequence converging $C^\alpha, 0 < \alpha < 1$, to a limit minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ whose leaves are smooth outside the origin. This regularity result implies that the curvature of M decays quadratically in terms of the radial function, which in turn is used to prove that such a surface cannot exist. Theorem 1 can be viewed as a geometric refinement of the local results of Colding and Minicozzi, after applying a standard blow-up argument at points of large curvature on an embedded minimal surface. In [24] we will apply Theorem 1 to obtain a bound on the number of ends and on the index of stability for a properly embedded minimal surface in $\mathbb{R}^3$ with finite topology and at least two ends. This bound only depends on the genus of the surface.

2 Preliminaries.

In this section we recall some of the basic definitions and theorems for properly embedded minimal surfaces M in $\mathbb{R}^3$ that will be essential in the proof of our main theorem. First recall the definition of the limit tangent plane at infinity for M. From the Weierstrass representation for minimal surfaces, one knows that the finite collection of ends of a complete embedded noncompact minimal surface $\Sigma \subset \mathbb{R}^3$ of finite total curvature and compact boundary is asymptotic to a finite collection of pairwise disjoint ends of planes and catenoids, each of which has a well-defined unit normal at infinity. The embeddedness of this collection of ends of planes and catenoids forces the limiting normals to the ends of Σ to be parallel. One defines *the limit tangent* plane of Σ to be the plane passing through the origin and orthogonal to the normals of Σ at infinity. Suppose that Σ is contained in a complement of M. One defines a limit tangent plane for M to be the limit tangent plane of Σ. In [1], it is shown that if M has at least two ends, then M has a unique limit tangent plane which we call *the limit tangent plane at infinity* for M. We say that the limit tangent plane at infinity is *horizontal* if it is the (x_1, x_2)-plane.

The main result in [14] is that if M has more than one end and horizontal limit tangent plane at infinity, then the ends of M can be ordered linearly by their "relative heights" over the (x_1, x_2)-plane and this ordering of the ends of M is a topological property of M, in the sense that if M is ambiently isotopic to another minimal surface M' with horizontal limit tangent plane at infinity, then the associated ordering of the ends of M' either agrees with or is opposite to the ordering coming from M.

Unless otherwise stated, we will assume that the limit tangent plane at infinity of M is horizontal, and so M is equipped with a particular ordering on its set of ends $\mathcal{E}(M)$. For any connected noncompact manifold M, its set of ends $\mathcal{E}(M)$ has a natural topology which makes it into a totally disconnected compact Hausdorff space that embeds topologically in $[0, 1]$. The limit points of $\mathcal{E}(M)$ are called *limit ends* of M. In the case that $M \subset \mathbb{R}^3$ is a properly embedded minimal surface with more than one end, the natural embedding of $\mathcal{E}(M)$ into $[0, 1]$ also preserves the geometric ordering of ends by relative heights. Hence, there exist unique maximal and minimal elements of $\mathcal{E}(M)$ for this ordering. The maximal element is called the *top* end of M. The minimal element is called the *bottom* end of M. Otherwise, the end is called a *middle* end of M.

The main theorem in [10] is that a limit end of M must be a top or bottom end of M. Hence, M can have at most two limit ends. The following related theorem appears in [23].

Theorem 2 *Suppose M is a properly embedded minimal surface in $\mathbb{R}^3$ with finite genus and horizontal limit tangent plane at infinity. If M has exactly one limit end e_∞, which is its top end, then each nonlimit end $e_n \in \mathcal{E}(M) = \{e_1, e_2, \ldots, e_\infty\}$ is asymptotic to a graphical annular end E_n of a vertical catenoid with negative logarithmic growth a_n satisfying* $a_1 \leq \ldots \leq a_n \leq a_{n+1} \leq \ldots < 0$.

3 Local simple connectivity of blow-downs.

In the proof by Meeks-Rosenberg [27] that the helicoid is the only properly embedded simply connected nonflat minimal surface in $\mathbb{R}^3$, it was essential to show that under a sequence of homothetic shrinkings of the surface, a subsequence converges to a minimal foliation $\mathcal{L}$ of $\mathbb{R}^3$ by parallel planes with singular set of convergence consisting of a connected Lipschitz curve $S(\mathcal{L})$ which intersects each planar leaf of $\mathcal{L}$ in a single point. The existence of the foliation $\mathcal{L}$ and singular curve $S(\mathcal{L})$ follows from Theorem 0.1 in [6].

Let $O \subset \mathbb{R}^3$ be an open set. A sequence $\{M(n) \subset O\}_{n\in\mathbb{N}}$ of minimal surfaces is said to be *locally simply connected* if for each point in O, there exists a ball $B \subset O$ centered at that point such that for n large, every component of $M(n) \cap B$ is a disk with boundary in the boundary of B. In the case $O = \mathbb{R}^3$, we say that a sequence $\{M(n)\}_n$ is *uniformly locally simply connected* (ULSC) if there exists $r > 0$ such that for each $p \in \mathbb{R}^3$ and for n large, the ball centered at p with radius r intersects $M(n)$ in components which are disks with boundary lying in the boundary of that ball.

The following result will be used in our proof of the main theorem of this section, which is Theorem 4. We will let $B(r)$ denote the ball of radius r centered at the origin, and K_Σ will stand for the Gauss curvature function of a surface $\Sigma \subset \mathbb{R}^3$.

Theorem 3 (Colding, Minicozzi [7]) *Let $M_n \subset B(R_n)$ be a* ULSC *sequence of embedded minimal planar domains such that $\partial M_n \subset \partial B(R_n)$, $R_n \to \infty$ and $M_n \cap B(2)$ contains a component which is not a disk for any n. If $\sup |K_{M_n \cap B(1)}| \to \infty$, then there exists a subsequence of the M_n (denoted in the same way) and two Lipschitz curves $S_1, S_2 \colon \mathbb{R} \to \mathbb{R}^3$ such that after a rotation of $\mathbb{R}^3$:*

1. *$x_3(S_k(t)) = t$ for all $t \in \mathbb{R}$, $k = 1, 2$.*

2. *Each M_n is horizontally locally graphical away from $S_1 \cup S_2$.*

3. *For each $\alpha \in (0, 1)$, $M_n - (S_1 \cup S_2)$ converges in the C^α-topology to the foliation $\mathcal{L}$ of $\mathbb{R}^3$ by horizontal planes.*

4. *$\sup |K_{M_n \cap B(S_k(t),r)}| \to \infty$ as $n \to \infty$, for all $t \in \mathbb{R}$, $r > 0$ and $k = 1, 2$.*

A sequence of possibly disconnected disjoint minimal graphs over the unit disk with bounded gradient and with nonempty limit set has a subsequence that converges to a minimal lamination of the open cylinder over that disk, whose leaves are graphical with the same gradient estimate. This fact, together with a standard diagonal argument has as a consequence the following lemma (see Theorem 4.39 in [29] or the proof of Theorem 1.6 in [27] for a similar analysis).

Lemma 1 *Let $O \subset \mathbb{R}^3$ be an open set and take a sequence $\{M(n) \subset O\}_n$ of embedded minimal surfaces (possibly disconnected) with locally bounded Gaussian curvature in the sense that for any closed ball $B \subset O$, the $M(n) \cap B$ have uniformly bounded Gaussian curvature, with the bound of the curvature depending only on B. If for every n, any divergent curve of finite length on $M(n)$ has limit point in the boundary of O, then a subsequence of the $M(n)$ converges to a $C^{1,\alpha}$ minimal lamination of O with the same local bounds for its Gaussian curvature that the ones of $M(n)$ satisfy.*

Throughout this paper, we will let $\mathbb{R}^3(*)$ denote the open set $\mathbb{R}^3 - \{(0,0,0)\}$.

Theorem 4 *Suppose that M satisfies the hypotheses in Theorem 2. For any sequence of positive numbers $\{\lambda_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \lambda_n = 0$, the sequence $M(n) = (\lambda_n M) \cap \mathbb{R}^3(*)$ is locally simply connected in $\mathbb{R}^3(*)$.*

Proof. Arguing by contradiction, suppose the sequence of surfaces fails to be locally simply connected in $\mathbb{R}^3(*)$. In this case, there exists a point $p \in \mathbb{R}^3(*)$ such that, after choosing a subsequence, the ball $B(p, \frac{1}{n})$ centered at p of radius $\frac{1}{n}$ intersects $M(n)$ in some nonsimply connected component. After possibly rescaling the family $M(n)$ by a fixed homothety factor, we may assume that $B(p,3) \subset \mathbb{R}^3(*)$. For n fixed, let $r_n \colon \overline{B(p,1)} \to (0,1]$ be defined for $x \in B(p,1)$ to have value $r_n(x) = 1$ if every component of $B(x,1) \cap M(n)$ is simply connected and otherwise, $r_n(x)$ is the radius of the largest open ball $B(x,r')$, $0 < r' < 1$, such that $B(x,r') \cap M(n)$ contains only simply connected components.

For each n, consider the function $\frac{d}{r_n} : \overline{B(p,1)} \to [0,\infty)$, where d is the extrinsic distance function to the boundary sphere $\partial B(p,1)$. Let $p(n) \in B(p,1)$ be a point where $\frac{d}{r_n}$ has its maximum value. Note that since $r_n(p) \to 0$ as $n \to \infty$, the maximum value of $\frac{d}{r_n}$ goes to infinity as $n \to \infty$. Now rescale the surface $M(n)$ to obtain the new surface $\widehat{M}(n) = \frac{1}{r_n(p(n))}(M(n) - p(n))$. By our choice of $p(n)$, given any ball B in $\mathbb{R}^3$ of radius less than $1/2$, $B \cap \widehat{M}(n)$ consists of only simply connected components for n large; the proof of this fact is straightforward and appears in the proof of Lemma 8 in [23]. On the other hand, by our choice of $p(n)$, the intersection of the closed unit ball $\overline{B(1)}$ centered at the origin with $\widehat{M}(n)$ contains a component which is not simply connected. It follows that the sequence $\{\widehat{M}(n)\}_n$ is ULSC in $\mathbb{R}^3$.

First we assume that the surfaces $\widehat{M}(n)$ have locally bounded Gaussian curvature in $\mathbb{R}^3$. In this case, Lemma 1 implies that, after choosing a subsequence, $\widehat{M}(n)$ converges to a $C^{1,\alpha}$-lamination $\mathcal{L}$ of $\mathbb{R}^3$. Since $\widehat{M}(n) \cap \overline{B(1)}$ contains a nonsimply connected component, the supremum of the norms of the second fundamental forms of the $\widehat{M}(n)$ in the ball $B(2)$ are bounded from below by some $\varepsilon > 0$. Hence, $\mathcal{L}$ contains at least one leaf L which is not flat. Now Theorem 1.6 in [27] and Theorem 5 in [23] give the following description of this leaf L:

- L is properly embedded in $\mathbb{R}^3$, in an open halfspace or in an open slab.
- If L has finite genus, then L is properly embedded in $\mathbb{R}^3$.
- If L is properly embedded in $\mathbb{R}^3$, then L is the only leaf in $\mathcal{L}$.

Note that L is complete and orientable since it separates the simply connected domain in which it is properly embedded. Since L is not a plane, L cannot be stable which implies that the convergence of the $\widehat{M}(n)$ to L in the region where L is properly embedded is of multiplicity one. Since for n large, the components of $M(n) \cap B(p, 1)$ have genus zero, a standard curve lifting argument implies that L has genus zero. Hence, L is properly embedded in $\mathbb{R}^3$ and is the only leaf in $\mathcal{L}$. As $\widehat{M}(n) \cap \overline{B(1)}$ has a component which is not simply connected, we deduce that L is not simply connected.

We have shown that L is a nonsimply connected properly embedded minimal planar domain in $\mathbb{R}^3$. It follows from the results in [9, 10, 18, 23] that L is either a catenoid, L has one limit end or L has two limit ends. We first check that L does not have two limit ends. If L has two limit ends, then the results in [23] imply that there is a plane P, not necessarily horizontal but parallel to the limit tangent plane at infinity of L, which intersects L in a simple closed curve γ that separates the top and bottom limit ends of L. Let $\gamma(n) \subset \widehat{M}(n)$ be a sequence of simple closed curves which converges smoothly to γ. Since for n large, $\gamma(n)$ bounds a domain $\Delta(n) \subset \widehat{M}(n)$ with a finite nonzero number of vertical catenoid ends of $\widehat{M}(n)$, the $\gamma(n)$ have vertical flux. It follows that L has vertical flux. Since L has vertical flux, Theorem 6 in [23] implies that the plane P is not horizontal. Since we are assuming that the ends of $\Delta(n)$ have negative logarithmic growth, the maximum principle implies that $x_3|_{\Delta(n)}$ attains its maximum somewhere on

$\gamma(n)$. Since $\Delta(n)$ converges smoothly to one of the components of $L-\gamma$, this component must lie below the height $\max x_3|_\gamma$. But both components of $L-\gamma$ contain planar ends which are parallel to P and so x_3 is not bounded on either component. This contradiction proves that L must be a catenoid or have one limit end. In both cases, L contains a catenoid-type end representative E whose boundary ∂E is a closed strictly convex curve contained in a plane Π and ∂E bounds a planar strictly convex open disk $D \subset \Pi$ such that $L \cap D = \emptyset$. As before, ∂E is the uniform limit of a sequence of simple closed planar curves $\widehat{\gamma}(n) \subset \widehat{M}(n) \cap \Pi$ whose fluxes are vertical. Hence the flux of E is also vertical and Π is horizontal or equivalently, L has horizontal limit tangent plane at infinity. Note that E must be either the top or the bottom end of L.

Since ∂E bounds a horizontal strictly convex disk $D \subset \Pi$, for n large $\widehat{\gamma}(n)$ bounds an open horizontal convex disk $\widehat{D}(n) \subset \Pi$. Since D is disjoint from L, it follows that $\widehat{D}(n) \cap \widehat{M}(n) = \emptyset$ for n large. Let $\alpha(n) = \frac{1}{\lambda n}[r_n(p(n))\widehat{\gamma}(n) + p(n)] \subset M$ be the related convex horizontal curves. After choosing a subsequence, we may suppose that the horizontal disks $D(n)$ that the $\alpha(n)$ bound lie in the same component of $\mathbb{R}^3 - M$.

Next we show that $\{\alpha(n)\}_n$ is a divergent sequence of curves in $\mathbb{R}^3$. Since ∂E is compact and $\widehat{\gamma}(n)$ converges uniformly to ∂E as $n \to \infty$, there exists $R > 0$ such that $\partial E, \widehat{\gamma}(n) \subset B(R)$ for each n. Thus $\|\lambda_n \alpha(n) - p(n)\| \leq \frac{d(p(n))R}{a(n)}$ where $a(n) = \frac{d(p(n))}{r_n(p(n))}$. As $d(p(n)) \leq 1$, $p(n) \in B(p,1)$ and $a(n) \to \infty$ as $n \to \infty$, we conclude that for n large $\lambda_n \alpha(n) \subset B(p,2)$. This property together with the fact that the origin lies outside $B(p,3)$ and $\lambda_n \to 0$ imply that $\{\alpha(n)\}_n$ diverges in $\mathbb{R}^3$ as desired.

Since each $\alpha(n)$ is compact we can extract a subsequence (also denoted in the same way) so that $\alpha(n) \cap \alpha(m) = \emptyset$ whenever $n \neq m$. As $\{\alpha(n)\}_n$ diverges and M has finite genus, $\alpha(n)$ must separate M for all n large enough. Let $\Omega(n) \subset M - \alpha(n)$ be the domain with finite topology such that $\partial\Omega(n) = \alpha(n)$. After gluing $\Omega(n)$ to the horizontal disk $D(n)$, one obtains a piecewise smooth properly embedded surface which separates $\mathbb{R}^3$. Since the ends of $\Omega(n)$ have negative logarithmic growth, the maximum principle implies that $\Omega(n) \subset \{x_3 < h_n\}$, where $h_n \in \mathbb{R}$ is the height of the horizontal plane containing $D(n)$. Furthermore, if $n \neq m$ then either $\Omega(n), \Omega(m)$ are disjoint or $\Omega(n) \subset \Omega(m)$ (or vice versa), and in this last case $h_n < h_m$ by the maximum principle.

Now fix a positive integer m and suppose that $\Omega(m)$ contains infinitely many domains $\Omega(n_k)$, $n_k \in \mathbb{N}$. Since $\Omega(m)$ has a finite number of ends all being asymptotic to half-catenoids with negative logarithmic growth and $\{\alpha(n_k)\}_k$ diverges in $\mathbb{R}^3$, we conclude that after passing to a subsequence, the Gauss map of M along $\alpha(n_k)$ is contained in an arbitrarily small neighborhood of the North or South pole of the sphere for all k large. This is impossible since the Gauss map of $\widehat{M}(n_k)$ along $\widehat{\gamma}(n_k)$ converges to the Gauss map of L along ∂E which is not constant. This contradiction shows that for m fixed, the domain $\Omega(m)$ can only contain a finite number of $\Omega(n)$, $n \in \mathbb{N}$.

If some $\Omega(n)$ is a planar domain which is not an annulus, then the López-Ros deformation argument [18, 29] applied to $\Omega(n)$ gives a contradiction. Thus, either $\Omega(n)$ is an annulus or it has positive genus. Next we prove that the first possibility cannot occur infinitely often. Again by contradiction, assume there exists a subsequence $\Omega(n_k)$ such that $\Omega(n_k)$ is an annulus for all k. Then an elementary separation argument shows that no ends of M lie below $\Omega(n_k)$ and so, $\Omega(n_k)$ is an end representative for the lowest annular end of M given in the Ordering Theorem. By the above discussion, we can assume that $\Omega(n_k) \subset \Omega(n_{k+1})$ for all k hence the union $\cup_k \Omega(n_k)$ is a properly embedded annulus without boundary in $\mathbb{R}^3$, which is contained in M. Since M is connected, this union coincides with M, which is impossible since M is assumed to have one limit end. It follows that for all n large, $\Omega(n)$ is never an annulus and thus, it has positive genus. Since each $\Omega(m)$ can only contain a finite number of $\Omega(n)$ and the genus of M is finite, we see that for all n large and some j arbitrarily large depending on n, it must be the case that $\Omega(n) \subset \Omega(n+j)$ and all the genus of $\Omega(n+j)$ is contained in $\Omega(n)$. Since M has infinitely many ends, there exists $j > n$ large such that the closure $\Sigma(n,j)$ of $\Omega(n+j) - \Omega(n)$ is a planar domain with a finite positive number of ends. Furthermore, the boundary of $\Sigma(n,j)$ consists of two convex horizontal curves in different planes which bound horizontal disks on the same side of M in $\mathbb{R}^3$ and $\Sigma(n,j)$ has vertical flux. In this situation one can also apply the López-Ros argument [29, 31] to obtain a contradiction. This contradiction proves that the sequence $\{\widehat{M}(n)\}_n$ does not have locally bounded curvature.

After a fixed homothety of these surfaces, we may assume that $\sup |K_{\widehat{M}(n)\cap B(1)}| \to \infty$ and $\widehat{M}(n) \cap B(2)$ contains a component which is not a disk for any n. Since for n large, the components of $M(n) \cap B(p,1)$ have genus zero, we deduce that for any ball centered at the origin and for any n sufficiently large, $\widehat{M}(n)$ intersects that ball in components which are planar domains. By Theorem 3, a subsequence of $\widehat{M}(n)$ converges to a foliation of $\mathbb{R}^3$ by parallel planes with singular set of convergence consisting of exactly two Lipschitz curves. By the regularity theorem in [20], $S(\mathcal{L})$ consists of two infinite straight lines S_1, S_2 orthogonal to $\mathcal{L}$.

For n large, one can construct homotopically nontrivial simple closed curves $\widehat{\gamma}_n \subset \widehat{M}(n)$ which converge with multiplicity two to a line segment l in the plane $\Pi \in \mathcal{L}$ passing through the origin and with end points $\Pi \cap (S_1 \cup S_2)$, such that the fluxes of the $\widehat{\gamma}_n$ converge to a vector $F \in \Pi$ orthogonal to the direction of l and the length of F is twice the length of l. The construction of $\widehat{\gamma}_n$ is carried out in detail in our previous paper [23].

Since for all n large $\widehat{\gamma}_n$ separates $\widehat{M}(n)$ in two proper noncompact domains, the flux vector to $\widehat{\gamma}_n$ is vertical by our previous arguments. Thus we will obtain a contradiction by proving that the planes in $\mathcal{L}$ are horizontal. Consider the finite topology domain $\widehat{\Sigma}(n) \subset \widehat{M}(n)$ bounded by $\widehat{\gamma}_n$. Note that $x_3|_{\widehat{\Sigma}(n)}$ attains its maximum value on $\widehat{\gamma}_n$. Take an increasing compact exhaustion of $\widehat{\Sigma}(n)$ by subdomains $\{\widehat{\Sigma}(n,m)\}_{m\in\mathbb{N}}$ with $\widehat{\gamma}_n \subset \partial\widehat{\Sigma}(n,m)$ for all m. After solving the Plateau problem with boundary $\partial\widehat{\Sigma}(n,m)$ in the region of $\mathbb{R}^3 - \widehat{M}(n)$ where $\widehat{\gamma}_n$ is not homologous to zero, we produce a sequence of embedded least-area surfaces $\widetilde{\Sigma}(n,m) \subset \mathbb{R}^3 - \widehat{M}(n)$ with $\partial\widetilde{\Sigma}(n,m) = \partial\widehat{\Sigma}(n,m)$, for all $m \in \mathbb{N}$. By the convex hull property, $x_3|_{\widetilde{\Sigma}(n,m)}$ has its maximum value on $\widehat{\gamma}_n$. Standard compactness and regularity theorems [33] insure that a subsequence of $\{\widetilde{\Sigma}(n,m)\}_m$ converges to a stable properly embedded orientable minimal surface $\widetilde{\Sigma}(n) \subset \mathbb{R}^3 - \widehat{M}(n)$. The surfaces $\widetilde{\Sigma}(n)$ have finite total curvature [12] with a finite number of ends which lie below the maximum value of $x_3|_{\widetilde{\Sigma}(n)}$. Hence, $x_3|_{\widetilde{\Sigma}(n)}$ has its maximum value on $\partial\widetilde{\Sigma}(n) = \widehat{\gamma}_n$. Since the $\widetilde{\Sigma}(n)$ have uniformly bounded curvature away from $\widehat{\gamma}_n$ and $\widetilde{\Sigma}(n)$ is proper and noncompact, the limit set of the $\widetilde{\Sigma}(n)$ must contain an end of at least one of the planar leaves of $\mathcal{L}$. Since x_3 restricted to $\widetilde{\Sigma}(n)$ has its maximum value on $\widehat{\gamma}_n$, it follows that the restriction of x_3 to the limit set of the $\widetilde{\Sigma}(n)$ has its maximum value on the line segment l. Therefore, Π is a horizontal plane. As remarked earlier, the fact Π is horizontal gives rise to a contradiction and completes the proof of Theorem 4. $\square$

4 Limit blow-down laminations.

For the remainder of this manuscript, we let M denote a properly embedded minimal surface in $\mathbb{R}^3$ satisfying the hypotheses and conclusions of Theorem 2 in Section 2. That is, M has finite genus, horizontal limit tangent plane at infinity, one limit end which is its top end and the remaining ends of M are asymptotic to catenoidal ends of negative logarithmic growths which respect the ordering of the middle ends. After a homothety and a translation of M, we will further assume that the bottom end e_1 of M, is asymptotic to the lower end of a vertical catenoid of logarithmic growth -1 and with axis the x_3-axis. We denote by $H(*) = \{x_3 \geq 0\} - \{(0,0,0)\}$. Given $\mu > 0$, we define the cones $\widetilde{C}_\mu = \{(x_1,x_2,x_3) \mid x_3 = \mu\sqrt{x_1^2+x_2^2}\}$ and $C_\mu = \widetilde{C}_\mu + (0,0,-1)$. Let X be the component of $\mathbb{R}^3 - C_\mu$ which lies below C_μ. We will devote this section to study the structure in X of the limit laminations obtained as blow-downs of M.

Lemma 2 *For C_μ sufficiently shallow (μ sufficiently small) and for any sequence $\{\lambda_n\}_n \subset \mathbb{R}^+$ converging to zero, the following statements hold:*

1. *A subsequence of the surfaces $M_X(n) = (\lambda_n M) \cap X$ converges to a minimal lamination $\mathcal{L}_X$ of $X \cap \{x_3 \geq 0\}$, with singular set of convergence being empty, whose leaves have almost horizontal tangent spaces, and with $\partial H(*) \cap X$ as one of its leaves.*

2. *For n sufficiently large, every component of $M_X(n)$ is noncompact, has compact boundary lying in C_μ and is a graph of negative logarithmic growth over its projection to the (x_1,x_2)-plane.*

3. *Every leaf of $\mathcal{L}_X$ is noncompact, has compact boundary contained in C_μ and is a graph of zero logarithmic growth over its projection to the (x_1,x_2)-plane.*

Proof. Let $\{\lambda_n\}_n \subset \mathbb{R}^+$ denote a sequence converging to zero. Since the bottom end of M is catenoidal of negative logarithmic growth, M under homothetic shrinking has limit set contained in the closed upper halfspace $\{x_3 \geq 0\}$ of $\mathbb{R}^3$. The curvature estimates in [6] imply for a sufficiently shallow cone C_μ and n sufficiently large, every point of $M_X(n) = \lambda_n M \cap X$ has an almost horizontal tangent plane (otherwise we could find points $q(n) \in M_X(n)$ with $\|q(n)\|$ arbitrarily large and $x_3(q(n))$ arbitrarily close to zero such that the tangent plane to $M_X(n)$ at $q(n)$ makes an angle bounded away from zero with the horizontal; now use that the sequence $\|q(n)\|^{-1} M_X(n)$ is locally simply connected in $\mathbb{R}^3(*)$ by Theorem 4 and apply the 1-sided curvature estimates in [6] to the sequence $\|q(n)\|^{-1} M_X(n)$ above the lower catenoidal end of $\lambda_n M$ shrunk by the factor $\|q(n)\|^{-1}$ in order to get a contradiction). The same curvature estimates imply that $M_X(n)$ has uniformly bounded curvature for n large. By Lemma 1, after passing to a subsequence, we may assume that the $M_X(n)$ converge to a minimal lamination $\mathcal{L}_X$ of $X \cap \{x_3 \geq 0\}$ with empty singular set of convergence with almost horizontal tangent spaces and with $\partial H(*) \cap X$ as a leaf. This proves item *1* in the statement of the lemma.

Let $G(n)$ be a component of $M_X(n)$. Next we prove that for n large, $G(n)$ is a graph over its projection to the (x_1, x_2)-plane. Since X is simply connected and $G(n)$ embeds properly in X with $\partial G(n) \subset \partial X$, we deduce that $G(n)$ separates X into two regions. Let $\pi(x_1, x_2, x_3) = (x_1, x_2, 0)$ be orthogonal projection from $\mathbb{R}^3$ to the (x_1, x_2)-plane. Suppose that for $x \in \pi(G(n))$, the fiber $\pi^{-1}(x) \cap G(n)$ consists of more than one point. In this case there exist two consecutive points in $\pi^{-1}(x) \cap G(n)$. Since $G(n)$ is connected and submerses onto the (x_1, x_2)-plane, it has a unique orientation induced by the vector $(0, 0, 1)$. But by the separation property of $G(n)$, consecutive points in the fiber $\pi^{-1}(x) \cap G(n)$ must have opposite orientations with respect to $(0, 0, 1)$. This contradiction implies that $G(n)$ is a graph. Since the boundary of $G(n)$ lies in C_μ, $G(n)$ lies in X and X is disjoint from the convex hull of C_μ, it follows that $G(n)$ does not lie in the convex hull of its boundary. By the convex hull property, $G(n)$ is not compact. Assume for the moment that $G(n)$ has compact boundary. Since $G(n)$ is a noncompact domain in $\lambda_n M$ with compact boundary, it must contain ends of $\lambda_n M$. As each of these ends is a catenoidal graph over the exterior of a disk in the (x_1, x_2)-plane and $G(n)$ is also a graph, we conclude that $G(n)$ represents exactly one end of $\lambda_n M$, from which we know it has negative logarithmic growth. Thus to prove part *2* of the lemma it only remains to show that $\partial G(n)$ is compact.

First suppose that $G(n)$ intersects the plane $\{x_3 = -2\}$. Since $G(n)$ is a graph, the portion $\widetilde{G}(n)$ of $G(n)$ that lies below $\{x_3 = -2\}$ must have at most logarithmic growth since it lies above or coincides with the graphical lower end of $\lambda_n M$. Since $\widetilde{G}(n)$ has bounded logarithmic growth and boundary values on a plane, Theorem 3.1 in [28] implies that every boundary component of $\widetilde{G}(n)$ is compact. Suppose that $\widetilde{G}(n)$ has at least two such boundary components Γ_1, Γ_2. Since for n large all compact nonseparating curves on $\lambda_n M$ pass close to the origin, they are not contained in $\{x_3 = -2\}$. In particular, Γ_1 separates $\lambda_n M$ and since $\lambda_n M$ has one limit end, Γ_1 bounds a proper subdomain Ω of $\lambda_n M$ with a finite number of catenoidal ends with negative logarithmic growth. By the maximum principle, $x_3|_\Omega$ attains its maximum value only along Γ_1. Clearly $\widetilde{G}(n) \subset \Omega$, hence Γ_2 is a curve interior to Ω where x_3 also attains its maximum value, a contradiction. This proves that $\widetilde{G}(n)$ is an annulus that projects bijectively onto the exterior of a Jordan curve $\Lambda \subset \{x_3 = -2\}$. Therefore, $G(n) \cap \{x_3 > -2\}$ projects into the interior of Λ in $\{x_3 = -2\}$ through the map $(x_1, x_2, x_3) \mapsto (x_1, x_2, -2)$. Since this last projection is proper when restricted to C_μ, we conclude that $\partial G(n)$ is compact.

Secondly assume that $G(n) \cap \{x_3 = -2\} = \emptyset$. Arguing by contradiction, suppose that $\partial G(n)$ is noncompact. First note that there exist points $q_k(n) \in G(n)$ such that $\|q_k(n)\| \to \infty$ and $\frac{|x_3(q_k(n)|}{\|q_k(n)\|} \to 0$ as $k \to \infty$ (this holds because otherwise, we could find $\mu' \in (0, \mu)$ such that $G(n)$ is contained in the closed region of $\mathbb{R}^3$ between C_μ and $C_{\mu'}$; since $G(n)$ lies in the convex hull of $C_{\mu'}$, Theorem 3.1 in [17] implies that $G(n)$ is contained in the convex hull of its boundary, which contradicts that $\partial G(n) \subset C_\mu$ but $G(n) \subset X$; one could also apply an easy modification of the original barrier argument used by Hoffman and Meeks [16] to show that a noncompact properly immersed minimal hypersurface in $\mathbb{R}^n$ with compact boundary contains a hyperplane in its convex hull in order to contradict the existence of $C_{\mu'}$). Consider the new homothetically shrunk graphs $G_k(n) = \frac{1}{\|q_k(n)\|} G(n)$. Note that the boundary of $G_k(n)$ is contained in a cone $C_\mu(k)$ parallel to C_μ that converges to $\widetilde{C}_\mu = C_\mu + (0,0,1)$ as $k \to \infty$. Moreover, $A_k(n) = G_k(n) \cap \{(x_1, x_2, x_3) \mid 1 \leq x_1^2 + x_2^2 \leq 2\}$ is a compact minimal graph with nonempty boundary. Since $\frac{1}{\|q_k(n)\|} q_k(n) \in A_k(n)$, $\partial G_k(n) \subset C_\mu(k)$ and the tangent planes to $G_k(n)$ are almost horizontal, we deduce that for k large enough $A_k(n) \cap \partial G_k(n) = \emptyset$. Therefore, $A_k(n)$ is a compact annulus for k large. By the 1-sided curvature estimates of Colding and Minicozzi, $A_k(n)$ converges smoothly as $k \to \infty$ to the flat annulus $\{(x_1, x_2, 0) \mid 1 \leq x_1^2 + x_2^2 \leq 2\}$. The related surface $\widetilde{G}_k(n) = G_k(n) \cap \{(x_1, x_2, x_3) \mid 1 \leq x_1^2 + x_2^2\}$ is a graph over a domain $\widetilde{\Omega}_k(n)$ in the (x_1, x_2)-plane. $\partial \widetilde{G}_k(n)$ consists of a graph $\partial_1(k,n)$ over the unit circle in $\partial \widetilde{\Omega}_k(n)$, together with a nonempty subset contained in $C_\mu(k)$ (note by our earlier assumptions, $\widetilde{G}_k(n)$ lies above the plane $\{x_3 = -2\}$ and $\partial \widetilde{G}_k(n)$ is not compact).

Since $\widetilde{G}_k(n)$ is a graph, the annulus $A_k(n)$ projects vertically onto $\{(x_1, x_2, 0) \mid 1 \leq x_1^2 + x_2^2 \leq 2\}$ and $A_k(n) \subset \widetilde{G}_k(n)$, we deduce that $\partial \widetilde{G}_k(n) - \partial_1(k,n)$ lies entirely outside the solid closed vertical cylinder of radius $\sqrt{2}$. In particular, for k large enough $\partial \widetilde{G}_k(n) - \partial_1(k,n)$ lies above the plane $\{x_3 = \frac{\mu\sqrt{2}}{2}\}$ (note that $x_3 = \mu\sqrt{2}$ is the height of the intersection of $\widetilde{C}_\mu = \lim_k C_\mu(k)$ with the vertical cylinder of radius $\sqrt{2}$). This implies that $\partial \widetilde{G}_k(n) - \partial_1(k,n)$ is separated in height from $\partial_1(k,n)$ by some positive constant. Let $p_k(n)$ be a point in $\partial \widetilde{G}_k(n) - \partial_1(k,n)$ where x_3 attains its minimum value. Define the domain $\widehat{G}_k(n) = \widetilde{G}_k(n) \cap \{(x_1, x_2, x_3) \mid x_3 \leq x_3(p_k(n))\}$. Then, the following properties hold:

- $\partial_1(k,n) \subset \partial \widehat{G}_k(n)$,
- If we define $\partial_2(k,n) = \partial \widehat{G}_k(n) - \partial_1(k,n)$, then $\partial_2(k,n)$ is nonempty and lies in the plane $\{x_3 = x_3(p_k(n))\}$.

- $\widehat{G}(n)$ lies in the closed slab $\{\min\left(x_3|_{\partial_1(k,n)}\right) \leq x_3 \leq x_3(p_k(n))\}$.

By Meeks' invariance of flux formula in [19] for properly immersed minimal surfaces in $\mathbb{R}^3$, the flux of ∇x_3 across $\partial_2(k,n) \subset \partial\widehat{G}_k(n)$ is equal to flux of ∇x_3 across $\partial_1(k,n)$ (flux here is considered to be a positive number). Note that the tangent plane to $\widehat{G}_k(n)$ at $p_k(n) \in \partial\widehat{G}_k(n)$ makes an angle $\theta_k(n)$ with the horizontal which is not less that the positive angle that the tangent spaces to the cone $\frac{1}{\|q_k(n)\|}C_\mu$ make with the horizontal. By the aforementioned uniform curvature estimates of $\widehat{G}_k(n)$ near $q_k(n)$, one sees that the flux of ∇x_3 across $\partial_2(k,n) \subset \partial\widehat{G}_k(n)$ is bounded from below by a positive constant that is independent of k. But the flux of ∇x_3 across $\partial_1(k,n)$ is converging to zero as $k \to \infty$, which contradicts the invariance of the flux property of ∇x_3. This contradiction shows that $\partial G(n)$ must be compact and finishes the proof of item *2* of the lemma.

Let G be a leaf of $\mathcal{L}_X$. Since the singular set of convergence of $\mathcal{L}_X$ is empty, G is a smooth limit of the graphs given in item *2*. Hence G is also a graph over its projection to the (x_1, x_2)-plane. Again G is noncompact by application of the convex hull property. Note that G does not have points at height $x_3 = -2$ because $G \subset H(*)$. In this situation, a straightforward modification of the arguments in the last paragraph shows that the boundary of G is compact. Since G is a graph with compact boundary over a noncompact region Ω of the (x_1, x_2)-plane, Ω must contain the complement of a compact set and therefore, G has just one end which has an annular representative with finite total curvature.

It remains to show that the logarithmic growth c of the end of G is zero. As G lies in the closed upper halfspace, $c \geq 0$. Now assume $c > 0$. Consider a relatively compact annular subdomain $\Omega' \subset \Omega$ such that the subgraph $G' \subset G$ over Ω' is very close to the intersection $\mathcal{A}$ of an upper vertical halfcatenoid with a horizontal slab S of width 1 and extremely high lower boundary plane. Since G is the smooth limit of graphical components $G(n) \subset M_X(n)$ and G' has compact closure, it follows that G' is the uniform limit of annular subgraphs $G'(n) \subset G(n)$ over Ω' and that $G'(n)$ are also arbitrarily close to $\mathcal{A}$ for n large enough. Since $G(n)$ is a graph, it has only one end which must be of catenoid-type end with negative logarithmic growth. By the maximum principle, the noncompact component of $G(n) - G'(n)$ must cut the cone C_μ along a possibly disconnected boundary $\Gamma(n)$ lying above the slab S. Let $q(n) \in \Gamma(n)$ be a point where $x_3|_{\Gamma(n)}$ attains its minimum. Note that the tangent plane to $\lambda_n M$ at $q(n)$ makes an angle $\theta(n)$ with the horizontal which is not less than the positive angle between the cone

C_μ and the horizontal. Let $\gamma(n)$ be the lower boundary curve of $G'(n)$ and $W(n) \subset G(n)$ the component of $G(n) - (\gamma(n) \cup \{x_3 > x_3(q(n))\})$ that contains $q(n)$ in its boundary. The angle condition on $\theta(n)$ together with the curvature estimates satisfied by $G(n)$ imply that the flux of $W(n)$ along the union $\alpha(n)$ of its compact boundary components at height $x_3(q(n))$ has arbitrarily large positive third component, for a sufficiently high choice of the slab S. Since the $\gamma(n)$ converge to a single closed curve in G, it follows that the third component of the flux of $W(n)$ along $\gamma(n)$ has an uniform lower bound. Finally, the Divergence Theorem applied to the harmonic function x_3 on $W(n)$ gives a contradiction, since each $W(n)$ has one catenoidal end with negative logarithmic growth tending to zero as $n \to \infty$. Thus the logarithmic growth of the end of G is zero. □

The proof of the next proposition was inspired by the proof of the removable singularities theorem by Gulliver and Lawson [15] for a properly embedded stable minimal surface in $\mathbb{R}^3(*)$, which implies that such a surface extends smoothly across the origin to be a plane in $\mathbb{R}^3$. Recall that a leaf L of a minimal lamination $\mathcal{L}$ is a *limit leaf* if for some $p \in L$, any small ball centered at p intersects $\mathcal{L}$ in an infinite number of disk components.

Proposition 1 *Let $\{\lambda_n\}_n \subset \mathbb{R}^+$ be a sequence converging to zero. Suppose that the sequence of surfaces $M(n) = (\lambda_n M) \cap \mathbb{R}^3(*)$ has locally bounded Gaussian curvature in $\mathbb{R}^3(*)$. Let $\mathcal{L}$ be a minimal lamination of $H(*)$ obtained as limit of a subsequence of the $M(n)$ (both exist by Lemma 1). If L is a limit leaf of $\mathcal{L}$ which is different from $\partial H(*)$, then L is a horizontal plane.*

Proof. Assume that L is not a horizontal plane and we will derive a contradiction. By the maximum principle, $L \subset \mathbb{R}^3(+) = \{(x_1, x_2, x_3) \mid x_3 > 0\}$. By Lemma 2, any component of $L \cap X$ is a graphical planar end, and hence orientable. Therefore if L is not orientable, then each such component of $L \cap X$ lifts to the oriented two-sheeted cover of L. Since L is a limit leaf of $\mathcal{L}$, then the oriented two-sheeted cover of L is stable, which, for the following arguments, allows us to assume that L is orientable. As L is stable, orientable and is not a plane, the results in [11] or [13] imply that L is not complete. Since L is a leaf of a lamination of $H(*)$, any proper arc $\alpha\colon [0, \infty) \to L$ of finite length must have $\lim_{t\to\infty} \alpha(t) = (0, 0, 0)$.

Consider the conformally related Riemannian metric $\widetilde{g} = \frac{(1+x_3)^2}{R^2} g$ on L where g is the Riemannian metric on L induced by the usual inner product and $R = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

We claim that $\widetilde{g}$ is complete; to see this, let $\beta = (\beta_1, \beta_2, \beta_3) \subset L$ be a divergent curve. We parametrize β by its arclength t with respect to the metric g, defined on an interval $I = [0, a)$ with $0 < a \leq \infty$. First assume that $\beta(t) \to \vec{0}$ as $t \to a^-$. The length of β with respect to $\widetilde{g}$ is given by

$$\int_0^a \frac{1+\beta_3}{\|\beta\|} dt \geq c \int_0^a \frac{dt}{\|\beta\|} \geq c \int_0^a \frac{-\langle \beta, \beta' \rangle}{\|\beta\|^2} dt = c \int_0^a (-\log \|\beta\|)' \, dt = \infty,$$

where c is a positive constant and we have used the Schwarz inequality in the second inequality above. Now assume that β diverges extrinsically to ∞ inside the region above the shallow cone C_μ given in Lemma 2. Thus $(1+\beta_3)^2 \geq \mu^2(\beta_1^2 + \beta_2^2)$ and the length of β with respect to $\widetilde{g}$ can be estimated by

$$\int_0^a \frac{1+\beta_3}{\|\beta\|} dt \geq \mu \int_0^a \frac{1+\beta_3}{\sqrt{1+(1+\mu^2)\beta_3^2}} dt.$$

Since the last integrand converges to $(1+\mu^2)^{-1/2}$ as $\beta_3 \to \infty$ and the length of β with respect to g is infinite, its length with respect to $\widetilde{g}$ is also infinite. Finally, suppose that β lies eventually in the component X of $\mathbb{R}^2 - C_\mu$ below C_μ, where C_μ is the shallow cone given in Lemma 2. This implies that β corresponds to one of the annular planar ends E of $L \cap X$ given in Lemma 2. Clearly $\frac{(1+x_3)^2}{R^2}|_E$ is bounded by below by $\frac{c}{r^2}|_E$, where $r = \sqrt{x_1^2 + x_2^2}$ and $c > 0$ is a suitable constant. Thus it suffices to check that $\widehat{g} = \frac{1}{r^2}\langle , \rangle$ is complete on $\Omega = \{(x_1, x_2) \mid x_1^2 + x_2^2 \geq 1\}$, where $\langle , \rangle$ stands for the usual flat metric in the (x_1, x_2)-plane. This last property follows from the fact that for a divergent curve $\alpha(r) = re^{i\theta(r)}$ in Ω parametrized by $r = |\alpha| \geq r_1$ where $\theta(r) \in \mathbb{R}$, its length with respect to $\widehat{g}$ is

$$\int_{r_1}^{\infty} \frac{\sqrt{1 + r^2(\theta')^2}}{r} dr \geq \int_{r_1}^{\infty} \frac{dr}{r} = \infty.$$

In summary $(L, \widetilde{g})$ is complete, as we claimed.

Let $\Delta, \widetilde{\Delta}$ denote the Laplace operators of $(L, g), (L, \widetilde{g})$, respectively, and let $K, \widetilde{K}$ denote the corresponding Gaussian curvature functions. It is well known that

$$\widetilde{\Delta} = \frac{R^2}{(1+x_3)^2}\Delta \qquad \text{and} \qquad \widetilde{K} = \frac{R^2}{(1+x_3)^2}\left(K - \Delta \ln \frac{1+x_3}{R}\right).$$

Since $\Delta \ln(R) = \frac{2(1-\|\nabla R\|^2)}{R^2} \geq 0$ and $\Delta \ln(1+x_3) = \frac{-\|\nabla x_3\|^2}{(1+x_3)^2} \leq 0$, it follows that $\widetilde{K} = \frac{R^2}{(1+x_3)^2}K + P$ where P is a nonnegative function. Since L is stable, the operator $-\Delta + 2K$

is positive semidefinite on (L, g). As $-\Delta + K \geq -\Delta + 2K$, it follows that $-\Delta + K$ is also positive semidefinite on (L, g), and so $\frac{R^2}{(1+x_3)^2}(-\Delta + K)$ is positive semidefinite on (L, g). Since $-\widetilde{\Delta} + \widetilde{K} \geq \frac{R^2}{(1+x_3)^2}(-\Delta + K)$, we conclude that $-\widetilde{\Delta} + \widetilde{K}$ is positive semidefinite on $(L, \widetilde{g})$. This property together with the completeness of $(L, \widetilde{g})$ imply that the universal covering of L is conformally $\mathbb{C}$ (Fischer-Colbrie and Schoen [13]). Since $x_3|_L$ is a nonconstant positive harmonic function on L, its lifting to the universal covering of L gives a positive harmonic function on $\mathbb{C}$. By Liouville's Theorem, x_3 must be constant, a contradiction. Now the proposition is proved. $\square$

After minor modifications, the proof of Proposition 1 also demonstrates that the following result holds.

Proposition 2 *If M is a stable orientable minimally immersed surface in an open halfspace $H \subset \mathbb{R}^3$ and there exists a point p_0 in the closure of H such that any divergent curve in M of finite length has p_0 as its limit point, then M is a plane parallel to ∂H, possibly punctured at p_0.*

5 The proof of Theorem 1.

Theorem 5 *A properly embedded minimal surface M in $\mathbb{R}^3$ with finite genus does not have one limit end.*

Proof. Take a sequence $\lambda_n \to 0$. We claim that the sequence of surfaces $M(n) = (\lambda_n M) \cap \mathbb{R}^3(*)$ has locally bounded Gaussian curvature in $\mathbb{R}^3(*)$. Arguing by contradiction, suppose there exists a point $p \in \mathbb{R}^3(*)$ such that as $n \to \infty$, the curvature of the $M(n)$ blows up as $n \to \infty$ in arbitrarily small neighborhoods of p. Since the limit set of $M(n)$ in contained in $\{x_3 \geq 0\}$, then $x_3(p) \geq 0$. From the proof of Lemma 2 we deduce that p cannot lie in X, which depends on the shallow cone C_μ defined there. A modification of this argument shows that $p \notin \partial H(*)$, and so, $x_3(p) > 0$. The following assertion follows from the arguments that go into the proof of Theorem 3 by Colding and Minicozzi [7].

Assertion 1 *There exists a smooth stable embedded minimal surface L in the limit set of the $M(n)$ (thus $L \subset \{x_3 \geq 0\}$) and a subsequence of the $M(n)$ (denoted in the same way) that satisfy the following properties:*

1. *L passes through p;*

2. *There exists a discrete subset of points $S_L \subset L$ such that for any $q \in L - S_L$, the $M(n)$ have locally bounded Gaussian curvature in a neighborhood U_q of q, but this property fails around any point of S_L;*

3. *Given $q \in L - S_L$, the $M(n)$ converge $C^\alpha, 0 < \alpha < 1$, to a minimal lamination $\mathcal{L}_q$ of the neighborhood U_q appearing in the preceding item and $L \cap U_q \subset \mathcal{L}_q$;*

4. *Any geodesic $\alpha\colon [0,1) \to L$ starting at p which cannot be extended through $t = 1$ has limiting value which must be point where the sequence $M(n)$ is not locally simply connected (such a limiting value can only be $(0,0,0)$ by Theorem 4).*

By Lemma 2, after passing to a subsequence we can assume that for $\mu > 0$ sufficiently small, $M_X(n) = (\lambda_n M) \cap X$ converges to a minimal lamination $\mathcal{L}_X$ of $X \cap \{x_3 \geq 0\}$. By item 3 above, given $q \in L - S_L$ we have $\mathcal{L}_q \cap X = \mathcal{L}_X \cap U_q$. In particular, $L \cap X \in \mathcal{L}_X$. By item 3 of Lemma 2, $L \cap X$ consists of planar ends of L. Since L is stable, a small modification of the proof of Proposition 1 implies that L is a horizontal plane. It follows from the proof of Theorem 3 that the surfaces $M(n)$ converge to the foliation of $H(*)$ by horizontal planes and the punctured plane $\partial H(*)$, with singular set of convergence $S(\mathcal{L})$ consisting of one or two Lipschitz curves. By Meeks' regularity theorem for $S(\mathcal{L})$ given in [21], $S(\mathcal{L})$ consists of vertical straight lines. Since the sequence $M(n)$ is locally simply connected in $\mathbb{R}^3(*)$ and $\mathcal{L} \subset H(*)$, the curvature estimates in [6] imply that the end points of each of these lines must be the origin. This means that $S(\mathcal{L})$ is the positive x_3-axis. It follows from results in [6] that for n large the cylinder $F = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1, -1 \leq x_3 \leq 1\}$ intersects $M(n)$ transversely in a double spiral curve. For n large this double spiral curve cannot intersect the bottom boundary circle of F, since the limit set of $M(n)$ lies in $\{x_3 \geq 0\}$. But since $M(n)$ is proper in $\mathbb{R}^3$, the double spiral curve $M(n) \cap F$ must exit the bottom boundary circle of F. This contradiction proves that the sequence $M(n)$ has locally bounded Gaussian curvature in $\mathbb{R}^3(*)$.

We next show that the absolute Gaussian curvature $|K_M|$ of M has at least quadratic decay in terms of the distance function R to the origin. If not, we can find points $p(n) \in M$ such that $\|p(n)\| \to \infty$ and $\frac{|K_M(p(n))|}{\|p(n)\|} \to \infty$ as $n \to \infty$. Then, the sequence $\{\lambda_n M\}_n$ with $\lambda_n = \|p(n)\|^{-1}$ does not have bounded curvature on the unit sphere in $\mathbb{R}^3$, contradicting that $\{\lambda_n M\}_n$ has locally bounded curvature in $\mathbb{R}^3(*)$.

Now choose a sequence of points $p'(n) \in M$ with $R(n) = ||p'(n)|| \to \infty$ as $n \to \infty$ such that the tangent plane to M at $p'(n)$ is vertical for each n, which is possible since M

has infinite total curvature. Consider the sequence $\{\lambda_n M\}_n$ with $\lambda_n = R(n)^{-1}$ for all n. Since the surfaces $\lambda_n M$ have locally bounded curvature in $\mathbb{R}^3(*)$, Lemma 1 implies that a subsequence (denoted also by $\{\lambda_n M\}_n$) converges to a minimal lamination $\mathcal{L}_1$ of $H(*)$. Note that $\mathcal{L}_1$ contains a leaf L_1 which has a vertical tangent plane at one of its points (which is the limit point of the sequence $\lambda_n p'(n)$ on the unit sphere). By Proposition 1, L_1 is not a limit leaf.

We now show that L_1 is properly embedded in $\mathbb{R}^3(+) = \{x_3 > 0\}$. If not, there is a limit leaf Π in $\mathcal{L}_1 - \partial H(*)$ in the closure of L_1. By Proposition 1, Π is a horizontal plane at height $2\delta > 0$. Since $|K_M|R^2 \leq c$ for a certain $c > 0$, this curvature estimate is invariant under rescaling and $\{\lambda_n M\}_n$ converges smoothly to $\mathcal{L}_1$, Lemma 1 gives that the curvature function K on the leaves of $\mathcal{L}_1$ also satisfies $|K|R^2 \leq c$. In particular, $\mathcal{L}_1$ has bounded curvature in the δ-neighborhood of Π and so, L_1 has bounded curvature in that δ-neighborhood of Π. This implies that for ε small, the tangent planes to L_1 in the ε-neighborhood of Π are almost horizontal. It follows that every component of L_1 in the closed ε-neighborhood of Π is a graph over its projection to Π. In particular, these components are proper in the closed ε-neighborhood of Π and have their boundaries on the boundary of the ε-neighborhood, which is impossible (see, for instance, the proof of the Halfspace Theorem [16]). This proves that L_1 is properly embedded in $\mathbb{R}^3(+)$.

Since L_1 is properly embedded in $\mathbb{R}^3(+)$ and the ends of $L_1 \cap X$ are all planar, then the heights of the graphical planar ends of L_1 form a discrete set of positive numbers. Therefore, there exists a horizontal plane $P \subset \{x_3 > 0\}$ that intersects L_1 transversely in a finite positive number of simple closed curves. Let $L_1(+)$ be one of the components of $L_1 - P$ which lies above P. We now check that $L_1(+)$ has infinite total curvature. Otherwise, $L_1(+)$ has a finite number of ends lying above P and so, each of these ends is asymptotic to a horizontal halfcatenoid or to a horizontal plane. By Lemma 2, $L(+)$ only has planar ends, hence $L_1(+)$ lies below the height of its highest planar end. This implies that the third coordinate function x_3 of $L_1(+)$ is a nonconstant bounded harmonic function that has constant values on $\partial L_1(+) \subset \Pi$, which is a contradiction since $L_1(+)$ is conformally a finitely punctured Riemann surface with boundary. Therefore, $L_1(+)$ has infinite total curvature.

Since $L_1(+)$ has infinite total curvature, its Gauss map takes on infinitely often some point v in the equator of the unit sphere. Hence, there exists a sequence of points $q(n) \in L_1(+)$ with normal vector v for any n. Since $L_1(+)$ is proper in the closed halfspace above P, then $\|q(n)\| \to \infty$ as $n \to \infty$. As the Gaussian curvature of $L_1(+)$ decays at least quadratically, the scalar fluxes $F_n = \int_{\alpha(n)} \frac{\partial x_3}{\partial \eta} ds$ of ∇x_3 across the components $\alpha(n)$ of $x_3^{-1}(x_3(q(n)) \subset L_1(+)$ which contains $q(n)$ (here η us the unit conormal vector along $\alpha(n)$ pointing outwards $L_1(+) \cap \{x_3 \leq x_3(q(n))\}$), form a sequence $\{F_n\}_n$ which diverges to infinity as $n \to \infty$ (because larger and larger geodesic disk neighborhoods of $q(n)$ approximate flat disks orthogonal to v). Since $L_1(+) \cap X$ consists of horizontal planar ends, the Divergence Theorem applied to the compact domain of $L_1(+)$ enclosed by the plane P, the horizontal plane containing $q(n)$ and a suitable vertical cylinder with axis the x_3-axis, implies that the scalar flux of ∇x_3 across $\partial L_1(+)$ is infinite. But the scalar flux of ∇x_3 across $\partial L_1(+)$ is at most equal to the length of $\partial L_1(+)$, which is finite. This contradiction completes the proof of Theorem 5. $\square$

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