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Properly embedded minimal surfaces with finite total curvature

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Preface

Among properly embedded minimal surfaces in Euclidean three space, those that have finite total curvature form a natural and important subclass. The first nontrivial examples, other than the plane and the Catenoid, were constructed only some years ago by Costa [9], and Hoffman and Meeks [16], [17]. These examples began the study of existence, uniqueness and structure theorems for minimal surfaces of finite total curvature, usually attending to their topology. Several methods compete to solve the main problems in this theory, although up to now, the structure of the space of such kind of surfaces with a fixed topology is not well understood. However, we dispose today of a certain number of partial results, and some of them will be explained in these notes.

On the other hand, there are other aspects of the theory which are not covered by these notes. We refer the interested reader to the following literature:

The classic book of Osserman [39] is considered nowadays as one of the obliged sources for the beginner. The texts by Meeks [30, 32], Hoffman and Karcher [14], López and Martín [27], and Colding and Minicozzi [7] review a large number of global results on minimal surfaces.

For the last progresses in constructions techniques of properly embedded minimal examples with finite total curvature, see Kapouleas [22], Pitts and Rubinstein [43] and Weber and Wolf [57], or Traizet [55] in the periodic case. Recent embedded examples with infinite total curvature can be found in Hoffman, Karcher and Wei [15] and Weber [56].

The analytical structure of the spaces of properly embedded minimal surfaces with (fixed) finite topology is studied in Pérez and Ros [42]. The paper by Mazzeo and Pollack [29] contains a comparative study between the theory of minimal surfaces and other noncompact geometric problems, like constant (nonzero) mean curvature surfaces.

Granada, March 2000 *Joaquín Pérez, Antonio Ros*

Contents

1. Background.

This Section is intended to introduce some basic material which will be used throughout the paper. First we review some properties of minimal surfaces in Euclidean three-space. *Weierstrass representation* reduces the study of these surfaces to one complex variable theory. This representation is strongly related to $\mathbb{R}^3$, and does not exist when considering other ambient spaces. *Complete minimal surfaces with finite total curvature* were first studied by Osserman [39], and nowadays play a fundamental role in the global theory of minimal surfaces. The *maximum principle* is another basic tool, specially useful with embedded surfaces. The *monotonicity formula* allows to control the area of a minimal surface inside a ball. *Stable minimal surfaces*, i.e. surfaces which minimize area up to second order, appear naturally as solutions of the *Plateau Problem* and have some nice and useful properties. Finally, we state one of the versions of the Plateau Problem, which is one of the central topics in the study of minimal surfaces.

1.0.1 Weierstrass Representation.

Let M be an oriented surface endowed with a Riemannian metric ds^2. We will denote respectively by $\nabla, \triangle$ the gradient and Laplacian operator associated to ds^2. Our Riemannian surface will be very often realized through an isometric immersion into Euclidean three-space $\psi : M \to \mathbb{R}^3$, thus Gauss equation holds for any pair of tangent vector fields to M,

$$X(d\psi(Y)) = d\psi(\nabla_X Y) + \langle AX, Y\rangle N,$$

where A is the *shape operator* and N the *Gauss map of* ψ. We also denote by k_1, k_2 the eigenvalues of A, so $K = k_1 k_2$ and $H = \frac{1}{2}(k_1 + k_2)$ are the *Gauss* and *mean curvature*, respectively. The immersion ψ is said to be *minimal* if H vanishes identically. From minimality one has $A^2 = -KI$, I being the identity on the tangent bundle. This equation implies $\langle dN(X), dN(Y)\rangle = -K\langle X, Y\rangle$ for arbitrary tangent vectors X, Y to M. In other words, N is a *conformal map*. After orientation of the 2-sphere $\mathbb{S}^2(1)$ with the outer normal, N turns out to reverse orientation, thus the stereographic projection from the North Pole of N gives rise to a meromorphic map g defined on the underlying Riemann surface determined by the conformal class of ds^2 in M.

The relation $\triangle\psi = 2HN$ reveals the equivalence between minimality and harmonicity of the immersion. In particular, it gives that minimal surfaces cannot be compact without boundary. This harmonicity allows us to consider the –locally well-defined up to translations– *conjugate minimal immersion* ψ^*, whose coordinate functions are harmonic conjugates of the ones of ψ. Thus, $\psi + i\psi^*$ is a holomorphic curve in $\mathbb{C}^3$ and $\Phi = d(\psi + i\psi^*)$ is a globally defined holomorphic differential on M, called the *Weierstrass form* of ψ. Putting $\Phi = (\phi_1, \phi_2, \phi_3)$, the conformality of ψ translates into $\sum_{k=1}^3 \phi_k^2 = 0$ and the nondegeneracy of the induced metric prevents $\sum_k |\phi_k|^2$ to vanish on M. Conversely, the real part of the integral of a $\mathbb{C}^3$-valued holomorphic differential $\Phi = (\phi_1, \phi_2, \phi_3)$ with $\sum_{k=1}^3 \phi_k^2 = 0$ defines locally a minimal surface in $\mathbb{R}^3$ which is unbranched if and only if $\sum_k |\phi_k|^2$ has no zeros on M.

In the sequel, we simply denote ϕ_3 by ϕ. The Weierstrass form can be written in terms of the Gauss map g and the height differential ϕ as $\Phi = (\frac{1}{2}(g^{-1} - g), \frac{i}{2}(g^{-1} + g), 1)\phi$, see [39]. Thus for any $z \in M$,

$$\psi(z) = \text{Real} \int^z \Phi \equiv \left(\overline{F}(z) - G(z), \text{Real} \int^z \phi \right) \in \mathbb{C} \times \mathbb{R} \equiv \mathbb{R}^3, \qquad (1.1)$$

where $F(z) = \frac{1}{2}\int^z g^{-1}\phi$ and $G = \frac{1}{2}\int^z g\phi$ (note that F, G need not to be well-defined on M, but $\overline{F} - G$ does). Conversely, if g is a meromorphic function and ϕ a holomorphic differential on M, then equation (1.1) gives rise to a well-defined minimal immersion if and only if the following compatibility conditions hold:

- Undegeneracy of ds^2. The only zeros of ϕ must occur at the zeros or poles of g, with the same multiplicity.
- Period problem. Real $\int_\gamma \Phi = 0$ for each 1-cycle $\gamma \subset M$.

The pair (g, ϕ) will be called the *Weierstrass data* of ψ. The easiest global examples are the plane ($M = \mathbb{C}, g(z) = 0, \phi = 0$), the Catenoid ($M = \mathbb{C} - \{0\}, g(z) = z, \phi = \frac{dz}{z}$), the Helicoid ($M = \mathbb{C}, g(z) = e^{iz}, \Phi = dz$) and the Enneper surface ($M = \mathbb{C}, g(z) = z, \phi = z\,dz$).

1.0.2 Finite Total Curvature.

Osserman [39] discovered that in the family of complete minimal surfaces we have a natural and important subclass which can be characterized by the property that the integral of the absolute Gauss curvature $C(M) = \int_M |K| dA$ is finite. This property is usually taken to define the family but nowadays we know that such surfaces can be characterized by several others interesting properties. The most important one is that they reduce via Weierstrass representation to the theory of compact Riemann surfaces.

Next we deal with the influence of the condition $C(M) < \infty$ on the behavior of a complete minimal surface M at infinity. The first consequence,

see Huber [19], is that M must be conformally equivalent to a finitely punctured compact Riemann surface $M = \overline{M} - \{p_1, \ldots, p_r\}$. The points $p_1, \ldots, p_r$ are usually referred as the *ends* of M. Thus the *genus* of M coincides with the genus of the compactification $\overline{M}$. Osserman [39] proved that both the Gauss map g and the height differential ϕ extend meromorphically to $\overline{M}$. In particular, g has a well-defined degree (which counts the spherical image of the Gauss map) and the normal vector at each end is well-defined. We also observe that in the family of minimal surfaces with finite total curvature, completeness of the metric is equivalent to properness of the immersion, see for instance [20].

All properties above extend to the case in which M is a complete minimal surface with finite total curvature and compact boundary. In this case, the compactification $\overline{M}$ is a compact Riemann surface with boundary and the ends $p_1, \ldots, p_r$ are interior points of $\overline{M}$.

The simplest examples of complete minimal surfaces with finite total curvature are the plane and the Catenoid. If we consider embedded surfaces, these two examples constitute the models at infinity for any surface in the family: each embedded end of a complete minimal surface with finite total curvature must be asymptotic to a halfcatenoid or to a plane (see [20] and [50]). In terms of the Weierstrass representation, this fact means that, up to a rotation in $\mathbb{R}^3$ so that the limit normal vector at the end is $(0, 0, -1)$, we can parameterize the end in the punctured disk $\{0 < |z| < \varepsilon\}$ with Weierstrass data

$$g(z) = z^k, \quad \phi = z^k \left(\frac{a}{z^2} + h(z) \right) dz, \tag{1.2}$$

where one of the following possibilities hold:

- Catenoidal end: $k = 1$ and $a \in \mathbb{R} - \{0\}$ (a is called the *logarithmic growth* of the end). In this case, the end is asymptotic to a (vertical) halfcatenoid and the level sets of the third coordinate in a neighborhood of the end look like horizontal circles of large radii.
- Planar end: $k \geq 2$ and $a \in \mathbb{C} - \{0\}$. Now the end is asymptotic to a (horizontal) plane at finite height, and the level set of the height function x_3 around the end at that height forms a equiangular system of $k-1$ curves crossing at infinity.

If we allow our surfaces to have self-intersections, then more complicated ends can appear, like the one in Enneper surface. On the contrary, if the whole surface M is embedded, then all its ends must be parallel, in the sense that the normal vector at the ends takes the same value up to a sign.

Finally, we dispose of a Gauss-Bonnet type formula in this setting, which relates some of the integers appeared in the discussion above [20]. In the particular case that M is a complete minimal surface with finite total curvature and embedded ends, it says that

$$\text{degree}(N) = \text{genus}(M) + r - 1. \tag{1.3}$$

Particularizing to a complete embedded minimal surface M in $\mathbb{R}^3$ with finite total curvature, we get that outside a big ball in space, M has a nice shape: there are a finite number of parallel ends and each end is asymptotic to a plane or to a halfcatenoid. We will assume in what follows that these ends are always horizontal.

A first fact that we deduce easily from this picture is that *the plane is the only complete embedded minimal surface with finite total curvature in* $\mathbb{R}^3$ *and just one end.* This follows because a surface M of this kind is parabolic and the third coordinate function $x_3 : M \to \mathbb{R}$ is harmonic and bounded above (or below). Thus x_3 must be constant and we get the result.

Another easy property is that *the Catenoid is the unique complete embedded minimal annulus with finite total curvature.* To show this fact, note that such an annulus M cannot have planar ends (a planar end would imply again that x_3 is bounded above or below), so the height differential ϕ has two simple poles at the ends and is holomorphic on M. As the compactified surface $\overline{M}$ obtained by attaching the two ends to M is a sphere, we deduce that ϕ does not vanish at M, hence the meromorphic Gauss map g misses the values $0, \infty$ in M. As g has a simple zero and a simple pole at the (catenoidal) ends, its degree must be one and M can be parameterized by the Weierstrass data $M = \mathbb{C} - \{0\}$, $g(z) = z$, $\phi = a\frac{dz}{z}$, a being a nonzero complex number. Finally, to solve the period problem a must be real, hence M is a Catenoid.

1.0.3 Maximum Principle.

This fundamental tool comes from the fact that minimal surfaces can be locally expressed as graphs $(x, y, f(x, y))$ of solutions of the quasilinear elliptic second order partial differential equation

$$(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy} = 0.$$

Briefly, the maximum principle can be stated as follows: if two minimal surfaces meet at an interior point and, locally around this point, one of the surfaces lies at one side of the other, then we conclude that both surfaces coincide around the contact point.This is a direct consequence of the Hopf maximum principle, see for instance [50]. There are several versions of this principle: the meeting point can be at the boundary of the surfaces, provided that both the tangent planes to the surfaces and the tangent lines to their boundaries coincide. In our context, an extremely useful version of the maximum principle due to Langevin, Meeks and Rosenberg [24, 35] tells us about the relative position of two such surfaces at infinity: the distance between two disjoint, connected, properly immersed minimal surfaces with compact boundary (possibly empty) in a complete flat 3-manifold, must be positive and it is attained at the boundary of at least one of the surfaces; if both boundaries are empty, the conclusion is that both surfaces are flat.

Proposition 1.0.1. *Let M be a compact minimal surface in $\mathbb{R}^3$ whose boundary is a Jordan curve which projects diffeomorphically onto a convex planar curve $\Gamma \subset \{x_3 = 0\}$. Then, M is a graph over the domain Ω enclosed by Γ.*

Proof. By using planes as barriers, we get that M must be contained in the solid cylinder $\overline{\Omega} \times \mathbb{R}$. Reasoning by contradiction, suppose that there exist $a, b \in M$ with the same orthogonal projection on Ω and with $x_3(a) > x_3(b)$. As M is compact, we can translate it up until the translated surface $M + t_0 e_3$ does not touch M, t_0 being positive. In particular, $x_3(a) < x_3(b) + t_0$ must hold. Now start displacing down continuously $M + t_0 e_3$, until arriving to a first contact point between $M + te_3$ and the original surface M for certain t. Clearly $t \geq x_3(a) - x_3(b) > 0$, hence the first contact point must occur at the interior of both surfaces, in contradiction with the maximum principle. Now our claim is proved and M is the graph of a function $u : \overline{\Omega} \to \mathbb{R}$. Using the boundary version of the maximum principle it can be shown that u is smooth everywhere, or equivalently that the normal vector of M is nowhere horizontal.

The maximum principle also makes possible to prove the following two uniqueness Theorems.

Theorem 1.0.1 (Hoffman & Meeks, [18]). *A properly immersed minimal surface in $\mathbb{R}^3$ which lies in a halfspace must be a plane.*

Theorem 1.0.2 (Schoen, [50]). *A properly embedded minimal surface with finite total curvature and two embedded ends must be a Catenoid.*

This second result follows from the Alexandrov reflexion technique which can be sketched as follows: Recall that the normal vectors at the ends are vertical. The starting point consists of taking a family of vertical planes and then considering the one-parameter family of minimal surfaces with boundary obtained by reflecting the part of M which is behind each one of these planes. Under our assumptions, we can guarantee the existence of a first contact point between the surface M and one of its reflected images. This contact may occur at the interior, at the boundary or at infinity. Then the maximum principle allows us to conclude that M is symmetric with respect to one of the planes in our family. This gives that M is symmetric with respect to all planes containing a fixed vertical axis and, so, it must be the Catenoid. For a complete proof, see [50].

1.0.4 Monotonicity Formula.

The next result has many applications in the study of minimal surfaces, and follows from application of the coarea formula, see for instance [7, 23].

Theorem 1.0.3. *Let M be a minimal surface properly immersed in a ball $B(p,R)$. Then, the function*

$$r \mapsto \frac{A(M \cap B(p,r))}{r^2}$$

is nondecreasing, $0 \leq r \leq R$.

As a direct consequence of the monotonicity formula, we can obtain a quadratic area growth estimate for complete minimal surfaces with finite total curvature, see for instance [17]. We state below the particular case of properly embedded minimal surfaces in the whole $\mathbb{R}^3$.

Proposition 1.0.2. *Let $M \subset \mathbb{R}^3$ be a properly embedded minimal surface with finite total curvature and r ends. Then $A(M \cap B(p,R)) \leq r\pi R^2$, for any $p \in M$ and $R > 0$.*

Proof. The monotonicity formula insures that $f(R) = R^{-2}A(M \cap B(p,R))$ is a nondecreasing function. For an embedded end of finite total curvature, the limit of f as $R \to +\infty$ is π, from where the Proposition follows directly.

1.0.5 Stability.

A (possibly noncompact) minimal surface M in $\mathbb{R}^3$ is called *stable* if for each relatively compact subdomain $\Omega \subset\subset M$ and respect to any nontrivial normal variation fixing $\partial\Omega$, the second derivative of the area functional is nonnegative. Equivalently, the first Dirichlet eigenvalue of the Jacobi operator $L = \Delta + |A|^2$ in Ω is nonnegative. Otherwise, M is called *unstable*.

The following result was obtained independently by do Carmo and Peng [2], Fischer-Colbrie and Schoen [13] and Pogorelov [44].

Theorem 1.0.4. *Let M be a complete and orientable minimal surface in $\mathbb{R}^3$. If M is stable, then it must be a plane.*

An important fact about stable surfaces it that they satisfy the following curvature estimate. It has been obtained by Schoen [51] (see also Ros [46] for a simpler proof).

Theorem 1.0.5. *There exists a constant $c > 0$ such that for any stable orientable minimal surface M in $\mathbb{R}^3$ and any $p \in M$, we have*

$$|K_M(p)| \leq c\rho_M(p)^2,$$

where $\rho_M(p)$ stands for geodesic distance between p and the boundary of M.

1.0.6 The Plateau Problem.

To finish this introductory review, we dedicate a few lines to the Plateau Problem. The simplest form of such question is finding a surface of least area spanned by a given closed Jordan curve. This problem goes back to the very beginning of the theory of minimal surfaces (in Lagrange's time), although it acquired its name in honor of the Belgian physicist J. A. F. Plateau, who described experimentally in nineteenth century the phenomena associated to rigid wires bounding soap films. A huge amount of literature is devoted to this problem, but we will reduce here to state a particular version, to be used later on.

Suppose that W is a compact Riemannian 3-manifold that embeds in the interior of another Riemannian 3-manifold. W is said to have *piecewise smooth mean convex boundary* if ∂W is a two-dimensional complex consisting of a finite number of smooth two-dimensional compact simplices with interior angles less than or equal to π, each one with nonnegative mean curvature with respect to the inward pointing normal[1]. The boundary of such 3-manifold is a good barrier for solving Plateau problems in N, in the following sense:

Theorem 1.0.6 ([32, 38, 53]). *Let W be a compact Riemannian 3-manifold with piecewise smooth mean convex boundary. Let Γ be a smooth collection of pairwise disjoint closed curves in ∂W, which bounds a compact orientable surface in W. Then, there exists an embedded orientable surface $\Sigma \subset W$ with $\partial\Sigma = \Gamma$ that minimizes area among all orientable surfaces with the same boundary (in particular, Σ is stable).*

[1] For our later purposes, it suffices to consider compact flat 3-manifolds whose boundary is a finite union of embedded smooth surfaces with nonnegative mean curvature with respect to the inward pointing normal, meeting with interior angles smaller than or equal to π.

2. Minimal Surfaces with Vertical Forces I.

In this Section we will study certain forces associated naturally to a minimal surface. Viewing minimal surfaces as soap films, there is a geometric invariant which describes the forces involved in these isolated balanced physical systems. This is a key notion, whose consequences are not completely understood yet.

The main result we will prove is that the Catenoid is the unique embedded nonflat complete minimal surface in $\mathbb{R}^3$ with finite total curvature and genus zero.

2.0.7 Basic Properties of Forces.

Minimal surfaces are mathematical models for soap films. These films can be characterized physically as objects which are surfaces and fluids at the same time, and remain in equilibrium under the action of the superficial tension. Imagine that we have a soap film constructed over three rigid wires like in Figure 2.1. The film pushes the wires because of the superficial tension, which we suppose homogeneously distributed along the surface. To preserve the equilibrium of the system we must assume the existence at each one of the wires of a force (represented by an arrow in the Figure) that, thanks to the fluidity and homogeneity of the film, is computed as the integral of the outward pointing conormal vector of the surface along the boundary component, i.e. the unit vector tangent to the surface but normal to the boundary curve. The balancing condition on this isolated physical system implies that the total force vanishes. There is a well-known simple soap film experiment, see Figure 2.2, which allows to visualize the superficial tension and the forces associated to it. Consider a planar closed rigid wire and imagine that we have a soap film with this boundary. Take a closed flexible thread and put it carefully on the soap film without double points. If we break with a pencil the part of the surface inside the thread, then it will take the shape of a perfect circle. Under the hypotheses of fluidity (i.e. the force density along a given curve is orthogonal to the curve) and homogeneity (i.e. the force density has constant length), Geometry predicts that the thread will become a curve of constant geodesic curvature. So, the experience confirms the validity of our model.

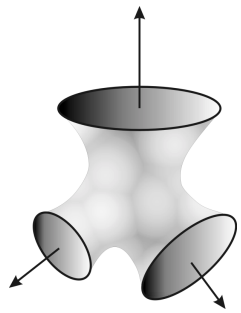

Fig. 2.1. Soap film bounded by three rigid wires

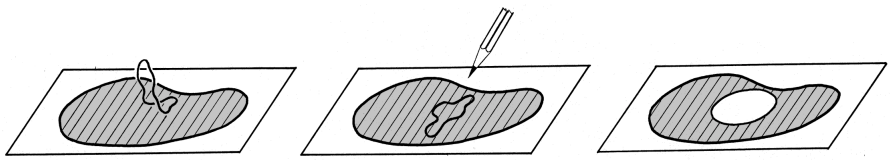

Fig. 2.2. The superficial tension experiment: a flexible boundary must be a circle

The notion of force is also useful when studying surfaces of constant (nonzero) mean curvature, see [21, 23].

Consider a conformal minimal immersion $\psi : M \to \mathbb{R}^3$. Given an oriented cycle $\gamma \subset M$, denote by $\eta = -d\psi(J\gamma')$ the conormal vector field along γ, where J stands for the complex structure of M and γ' is the derivative of γ respect to its arclength. The *force of* ψ *along* γ is defined as

$$\mathrm{Force}(\gamma) = \mathrm{Force}(\psi, \gamma) = \int_\gamma \eta \, ds.$$

When γ is the boundary of a regular domain $\Omega \subset M$, divergence Theorem gives $\mathrm{Force}(\gamma) = \int_{\partial\Omega} \eta ds = \int_\Omega \triangle\psi \, dA = 0$, thus $\mathrm{Force}(\gamma)$ does not depend on the curve in a fixed homology class. In fact, it can be viewed as the $\mathbb{R}^3$-valued cohomology class determined by the closed differential $-(d\psi) \circ J$. In particular, the force of any annular end of a minimal surface is well-defined as the force along a positively oriented generator of its homology. On the other hand, if $\Phi = d(\psi + i\psi^*)$ is the Weierstrass form of ψ (thus ψ^* is the conjugate minimal immersion, not globally well-defined), then the force of ψ along γ coincides with the period of ψ^* along the same curve:

$$i\,\mathrm{Force}(\psi, \gamma) = -i \int_\gamma d\psi(J\gamma') ds = i \int_\gamma d\psi^*(\gamma') \, ds = \mathrm{Imag} \int_\gamma \Phi. \tag{2.1}$$

Direct computation from equation (1.2) gives that the force of a planar end is zero, and the force of a catenoidal end is (up to a sign) $2\pi a N_\infty$, where a is the logarithmic growth and N_∞ the limit normal vector.

2.1 Vertical Forces.

In these notes we will prove in several situations that, under suitable global restrictions, the condition on an embedded minimal surface to have all its forces in just one direction is extremely restrictive. Let us start with another soap film experience [49], described in Figure 2.3. Consider again a planar soap film bounded by a convex rigid wire. Take two small planar convex rigid wires, suspended above the plane by two vertical flexible threads. Drop carefully both small wires on the soap film and break with a pencil the two disks in the surface inside the small curves. Now we have a planar 2-holed soap film, and the three forces at the boundary components vanish. Pull up a bit the two threads. Then, we get a soap film with nonzero forces. At the small curves, these forces point necessarily to the direction of the threads (flexible threads only support tangent forces). If our film looks like the one in the third picture, then we will have constructed a minimal surface with vertical forces and higher connectivity than an annulus. But it is clear that what we will truly obtain is something like in the fourth picture. So, what this experience shows is that a film with vertical forces like the third one in Figure 2.3 cannot exist (of course, we can produce this soap film if we assume that our threads are rigid, but in that case forces are not vertical).

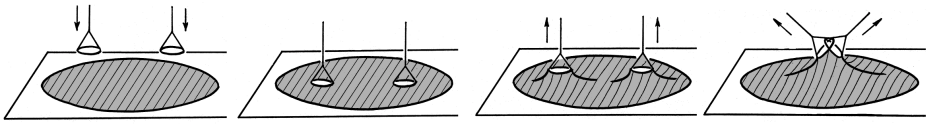

Fig. 2.3. Vertical forces experiment

We now investigate how the condition of having all the forces in just one direction (say vertical) affects to the minimal surface.

Using the Weierstrass data (g, ϕ), equation (2.1) reduces to

$$\text{Force}(\gamma) = \text{Imag} \int_\gamma (\frac{1}{2}(g^{-1} - g), \frac{i}{2}(g^{-1} + g), 1)\phi.$$

Thus, Force(γ) is vertical if and only if $\int_\gamma g\phi = \int_\gamma g^{-1}\phi = 0$. This fact occurs for any cycle $\gamma \subset M$ (in which case we will say that ψ *has vertical forces*) if and only if both $g\phi, g^{-1}\phi$ are exact holomorphic differentials on M, i.e. the maps F, G defined in (1.1) are univalent.

2.1.1 The Deformation.

The condition for a minimal immersion $\psi : M \to \mathbb{R}^3$ to have vertical forces does not guarantee that the conjugate minimal immersion ψ^* is well-defined

on M: both $g\phi, g^{-1}\phi$ are exact but ϕ need not to be exact. Nevertheless, verticality of the forces of ψ is equivalent to the existence of a one-parameter deformation via Weierstrass representation: for each positive number λ, consider on M the meromorphic map $g_\lambda = \lambda g$. Then, the pair (g_λ, ϕ) determines a well-defined map $\psi_\lambda : M \to \mathbb{R}^3$ by

$$\psi_\lambda(z) = \left(\frac{1}{2} \left(\overline{\int^z g_\lambda^{-1}\phi} - \int^z g_\lambda \phi \right), \mathrm{Real} \int^z \phi \right) = \left(\frac{1}{\lambda}\overline{F} - \lambda G, x_3 \right),$$

where x_3 is the third coordinate function of ψ. The deformation $\{\psi_\lambda\}_{\lambda>0}$ seems to be firstly considered by Goursat [12]. On the Catenoid (which has vertical forces), this deformation gives again the Catenoid after the change of parameter $\xi = \lambda z$:

$$\begin{aligned} \mathrm{Catenoid}_\lambda &= \left(M = \mathbb{C} - \{0\}, g_\lambda(z) = \lambda z, \phi = \frac{dz}{z} \right) \\ &= \left(g_\lambda(\xi) = \xi, \phi = \frac{d\xi}{\xi} \right) = \mathrm{Catenoid}. \end{aligned}$$

We now study some basic properties of the λ-deformation. From now on, $\psi : M \to \mathbb{R}^3$ will denote a nonflat minimal immersion with vertical forces. Firstly observe that the third coordinate function and the set of points in M where the normal vector is vertical do not depend on λ. Moreover, as the induced metric and the Gauss curvature of ψ are respectively given by

$$ds^2 = \frac{1}{4}(|g|^{-1} + |g|)^2|\phi|^2, \qquad K = -\frac{4}{(|g|^{-1} + |g|)^4}\frac{|dg/g|^2}{|\phi|^2},$$

it follows easily that if ds^2 is complete (resp. ψ has finite total curvature) then the same holds for ds^2_λ (resp. for ψ_λ). Moreover, if ψ has finite total curvature, then the order of zeros and poles of the extended Gauss map γ is also preserved, thus planar ends transform into planar ends with the same height, and catenoidal ends transform into catenoidal ends with the same logarithmic growth.

The next two Lemmas state key properties of the λ-deformation in terms of embeddedness.

Lemma 2.1.1. *If $p \in M$ is a point where the normal vector is vertical, then for any neighborhood D of p there exists $\lambda > 0$ such that $\psi_\lambda|_D$ is not one-to-one.*

Proof. Assume $N(p) = (0, 0, -1)$ and take a conformal coordinate z centered at p such that the Weierstrass data of ψ are written as $g(z) = z^k$, $\phi = z^k(a + zh(z))\,dz$ in $|z| < \varepsilon$, k being a positive integer, a a nonzero complex number and h a holomorphic function. In order to study ψ around p we

consider the new conformal coordinate $\xi = \lambda^{1/k} z$ defined on $|\xi| < \lambda^{1/k}\varepsilon$. Then, ψ_λ is determined by

$$g_\lambda(\xi) = \xi^k, \qquad \phi_\lambda = \frac{\xi^k}{\lambda^{1+\frac{1}{k}}}\left(a + \frac{\xi}{\lambda^{1/k}} h\left(\frac{\xi}{\lambda^{1/k}}\right)\right) d\xi.$$

Now we expand homothetically ψ_λ with factor $\lambda^{1+\frac{1}{k}}$, obtaining a new minimal immersion $\lambda^{1+\frac{1}{k}}\psi_\lambda$ with Weierstrass data $(g_\lambda, \lambda^{1+\frac{1}{k}}\phi_\lambda)$. When $\lambda \to +\infty$, $\lambda^{1+\frac{1}{k}}\psi_\lambda$ converges uniformly on compact sets of $\mathbb{C}$ to the minimal immersion whose Weierstrass data are $g_\infty(\xi) = \xi^k$, $\phi_\infty = a\xi^k\, d\xi$, $\xi \in \mathbb{C}$. This limit surface is complete and has a nonembedded end at infinity, thus it has transversal self-intersections. So, the same holds for ψ_λ with λ large enough. If $N(p) = (0,0,1)$ the proof is analogous.

Lemma 2.1.2. *If $\psi : M \to \mathbb{R}^3$ has a (horizontal) planar end, then for any representative D of this end there exists $\lambda > 0$ such that $\psi_\lambda|_D$ is not one-to-one.*

Proof. A planar end is represented by the complex data $g(z) = z^k$, $\phi = z^k\left(\frac{a}{z^2} + h(z)\right) dz$, $0 < |z| < \varepsilon$, where $k \geq 2$ is an integer, $a \in \mathbb{C} - \{0\}$ and h is a holomorphic function. The same argument as in the proof of Lemma 2.1.1 applied to these data gives a limit surface determined by $g_\infty(\xi) = \xi^k$, $\phi_\infty = \frac{a}{\xi^{2-k}}\, d\xi$ with $\xi \in \mathbb{C} - \{0\}$. This surface has a nonembedded end at infinity and we conclude as before.

2.1.2 A Characterization of the Catenoid.

We are ready to prove the main results of this Section, Theorem 2.1.1 and Corollary 2.1.1. They were firstly obtained by López and Ros [28], and reproved by Pérez and Ros [41]. Here we will follow the arguments in [41].

Proposition 2.1.1. *If $\psi : M \to \mathbb{R}^3$ is a properly embedded minimal surface with finite total curvature and vertical forces, then ψ_λ is embedded for any $\lambda > 0$.*

Proof. Define $\mathcal{L} = \{\lambda > 0 \ / \ \psi_\lambda \text{ is an embedding }\}$, thus $1 \in \mathcal{L}$.

Let us see that $\mathcal{L}$ is open: If $\lambda_0 \in \mathcal{L}$, then ψ_{λ_0} is embedded and, in particular, its ends are ordered by heights. We have seen that planar ends transform into planar ends with the same asymptotic plane through the deformation, and that catenoidal ends go to catenoidal ends with the same logarithmic growth. In other words, the asymptotic halfcatenoid to each one of these ends changes by translation but not by homotheties. Moreover, the distance between ends depends continuously on λ. It follows that ψ_λ is embedded outside a compact set for λ close to λ_0. On the other hand, the dependence of (g_λ, ϕ) respect to λ gives uniform convergence on compact sets of ψ_λ to ψ_{λ_0}

as $\lambda \to \lambda_0$. As consequence, ψ_λ is embedded inside a compact set for λ close to λ_0. Both facts imply the openness of $\mathcal{L}$.

The Proposition will be proved if we check that $\mathcal{L}$ is closed. Take a sequence $\{\lambda_k\}_k \subset \mathcal{L}$ converging to $\lambda_0 > 0$. The embeddedness of ψ_{λ_k} insures that ψ_{λ_0} has no transversal self-intersections. Reasoning by contradiction, if ψ_{λ_0} is not an embedding, then the maximum principle implies that ψ_{λ_0} is a covering (with more than one sheet) of its image set $\psi_{\lambda_0}(M) \subset \mathbb{R}^3$, and this last one is a properly embedded minimal surface. Moreover this covering is finitely sheeted, because our surfaces have finite total curvature. Thanks to the maximum principle at infinity we can take an embedded regular neighborhood U of $\psi_{\lambda_0}(M)$. Let $\pi : U \to \psi_{\lambda_0}(M)$ and $l : U \to \mathbb{R}$ be the orthogonal projection and the oriented distance to the central surface $\psi_{\lambda_0}(M)$, respectively. As $\{\lambda_k\}_k$ converges to λ_0, we have $\psi_{\lambda_k}(M) \subset U$ for k large enough thus $\pi \circ \psi_{\lambda_k} : M \to \psi_{\lambda_0}(M)$ is a proper local diffeomorphism and, hence, a finite covering. As ψ_{λ_k} is embedded, $l \circ \psi_{\lambda_k}$ is a continuous function that separates the points in the fibers of the covering $\pi \circ \psi_{\lambda_k}$, thus this covering has only one sheet. Finally, the uniform convergence of $\pi \circ \psi_{\lambda_k}$ to $\pi \circ \psi_{\lambda_0} = \psi_{\lambda_0}$ gives that $\psi_{\lambda_0} : M \to \psi_{\lambda_0}(M)$ has also one sheet, a contradiction.

Theorem 2.1.1 ([28, 41]). *The unique properly embedded minimal surfaces in $\mathbb{R}^3$ with finite total curvature and vertical forces are planes and Catenoids.*

Proof. Suppose that $\psi : M \to \mathbb{R}^3$ is a nonflat proper minimal embedding with finite total curvature and vertical forces. From Lemmas 2.1.1, 2.1.2 and Proposition 2.1.1, it follows that M has neither points with vertical normal vector nor planar ends. Thus, the vertical coordinate of M is proper and has no critical points. This implies that M is an annulus and consequently (see Subsection 1.0.2), it must be a Catenoid.

As a consequence of the above theorem, we can give all the surfaces in our family with the smallest genus. The only observation to be made is that in the genus zero setting, all forces are vertical because the curves around the ends have necessarily vertical forces.

Corollary 2.1.1 ([28, 41]). *The plane and the Catenoid are the only properly embedded minimal surfaces with finite total curvature and genus zero.*

The main ingredients in Alexandrov reflection technique and in the λ-deformation are the existence of a simple 1-parameter family of minimal surfaces associated to the given surface and the maximum principle for minimal surfaces. An important difference between Theorem 1.0.2 (proved by using Alexandrov technique) and Theorem 2.1.1 is that Schoen's result does not require embeddedness of the surface, only embedded ends are needed. However, the uniqueness of the Catenoid among surfaces with (nonzero) vertical forces holds only for embedded surfaces and does not extend to immersed ones: there are genus zero surfaces with more than six ends which are embedded outside a compact set, i.e. they have the same asymptotic behavior as an

embedded surface, see [26]. There are also genus zero surfaces with two horizontal catenoidal ends and an almost horizontal planar end which are almost embedded: the only problem is that the planar end intersects the other ends, see [14]. Moreover, the planar end can be taken arbitrarily horizontal but the limit surface does not exist, or, more precisely, the limit surface consists of a pair of vertical Catenoids.

A basic conjecture of Hoffman and Meeks asserts that for any given genus, there are no embedded minimal examples with that genus and a large number of ends. More precisely, they conjecture that for any surface in our family, the number of ends is not greater than the genus plus two. All known examples up to date verify this restriction, and Corollary 2.1.1 says that the conjecture is true in the genus zero case.

2.2 Other Results on Vertical Forces.

2.2.1 A Property of Planar Ends.

We have seen that the extended Gauss map is unbranched at a catenoidal end and has a branch point when the end is planar. In this last case, the simplest case would be having branch order equals one, that is $g(z) = z^2$ or $g(z) = z^{-2}$ locally. Geometrically, the intersection of such an end with the asymptotic tangent plane consists of two divergent curves asymptotic to opposite rays in that plane. Up to now, no examples of properly embedded minimal surfaces with finite total curvature with a planar end of minimum branching order have appeared. It remains an open question (see Hoffman and Karcher [14]) to decide if this situation is possible. However, there exist classical examples with this property when we allow the total curvature to be infinite, as in Riemann minimal examples. The next result, obtained by Choe and Soret, gives a partial answer to this question.

Theorem 2.2.1 ([4]). *Let M be a properly embedded minimal surface with finite total curvature in $\mathbb{R}^3$ and Π a plane. If $M \cap \Pi$ is a line, then M is flat.*

Proof. As M has finite total curvature, it is conformally equivalent to a compact Riemann surface $\overline{M}$ minus a finite number of punctures, and the Weierstrass data (g, ϕ) of M extend meromorphically to $\overline{M}$. Note that the top and bottom ends of M are necessarily of catenoidal type. Rotate M so that its ends are horizontal. As $M \cap \Pi$ is a line and M behaves at infinity as two halfcatenoids, we deduce that Π is a horizontal plane. Hence we can suppose that $\Pi = \{x_3 = 0\}$ and $M \cap \Pi = \{x_1 = x_3 = 0\}$, line which in the sequel will be denoted by L. As the level set of M at height zero is just a straight line, it follows that M has a planar end p asymptotic to $\{x_3 = 0\}$, where g has branching order one. Without loss of generality, we can assume $g(p) = 0^{2)}$. The maximum principle at infinity implies that p is the unique

planar end of M at height $x_3 = 0$. Identify L with the corresponding curve in the compactification $\overline{M}$ that crosses p, and put $\overline{L} = L \cup \{p\}$. Note that the height function x_3 is a finite differentiable function in a neighborhood of $\overline{L}$ in $\overline{M}$. On the other hand, g is real and does not take the values $0, \infty$ along L. This fact together with the nondegeneracy of the induced metric imply that ϕ has no zeros on L. As $\phi = \frac{\partial x_3}{\partial z}\, dz$, it follows that dx_3 does not vanish on L. Moreover, as g has a double zero at p and the end is planar, necessarily ϕ does not vanish at p and the same holds for dx_3. By continuity, there exists $\varepsilon > 0$ such that dx_3 does not vanish and $0 < |g| < +\infty$ on $E := \overline{M} \cap \{|x_3| \leq \varepsilon\}$. In particular, E is conformally an annulus $\{z \in \mathbb{C} \ / \ r^{-1} \leq |z| \leq r\}$ with $\log r = \varepsilon$, $p = 1$, $\overline{L} = \{|z| = 1\}$, $x_3(z) = \log |z|$ and $\phi = \frac{dz}{z}$.

As $M \cap \{x_3 = 0\}$ consists only on L, we deduce that $\{|z| = 1\}$ separates M and so the same holds for $\{|z| = r\}$. Hence $\{|z| = r\}$ is homologous in M to a combination of closed curves around the ends of M. In particular, the force of M along $\{|z| = r\}$ is vertical, or equivalently,

$$\int_{\{|z|=r\}} g\phi = \int_{\{|z|=r\}} g^{-1}\phi = 0.$$

As $g\phi$ is a holomorphic differential on E, we deduce that $\int_{\{|z|=1\}} g\phi = 0$. But this integral can be computed as

$$\int_{\{|z|=1\}} g\phi = \int_0^{2\pi} g(e^{i\theta}) \frac{de^{i\theta}}{e^{i\theta}} = i \int_0^{2\pi} g(e^{i\theta})\, d\theta,$$

which cannot be zero because g is real with constant (nonzero) sign along L. This contradiction proves the Theorem.

Remark 2.2.1. The proof above works without changes if $M \cap \Pi$ is not a line but a global graph over a line $L \subset \Pi$ which is asymptotic to L at infinity.

2.2.2 Singly Periodic Minimal Surfaces.

A connected, proper minimal surface $\widetilde{\psi} : \widetilde{M} \to \mathbb{R}^3$ is said to be *periodic* if it is invariant by a group G of isometries that acts freely on $\mathbb{R}^3$. The usual approach to periodic minimal surfaces consists of studying them in the quotient space $\mathbb{R}^3/G$, which is a connected, complete, nonsimply connected, flat 3-manifold. It follows from the classification of flat 3-manifolds that $\mathbb{R}^3/G$ is finitely covered by $\mathbb{R}^3/S_\theta$, $\mathbb{T}^2 \times \mathbb{R}$ or $\mathbb{T}^3$, where S_θ is a right-hand screw motion obtained by rotation around the positive x_3-axis by angle $0 \leq \theta < 2\pi$ followed by a nontrivial translation along the x_3-axis, $\mathbb{T}^2$ is a flat 2-torus and $\mathbb{T}^3$ is a flat 3-torus. Any periodic minimal surface $\widetilde{\psi} : \widetilde{M} \to \mathbb{R}^3$ induces in a natural way a properly embedded minimal surface $\psi : M = \widetilde{M}/G \to \mathbb{R}^3/G$. Conversely, any proper nonflat minimal embedding $\psi : M \to \mathbb{R}^3/G$ lifts to a connected periodic minimal surface in $\mathbb{R}^3$ by the strong halfspace Theorem

of Hoffman and Meeks [18]. Thus, the theory of periodic minimal surfaces reduces to the study of properly embedded minimal surfaces in $\mathbb{R}^3/S_\theta$, $\mathbb{T}^2 \times \mathbb{R}$ or $\mathbb{T}^3$. Surfaces in these ambient spaces are respectively called *singly, doubly* and *triply periodic minimal surfaces.* As $\mathbb{T}^3$ is compact, triply periodic minimal surfaces are also compact. In the noncompact setting, it holds a strong relationship between the topology and the total curvature of a minimal surface:

Theorem 2.2.2 (Meeks,Rosenberg [34, 36]). *A properly embedded minimal surface M in $\mathbb{R}^3/S_\theta$ or $\mathbb{T}^2 \times \mathbb{R}$ has finite total curvature if and only if it has finite topology.*

The Helicoid shows that Theorem 2.2.2 does not hold when the ambient space is $\mathbb{R}^3$.

Next we will obtain some results for singly periodic minimal surfaces, based in the λ-deformation. It is also proved in [36] that the behavior at infinity of a properly embedded minimal surface $\psi : M \to \mathbb{R}^3/S_\theta$ is one of the followings:

1. All the ends of ψ are asymptotic to nonvertical parallel planes, as in the Riemann minimal examples described below. These ends lift to planar ends in $\mathbb{R}^3$. If $\theta \neq 0$, then the ends are necessarily horizontal.
2. All the ends of ψ are asymptotic to flat vertical annuli, like in Scherk's simply periodic surface $(M = \mathbb{C} - \{1, -1, i, -i\}, g(z) = z, \phi = i\frac{4z}{z^4-1}\,dz)$, see Figure 2.4. For this reason, this type of ends are usually called *Scherk type ends.* This case occurs only if θ is a rational number.

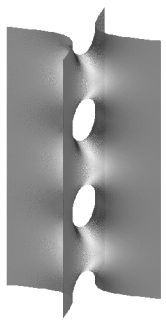

Fig. 2.4. Singly periodic Scherk's surface

3. All the ends of ψ are asymptotic to ends of helicoids. These helicoids have the same slope up to sign, the same winding number and their limit tangent planes at the ends are horizontal. In this case we will say that the ends are of *helicoidal type*.

The lifting to $\mathbb{R}^3$ of a properly embedded minimal surface of genus zero[1] and a finite number of planar ends in $\mathbb{R}^3/S_\theta$ is a properly embedded minimal surface in $\mathbb{R}^3$ with a finite number of ends, all of planar type. Thus, a coordinate function must be bounded on the lifted surface, and consequently, it must be constant. This elementary argument shows that the only properly embedded minimal surface of genus zero and a finite number of planar ends in $\mathbb{R}^3/S_\theta$ is the plane. Hence, the simplest nontrivial classification problem for this type of ends appears when the genus of M is one. The expected answer to this problem is that the only surfaces of this kind are the Riemann minimal examples. These surfaces consist of a 1-parameter family $\{R_t\}_{t>0}$, each one with the following properties: R_t is foliated by circles and straight lines in horizontal planes[2], it intersects horizontal planes in straight lines at precisely integer heights, with the radii of the circles going to infinity near the lines, it is invariant under reflection in the (x_1, x_3)-plane and by translation of vector $\mathbf{T}_t = (t, 0, 2)$, and it is conformally a vertical cylinder in $\mathbb{R}^3$ punctured at integer heights. In the quotient space $\mathbb{R}^3/\mathbf{T}_t$, the surface $R_t/\mathbf{T}_t$ has genus one and two ends asymptotic to planes (note that in the notation $\mathbb{R}^3/S_\theta$ with $\theta = 0$ it is assumed that the translation vector is vertical; nevertheless in our presentation of R_t, the translation vector is not vertical and the planar ends are horizontal), see Figure 2.5. After a long list of partial results, the

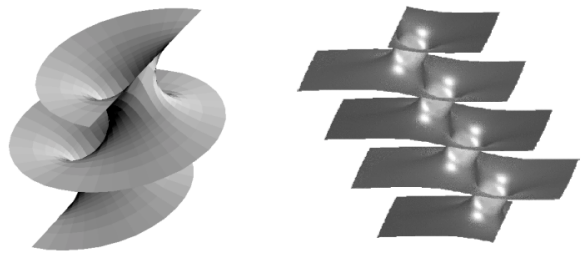

Fig. 2.5. Two views of a Riemann minimal example

complete classification of singly periodic minimal tori with a finite number of planar ends was finally proved by Meeks, Pérez and Ros in [33]. Next we will give a partial result in this problem.

[1] As in the case of surfaces with finite total curvature in $\mathbb{R}^3$, we define the *genus* of a minimal surface M with finite topology in $\mathbb{R}^3/S_\theta$ or $\mathbb{T}^2 \times \mathbb{R}$ as the genus of its compactification.

[2] In fact, Riemann [45] showed these surfaces within the classification of all minimal surfaces in $\mathbb{R}^3$ that are foliated by circles and straight lines in horizontal planes. He proved that the only surfaces in this family are planes, Catenoids, Helicoids and the 1-parameter family $\{R_t\}_{t>0}$.

Theorem 2.2.3 ([41]). *There are no properly embedded minimal tori with a finite number of planar ends in* $\mathbb{R}^3/S_\theta$, $0 < \theta < 2\pi$.

Proof. Let $\psi : M \to \mathbb{R}^3/S_\theta$ be a proper minimal embedding with a finite number of planar ends, M being a finitely punctured torus. As above, let us call $\widetilde{\psi} : \widetilde{M} \to \mathbb{R}^3$ to the lifted singly periodic surface. Embeddedness implies that the ends are horizontal. Moreover, the infinite cyclic covering map $\widetilde{M} \to M$ extends in an unbranched way through the ends. Hence, $\widetilde{M}$ is conformally the cylinder $\mathbb{C} - \{0\}$ minus a infinite number of punctures corresponding to the planar ends of $\widetilde{\psi}$. As the third coordinate of $\widetilde{\psi}$ is well-defined up to translations on M, it follows that the holomorphic differential $\phi = \frac{\partial x_3}{dz}$ is globally well-defined on M and it extends holomorphically to the torus [36]. But the holomorphic differential is unique on a given torus, thus the third coordinate function of $\widetilde{\psi}$ must is given by $x_3 = \log|z|$ and its height differential must be $\phi = \frac{dz}{z}$ on $\mathbb{C} - \{0\}$.

We claim that $\widetilde{\psi}$ has vertical forces. In order to prove this fact, we only need to consider the force along the homology generator of $\mathbb{C} - \{0\}$ (the force vanishes at planar ends). This generator can be represented by any horizontal section Γ of $\widetilde{\psi}(\widetilde{M})$ whose height does not coincide with the one of any planar end. Denote by S the conformal transformation of M induced by S_θ, i.e. $S_\theta \circ \widetilde{\psi} = \widetilde{\psi} \circ S$. Then, $S(\Gamma)$ is another planar section and $\Gamma \cup S(\Gamma)$ bounds a domain in the compactified surface, hence the force of $\widetilde{\psi}$ along $S(\Gamma)$ coincides with the one along Γ. On the other hand, the conormal vector fields along the curves Γ and $S(\Gamma)$ differ by rotation R_θ of angle θ around the x_3-axis, thus the definition of force implies that $\text{Force}(S(\Gamma)) = R_\theta\,(\text{Force}(\Gamma))$. As $0 < \theta < 2\pi$, we deduce that the force along Γ must be vertical and our claim is proved.

As $\widetilde{\psi}$ has vertical forces, the deformation $\{\widetilde{\psi}_\lambda\}_{\lambda>0}$ is well-defined on $\widetilde{M}$. Similar arguments to the ones in the proof of Proposition 2.1.1 show that $\widetilde{\psi}_\lambda$ is embedded for all $\lambda > 0$, so we have a contradiction with Lemma 2.1.2.

The λ-deformation is also useful for singly periodic surfaces with helicoidal type ends:

Theorem 2.2.4 ([40]). *The Helicoid is the unique properly embedded minimal surface in* $\mathbb{R}^3/S_\theta$, $0 \le \theta < 2\pi$, *with finite topology, helicoidal type ends and whose lifting to* $\mathbb{R}^3$ *has vertical forces.*

Proof. Take a proper minimal embedding $\psi : M \to \mathbb{R}^3/S_\theta$ with finite topology and helicoidal type ends, and label its periodic lifting by $\widetilde{\psi} : \widetilde{M} \to \mathbb{R}^3$. The invariance of $\widetilde{\psi}$ translates into $S_\theta \circ \widetilde{\psi} = \widetilde{\psi} \circ S$, for a conformal transformation S of $\widetilde{M}$. Let us denote by (g, ϕ) the Weierstrass data of $\widetilde{\psi}$. As the third coordinate x_3 of $\widetilde{\psi}$ satisfies $x_3 + c = x_3 \circ S$ for a fixed constant $c \in \mathbb{R} - \{0\}$, it follows that ϕ is well-defined on $M = \widetilde{M}/S$. Nevertheless, the Gauss map g differs in a rotation by angle θ around the x_3-axis at points

related by S. Meeks and Rosenberg [36] described conformally any helicoidal type end E of ψ as follows: E can be conformally parametrized by a punctured disk $D^*(\varepsilon) = \{z \in \mathbb{C} \ / \ 0 < |z| \leq \varepsilon\}$, the height differential ϕ can be written on $D^*(\varepsilon)$ as $\phi = \left(-\frac{i\beta}{z} + f(z)\right) dz, \quad 0 < |z| \leq \varepsilon$, where β is a nonzero real number called the *slope* of the end, f is a holomorphic function in $D(\varepsilon) = D^*(\varepsilon) \cup \{0\}$, and

i) If $\theta > 0$, the Gauss map g is multivalued on E but it can be continuously extended to $z = 0$, with vertical limit normal vector. If $g(0) = 0$, we can write $g(z) = z^{k+a}$, k being a nonnegative integer and $a = \frac{\theta}{2\pi}$. If $g(0) = \infty$, then the expression of g is $g(z) = z^{-(k+a)}$, with k, a as above.

ii) If $\theta = 0$, g is univalent on A and the remaining assertions in case *i)* hold with $a = 0$.

The trace of the end on a vertical cylinder C_R of very large radius R is extremely close to a helix of slope β that rotates an angle $2\pi(k + a)$ when $0 \leq \arg(z) < 2\pi$. Thus, all the slopes at the ends ψ are equal up to sign —two slopes coincide if and only if the vertical limit normal vector at the ends are the same— and the number k in the expression of g above does not depend on the end.

Assume that $\widetilde{\psi}$ has vertical forces. Thus, the deformation $\{\widetilde{\psi}\}_{\lambda>0}$ is well-defined on $\widetilde{M}$.

Assertion 1: *All the $\widetilde{\psi}_\lambda$'s are invariant by the same S_θ.*

Consider a helicoidal type end E of ψ, parametrized by $D^*(\varepsilon)$ and with Weierstrass data (g, ϕ) as before (we assume $g(0) = 0$). Using the new conformal parameter $w = \lambda^{\frac{1}{k+a}} z$, it follows that

$$g_\lambda(w) = w^{k+a}, \qquad \phi_\lambda = \phi = \left(-\frac{i\beta}{w} + \lambda^{\frac{-1}{k+a}} f(\lambda^{\frac{-1}{k+a}} w)\right) dw,$$

thus (g_λ, ϕ) is the Weierstrass data of a helicoidal type end in the same space $\mathbb{R}^3/S_\theta$, which horizontal limit tangent plane and the same slope β as in the original case $\lambda = 1$. This fact suffices to prove our assertion.

Assertion 2: *$\widetilde{\psi}_\lambda$ is an embedding, $\forall \lambda > 0$.*

Again the proof is analogous to the arguments in Proposition 2.1.1, but reasoning in the quotient space $\mathbb{R}^3/S_\theta$ (where maximum principle at infinity still holds).

Now we finish the proof of Theorem 2.1.1: As $\widetilde{\psi}_\lambda$ is an embedding for all $\lambda > 0$, Lemma 2.1.1 guarantees that the (possibly multivalued) Gauss map of ψ does not take vertical values on M. As ϕ has a simple pole at each end of ψ, we conclude that ϕ has no zeros on the compactified surface $\overline{M}$ obtained by attaching the punctures to M. As the Euler characteristic of $\overline{M}$ is $\chi(\overline{M}) = \#\{\text{poles of } \phi\} - \#\{\text{zeros of } \phi\}$, $\chi(\overline{M})$ must be positive and so, $\overline{M}$ is topologically a sphere and ϕ has just two poles, or equivalently, ψ has two

ends. Finally, Toubiana [54] characterized the Helicoid as the only properly embedded minimal surface in $\mathbb{R}^3/S_0$ with genus zero and two helicoidal type ends, and later Meeks and Rosenberg [36] extended this result to the case $0 < \theta < 2\pi$, so our Theorem follows from both characterizations.

Corollary 2.2.1 ([41]). *The Helicoid is the unique properly embedded minimal surface in $\mathbb{R}^3/S_0$ with genus zero and any finite number of helicoidal type ends.*

Proof. By Theorem 2.2.4, it suffices to prove that if $\psi : M \to \mathbb{R}^3/S_0$ is a proper minimal embedding with genus zero and a finite number of helicoidal type ends, then the singly-periodic lifting $\widetilde{\psi} : \widetilde{M} \to \mathbb{R}^3$ of ψ has vertical forces. Let (g, ϕ) be the Weierstrass representation of $\widetilde{\psi}$. As $\widetilde{\psi}$ is invariant by a translation, we have that not only ϕ is well-defined on M, but also g. The above description of helicoidal type ends shows that $g^{-1}\phi$ is holomorphic on the compactification $\overline{M} = \mathbb{C} \cup \{\infty\}$ except at the ends p_i where $g(p_i) = 0$, where it has a pole without residue. Symmetrically, $g\phi$ has no residue at the ends p_i with $g(p_i) = \infty$ and is holomorphic everywhere else. The genus zero hypothesis implies that both $g^{-1}\phi$ and $g\phi$ are exact one-forms on M and consequently, they are also exact on $\widetilde{M}$. This fact is equivalent to the verticality of the forces of $\widetilde{\psi}$.

We will finish this Subsection by proving another characterization of the Helicoid, now in terms of rigidity. There are two natural notions of rigidity for minimal surfaces:

Definition 2.2.1. *A properly embedded minimal surface $M \subset \mathbb{R}^3$ is said to be (minimally)* strongly rigid *if the inclusion map represents the unique isometric minimal immersion of such a surface up to a rigid motion in $\mathbb{R}^3$. Under the same conditions, M is called* weakly rigid *if every intrinsic isometry extends to an isometry of $\mathbb{R}^3$.*

Strong rigidity implies weak rigidity but the converse fails, as demonstrates the Helicoid. Meeks [31] has conjectured that any nonsimply connected properly embedded minimal surface in $\mathbb{R}^3$ is strongly rigid. On the other hand, every minimal surface can be locally and isometrically deformed by its associate surfaces (defined by the Weierstrass data $(g_\theta(z) = g(z), \phi_\theta = e^{i\theta}\phi$ if (g, ϕ) represents the original surface), thus strong rigidity questions must be understood globally. In this setting, Choi, Meeks and White [5] proved that any properly embedded minimal surface with more than one end is strongly rigid. This property does not extend to the one-ended case, because of the Helicoid. Meeks and Rosenberg [31, 34] obtained strong rigidity if the symmetry group of the surface contains two linearly independent translations, hence doubly and triply periodic properly embedded minimal surfaces are strongly rigid. Again the Helicoid shows that this statement fails to hold if we only impose that the symmetry group contains an infinite cyclic group.

In this singly-periodic case, Meeks [31] proved that when the induced quotient surface has finite topology, then the lifted surface in $\mathbb{R}^3$ is weakly rigid (in particular, the Helicoid is weakly rigid but it is not strongly rigid). Next result characterizes the Helicoid as the unique surface in this family where (strong) rigidity fails. More precisely,

Theorem 2.2.5 ([31, 34, 40]). *Let $\widetilde{M} \subset \mathbb{R}^3$ be a nonflat, properly embedded minimal surface invariant by an infinite discrete group G of isometries of $\mathbb{R}^3$, such that $\widetilde{M}/G$ has finite topology. Then, $\widetilde{M}$ is strongly rigid or it is the Helicoid.*

Proof. The Calabi-Lawson characterization [1, 25] insures that whenever $\psi_1, \psi_2 : \widetilde{M} \to \mathbb{R}^3$ are isometric minimal immersions, ψ_2 must be congruent to an associate surface of ψ_1. In particular, if $\widetilde{M} \subset \mathbb{R}^3$ is supposed to be nonrigid, then its associate immersions are well-defined on M. This fact can be expressed equivalently by saying that the three components of the Weierstrass form of $\widetilde{M}$ are exact, or that the force of $\widetilde{M}$ along any closed curve is zero.

Firstly suppose that $\widetilde{M}$ is doubly or triply periodic, so it is invariant by a group G of translations that contains a subgroup G_1 of rank two. Up to a rotation, we can assume that G_1 is generated by two independent translations in the (x_1, x_2)-plane. Thus, $M_1 = \widetilde{M}/G_1$ is a nonflat properly embedded minimal surface (possibly with infinite topology) in $\mathbb{R}^3/G_1 = \mathbb{T}^2 \times \mathbb{R}$ and the third coordinate x_3 is well-defined on M_1. We claim that M_1 cannot be an annulus. This claim will be proved if we check that the induced mapping $i_* : \pi_1(M_1) \to \pi_1(\mathbb{T}^2 \times \mathbb{R})$ is surjective. With this aim, take a loop $\gamma \subset \mathbb{T}^2 \times \mathbb{R}$ with base point in M_1. The lifting $\widetilde{\gamma} \subset \mathbb{R}^3$ of γ is an open arc starting and ending at points in $\widetilde{M}$ that differ in a translation in G_1. As $\widetilde{M}$ is connected, there is an arc $\widetilde{\alpha} \subset \widetilde{M}$ with the same ends points that $\widetilde{\gamma}$. As $\pi_1(\mathbb{R}^3) = 1$, $\widetilde{\alpha}$ and $\widetilde{\gamma}$ must be homotopic. Coming back to the quotient surface M_1, it follows that γ must be homotopic to the loop $\alpha \subset M_1$ obtained after projection of $\widetilde{\alpha}$. Now our claim is proved.

As M_1 is connected, is not an annulus and $x_3 : M_1 \to \mathbb{R}$ is proper, it follows that x_3 must have at least one critical point $p_1 \in M_1$, whose height can be assumed to be zero. The intersection $\Gamma = M_1 \cap (\mathbb{T}^2 \times \{0\})$ is a singular analytic 1-dimensional set and p_1 is one of its singularities, where Γ forms an equiangular system of curves. We now claim that at least one of the components of $(\mathbb{T}^2 \times \{0\}) - \Gamma$ must be a disk. Using Gauss-Bonnet formula, the sum of the Euler characteristics of the domains in $(\mathbb{T}^2 \times \{0\}) - \Gamma$ equals the sum of the external angles of Γ at its singularities. Each singular point in Γ contributes at least with 2π, hence at least one domain in $(\mathbb{T}^2 \times \{0\}) - \Gamma$, say D, has positive Euler characteristic, which proves the claim. As D lifts to a disk in $\mathbb{R}^3$, it follows that the boundary ∂D is a closed curve in M that lifts to a closed oriented curve $\beta \subset \mathbb{R}^3$. We claim that the third component of the conormal vector of $\widetilde{M}$ along β does not change sign: This is clear around the

points of β where $\widetilde{M}$ and $x_3 = 0$ meet transversally. At the points where they meet tangentially, it follows from the local picture of $\widetilde{M} \cap \{x_3 = 0\}$ around the point. We deduce that the force of $\widetilde{M}$ along β must be nonzero, which finishes the proof in the doubly and triply periodic cases.

If $\widetilde{M}$ is singly periodic with planar ends, choose a plane $\Pi \subset \mathbb{R}^3$ parallel to the ends, whose relative height respect to the limit normal direction does not coincide with the height of any end. Thus, $\widetilde{M} \cap \Pi$ must be compact. We can also suppose that Π is transversal to $\widetilde{M}$, after a slight parallel translation. Then, $\widetilde{M} \cap \Pi$ is a finite union of compact planar curves which separates $\widetilde{M}$, so the conormal vector along this cycle points into one of the halfspaces determined by Π. Any component of $\widetilde{M} \cap \Pi$ will be valid as the closed curve in $\widetilde{M}$ with nonzero force that we are looking for.

If $\widetilde{M}$ is singly periodic with (vertical) Scherk type ends, choose a vertical plane $\Pi \subset \mathbb{R}^3$ not parallel to any end, tangent to $\widetilde{M}$ at least at a point $\widetilde{p}$. The quotient Π/G is an annulus and $\Gamma = M \cap (\Pi/G)$ is a compact singular analytic 1-dimensional set inside this annulus, $\widetilde{p} + G$ being one of the singularities of Γ. An Euler characteristic argument as before shows that Γ contains a cycle β which is null homotopic in Π/G. This cycle β lifts to a closed curve in $\widetilde{M}$ with nonzero force.

Finally, suppose that $\widetilde{M}$ is singly periodic with helicoidal type ends and it is strongly rigid. As the force of $\widetilde{M}$ along any closed curve is zero, in particular $\widetilde{M}$ has vertical forces. In this setting, Theorem 2.2.4 implies that $\widetilde{M}$ is the Helicoid. Now that proof is complete.

3. Minimal Surfaces with Vertical Forces II.

In this Section we continue studying embedded minimal surfaces by means of their force distribution. The principal tool in this analysis is again the λ-deformation, with a key difference if we compare with the arguments in Section 2: an embedded minimal surface M in $\mathbb{R}^3$ with compact boundary and vertical forces certainly admits the λ-deformation, but we can no longer expect the deformed surfaces M_λ to be free of selfintersections in $\mathbb{R}^3$. Nevertheless, one can impose suitable conditions to M to guarantee that M_λ is embedded in a certain flat 3-manifold (which in turn is immersed in $\mathbb{R}^3$), where the main part of the machinery developed in Section 2 remains valid. These arguments will allow us to make precise the idea that verticality of the forces can only coexist with annular topology. The results below are proved in Ros [48] in a situation somewhat more general. In our setting, the ideas involved become more transparent.

Along this Section, the open (resp. closed) Euclidean ball of radius R centered at the origin will be simply denoted by $B(R)$ (resp. $\overline{B}(R)$).

3.1 Immersed 3-manifolds.

Lemma 3.1.1. *Let W^3 be a 3-dimensional region properly immersed in an open ball $B \subset \mathbb{R}^3$, with a finite number of boundary components, $\partial W = S_1 \cup \ldots \cup S_m$. If each component S_i is embedded, then W is embedded.*

Proof. As W is properly immersed in B, it follows that each boundary component S_i is properly embedded in B, thus it must separate B in two regions. Moreover, W is locally at one side of S_i thus W lies locally near S_i inside one of the components of $B - S_i$. Denote by V_i the component of $B - S_i$ which lies at the opposite side of S_i respect to W, see Figure 3.1. Define $\widetilde{W}$ as the 3-manifold without boundary obtained by gluing W with each V_i along S_i, $1 \leq i \leq m$. Note that the proper immersion $\phi : W \to B$ extends naturally to a proper immersion $\widetilde{\phi} : \widetilde{W} \to B$. As a proper immersion between manifolds without boundary with the same dimension is a finite covering map, it follows from the fact that B is simply connected that $\widetilde{\phi}$ is a global diffeomorphism. In particular ϕ is an embedding, as we claimed.

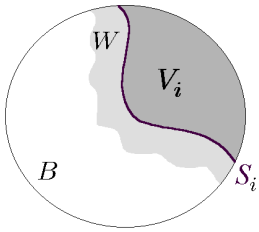

Fig. 3.1. A component S_i of ∂W together with the region V_i.

Proposition 3.1.1. *Let W^3 be a region properly immersed in a ball $B \subset \mathbb{R}^3$, with a finite number of boundary components, $\partial W = S_1 \cup \ldots \cup S_m$ such that $S_2, \ldots, S_m$ are embedded. Assume that S_1 is the interior of a compact surface $\overline{S}_1$ with connected boundary and that the immersion of S_1 into B extends to an immersion of $\overline{S}_1$ into $\overline{B}$. If the immersed curve ∂S_1 is close enough (in the smooth sense) to a n-sheeted covering of an equator in ∂B, then $n = 1$.*

Proof. By using the gluing argument in the proof of Lemma 3.1.1 we can remove the embedded boundary components and, therefore, we will assume that $\partial W = S_1$. Normalize our situation so that $B = B(1+\varepsilon)$ where ε is an small positive number. Let $\widetilde{M}$ be the pullback in W of the unit sphere $\partial B(1) \subset B$. Our hypotheses imply that $\Gamma = S_1 \cap \widetilde{M}$ is a Jordan curve whose immersed image in B is extremely close to a n-sheeted covering of an equator in $\partial B(1)$, which can be assumed to be horizontal. Let M be the connected component of $\widetilde{M}$ with $\partial M = \Gamma$. As M immerses into the unit sphere, it must be orientable and the Gauss-Bonnet Theorem applied to M with the spherical metric gives

$$2\pi\chi(M) = A(M) + \int_\Gamma k_g \, ds \geq \int_\Gamma k_g \, ds,$$

where $A(M)$ is the area of M and k_g the geodesic curvature of Γ. By hypothesis, the integral of the geodesic curvature is very small, hence $\chi(M)$ is nonnegative, which forces M to be a disk. Moreover, from the above formula one gets that $A(M)$ is close to 2π. The openness of the immersed image of M in $\partial B(1) - \Gamma$ implies that M omits one of the poles $(0, 0, \pm 1)$ of $\partial B(1)$ (otherwise M should cover almost all the sphere). After stereographic projection from the omitted pole, we get an immersion $\psi : M \to \mathbb{R}^2$ from the disk $M \equiv \{z \in \mathbb{C} \ : \ |z| \leq 1\}$ in the plane, whose boundary is close to a covering of the unit circle. The family of curves $\psi(\{|z| = t\})$ with $0 < t \leq 1$ gives a regular homotopy between the boundary of M and the curve $\psi(\{|z| = \delta\})$, which is embedded for δ small enough. Thus $n = 1$.

Corollary 3.1.1. *Let M a complete one-ended minimal surface in $\mathbb{R}^3$ with finite total curvature, and W be a 3-manifold with $\partial W = M$. If the immersion of M extends to a proper immersion of W in $\mathbb{R}^3$, then M is a plane.*

Proof. For R large, consider the part of W which is mapped into the ball $B(R)$. As M has a nice shape at infinity, after contraction by factor R^{-1} we get a figure which verifies the hypotheses of Proposition 3.1.1. Therefore, the winding number of the end is 1 and so, the end is embedded (in particular, asymptotic to a plane or a Catenoid), which implies that M is a plane.

3.2 Topological Uniqueness

Let $\psi : M \to \mathbb{R}^3$ be a minimal surface with boundary and vertical forces. So, the λ-deformation applies to ψ producing a family $\{\psi_\lambda : M \to \mathbb{R}^3\}_{\lambda>0}$ of minimal surfaces. Suppose that the plane $\{x_3 = 0\}$ meets $\psi(M)$ transversally along a boundary component Γ. Denote by Γ_λ the corresponding curve in $\psi_\lambda(M) \cap \{x_3 = 0\}$ (clearly, $\psi_\lambda(M)$ is also transversal to $\{x_3 = 0\}$ along Γ_λ).

Lemma 3.2.1. *In the above situation,*

1. *If Γ is a straight line segment, then Γ_λ is a segment parallel to Γ.*
2. *If Γ is convex, then Γ_λ is convex.*
3. *If $\psi(M)$ meets $\{x_3 = 0\}$ with constant angle along Γ, then the same holds for $\psi(M_\lambda)$ (with angle depending on λ). Moreover, Γ and Γ_λ are homothetic.*

Proof. Denote respectively by ν, ν_λ the normal vectors to Γ, Γ_λ as planar curves. These are horizontal unit vector fields pointing to the same direction as g and g_λ, viewed as plane vectors. As $g_\lambda = \lambda g$ it follows that $\nu_\lambda = \nu$, from where *1,2* follow directly. Concerning *3*, as $\psi(M)$ meets $\{x_3 = 0\}$ with constant angle along Γ, we deduce that $|g|$ is constant along Γ, thus $|g_\lambda| = \lambda|g|$ is also constant. Finally, the length elements along Γ, Γ_λ are respectively given by $ds = \frac{1}{2}(|g|^{-1} + |g|)|\phi|$, $ds_\lambda = \frac{1}{2}(\lambda^{-1}|g|^{-1} + \lambda|g|)|\phi|$, so Γ and Γ_λ are homothetic and the Lemma is proved.

We now state the main statement of this Section. For a more general result, see Theorem 2 in [48].

Theorem 3.2.1. *Let M be a compact nonflat minimal surface inside the horizontal slab $Q = \{0 \le x_3 \le 1\}$, whose boundary consists of a finite number $\Gamma_1, \ldots, \Gamma_k$ of convex curves contained in the boundary of Q. Denote by D_i the planar disk enclosed by Γ_i, $1 \le i \le k$. If $\overline{M} = M \cup D_1 \cup \ldots \cup D_k$ is an embedded surface and M has vertical forces, then M must be an annulus (thus $k = 2$).*

Proof. By the interior maximum principle, M meets the boundary of Q only along ∂M, and this intersection is transversal by the maximum principle at the boundary. On the other hand, the embeddedness of $\overline{M}$ implies that $\overline{M}$ divides $\mathbb{R}^3$ in two connected components. Denote by W the compact connected component of $\mathbb{R}^3 - \overline{M}$. Clearly, W is a flat 3-dimensional manifold

isometrically embedded in $\mathbb{R}^3$, with piecewise smooth boundary $\overline{M}$. As M has vertical forces, the deformation $\psi_\lambda : M \to \mathbb{R}^3$ is well-defined, for each $\lambda > 0$ (ψ_1 is the inclusion map). For any λ, $\psi_\lambda(M)$ is an immersed minimal surface contained in Q, with boundary $\psi_\lambda(\Gamma_1) \cup \ldots \cup \psi_\lambda(\Gamma_k)$. By Lemma 3.2.1, each $\psi_\lambda(\Gamma_i)$ is a convex curve at the same height as Γ_i. Denote by $D_{i,\lambda}$ the planar disk enclosed by $\psi_\lambda(\Gamma_i)$, $1 \leq i \leq k$. As before, the maximum principle implies that $\psi_\lambda(M)$ meets the boundary of Q only along the curves $\psi_\lambda(\Gamma_i)$, this intersection being transversal (note that two disks $D_{i,\lambda}, D_{j,\lambda}$ at the same height may overlap, as in Figure 3.2). We label $\overline{M}_\lambda$ to the piecewise smooth immersed surface $\psi_\lambda(M) \cup D_{1,\lambda} \cup \ldots \cup D_{k,\lambda}$ and $\psi_\lambda : \overline{M}_\lambda \to \mathbb{R}^3$ to the extended immersion.

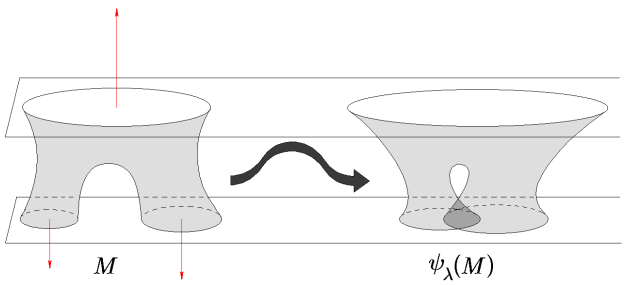

Fig. 3.2. After some time, two disks $D_{i,\lambda}, D_{j,\lambda}$ may overlap.

Denote by $\mathcal{L}$ the set of positive numbers λ for which there exists a compact flat 3-manifold W_λ with piecewise smooth boundary $\partial W_\lambda = \overline{M}_\lambda$ endowed with an isometric immersion $\phi_\lambda : W_\lambda \to \mathbb{R}^3$ which extends $\psi_\lambda : \overline{M}_\lambda \to \mathbb{R}^3$. Clearly, $1 \in \mathcal{L}$ and ϕ_1 is the inclusion map of $W_1 = W$ into $\mathbb{R}^3$.

Step 1: *Given $\lambda \in \mathcal{L}$, there exist*

i) an immersed 3-manifold $\phi_\lambda : \widetilde{W}_\lambda \to \mathbb{R}^3$ with $\partial\widetilde{W}_\lambda = \emptyset$ and $W_\lambda \subset \widetilde{W}_\lambda$, extending the immersion $\phi_\lambda : W_\lambda \to \mathbb{R}^3$, and

ii) $\varepsilon = \varepsilon(\lambda) > 0$ depending only on an upper bound of λ and $\frac{1}{\lambda}$, such that whenever $|\mu - \lambda| < \varepsilon$ the immersion $\psi_\mu : \overline{M}_\mu \to \mathbb{R}^3$ lifts to $\widetilde{W}_\lambda$, i. e. there is an immersion $f_\mu : \overline{M}_\mu \to \widetilde{W}_\lambda$ with $\psi_\mu = \phi_\lambda \circ f_\mu$. Moreover, f_μ depends smoothly on μ.

Proof of Step 1. Define $\widetilde{W}_\lambda$ as the 3-manifold obtained by gluing W_λ with an immersed half-tubular neighborhood of the immersed surface $\psi_\lambda : \overline{M}_\lambda \to \mathbb{R}^3$. This construction and the corresponding extension of ϕ_λ to $\widetilde{W}_\lambda$ is clear when $\overline{M}_\lambda$ is smooth, see Figure 3.3 (a). In the piecewise smooth case the same idea works, because each point p in the nonsmooth part of ∂W_λ has a quarterball shaped neighborhood V in W_λ such that the immersion $\psi_\lambda : \partial V \cap M_\lambda \to \mathbb{R}^3$ extends to an immersion of a whole disk around p (see Figure 3.3 (b)), which in turn has an embedded neighborhood in $\mathbb{R}^3$. This proves *i)*.

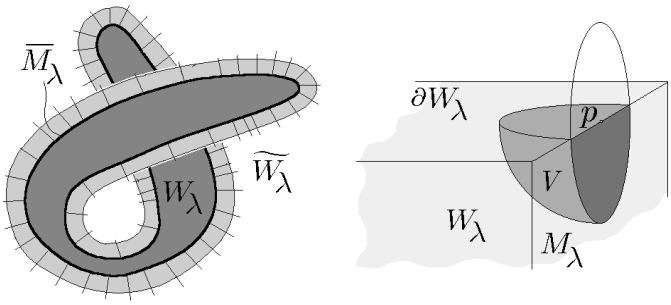

Fig. 3.3. (a) (b)

Thanks to the smoothness of ψ_λ, the compactness of $\overline{M}_\lambda$ and the embeddedness of the disks $D_{i,\lambda}$, it follows that the distance from W_λ to the boundary of $\widetilde{W}_\lambda$ can be chosen depending only on an upper bound of λ and $\frac{1}{\lambda}$. So the claim in item *ii)* follows directly, and Step 1 is finished.

Step 2: *For all $\lambda > 0$, $\overline{M}_\lambda$ encloses a compact region W_λ immersed in $\mathbb{R}^3$, i.e. $\mathcal{L} = \{\lambda > 0\}$.*

Proof of Step 2. Firstly we demonstrate that given $\lambda \in \mathcal{L}$ and $\mu > 0$ with $|\lambda - \mu| < \varepsilon(\lambda)$, the lifting f_μ is an embedding. This is clear if μ is close enough to λ. By construction of $\widetilde{W}_\lambda$, f_μ is always injective when restricted to the planar part $S = D_{1,\mu} \cup \ldots \cup D_{k,\mu}$ of $\overline{M}_\mu$. Moreover $f_\mu(S)$ and $f_\mu(M)$ meet only at their boundary (otherwise an interior point of the minimal surface $\psi_\mu(M)$ immersed in $\mathbb{R}^3$ should touch the boundary of the slab Q, which contradicts the maximum principle). Assume, reasoning by contradiction, that for some μ, $\lambda < \mu < \lambda + \varepsilon$, f_μ is not an embedding. Assume also that μ is the smallest one with this property. Therefore $f_\mu(M)$ will have a first contact point at its interior. In this situation the maximum principle, see Subsection 1.0.3, implies that any point of the image set $f_\mu(M)$ is a multiple point of f_μ. As this contradicts the fact that f_μ is injective on ∂M, we get that f_μ is an embedding, $\lambda < \mu < \lambda + \varepsilon$. The same argument works when $\lambda - \varepsilon < \mu < \lambda$.

Secondly, as $f_\mu(\overline{M}_\mu)$ is a closed surface embedded in $\widetilde{W}_\lambda$ which can be continuously deformed into $f_\lambda(\overline{M}_\lambda)$ and this last surface encloses a region in $\widetilde{W}_\lambda$, it follows that $f_\mu(\overline{M}_\mu)$ also encloses a region W_μ in $\widetilde{W}_\lambda$. We define the immersion $\phi_\mu : W_\mu \to \mathbb{R}^3$ as the restriction to W_μ of $\phi_\lambda : \widetilde{W}_\lambda \to \mathbb{R}^3$. Thus, we conclude that $|\mu - \lambda| < \varepsilon$ implies $\mu \in \mathcal{L}$.

Finally, the fact that ε depends only on an upper bound of λ and $\frac{1}{\lambda}$ allows us to show that $\mathcal{L} = \{\lambda > 0\}$. This finishes the proof of Step 2.

Step 3: *M has no points with vertical normal vector.*

Proof of Step 3. Reasoning by contradiction, suppose that there exists a point $p \in M$ with vertical normal vector, say $N(p) = (0, 0, -1)$. Up to a translation,

we will assume $p = 0 \in \mathbb{R}^3$. Take a conformal coordinate z around p such that the Weierstrass data of M are written $g(z) = z^k$, $\phi = z^k(a+zh(z))\,dz$, k being a positive integer and $a \in \mathbb{C} - \{0\}$. In the proof of Lemma 2.1.1 we saw that the rescaled images $\lambda^{1+\frac{1}{k}}\psi_\lambda$ of a neighborhood of p converge on compact sets of $\mathbb{C}$ to the Enneper-type minimal surface E given by $g(\xi) = \xi^k$, $\phi = a\xi^k\,d\xi$, $\xi \in \mathbb{C}$. We will work with the homothetic images $M'_\lambda = \lambda^{1+\frac{1}{k}}\psi_\lambda(M)$ and $W'_\lambda = \lambda^{1+\frac{1}{k}}W_\lambda$.

Take $R > 0$ large and $R' > 5R$. The asymptotic geometry of E insures that the curve $\frac{1}{R}(E \cap \partial\overline{B}(R))$ can be taken arbitrarily close to a $(2k+1)$-sheeted covering of the circle $x_1^2 + x_2^2 = 1, x_3 = 0$. As E is unstable by Theorem 1.0.4, we can also assume that $E \cap \overline{B}(R')$ is unstable.

Denote by S the connected component of $M'_\lambda \cap \overline{B}(R')$ which passes through the origin (thus S depends on λ, R'). Fix $\lambda > 0$ large enough so that S is extremely close to $E \cap \overline{B}(R')$, in particular S is unstable. As ∂S is a nulhomologous 1-cycle embedded in $\partial W'_\lambda$, Theorem 1.0.6 insures that there exists an embedded least-area surface $\Sigma \subset W'_\lambda$ with boundary $\partial\Sigma = \partial S$ (again Σ depends on λ, R'). As Σ is stable but S is unstable, it follows from the maximum principle that both surfaces meet only at their boundary. As the boundary of W'_λ is connected and mean convex, it follows from Meeks [30] that W'_λ is a handlebody. Therefore, $\Sigma \cup S$ bounds a compact region $V \subset W'_\lambda$ with piecewise smooth boundary. Note that Σ is no longer embedded when viewed into $\mathbb{R}^3$, because W'_λ immerses into $\mathbb{R}^3$.

We claim that for R and R' large enough, any component of $\Sigma \cap \overline{B}(R)$ is embedded. This property clearly holds if for arbitrary $q \in \Sigma \cap \overline{B}(R)$, the component of $\Sigma \cap B(5R)$ through q contains a graph over a disk of radius $2R$: As Σ is stable, the length of its second fundamental form can be bounded above by $|A_\Sigma|(q) \leq C\,d(q, \partial B(R'))^{-1}$ for arbitrary $q \in \Sigma$ and some universal constant C, see Theorem 1.0.5. Hence, $|A_\Sigma| \leq \frac{C}{R'-5R}$ in $\Sigma \cap B(5R)$. In this setting, the Uniform Graph Lemma (Lemma 4.1.1 below) insures that that for arbitrary $q \in \Sigma \cap B(5R)$, the component of $\Sigma \cap B(5R)$ through q contains a graph over a disk in its tangent plane of radius $r(q) = \min\left\{\frac{R'-5R}{4C}, 2R\right\}$, because the euclidean distance from q to $\partial B(5R)$ is al least $4R$. If we assume from the beginning of the proof that $R' > (5+8C)R$, then the above minimum is not less than $2R$ and our claim holds.

Note also that, possibly after a perturbation of R, we can suppose that Σ cuts $\partial\overline{B}(R)$ transversally, so the number of components of $\Sigma \cap \partial\overline{B}(R)$ is finite. Finally, the region $V \cap B(R)$ is properly immersed in the ball $B(R)$, with a finite number of boundary components. The component $S \cap \overline{B}(R)$ is a properly immersed closed disk. Moreover, after rescaling by the factor $\frac{1}{R}$, the boundary of this disk can be taken arbitrarily close to a $(2k+1)$-sheeted covering of the horizontal equator in $\partial B(1)$. The remaining boundary components of $V \cap B(R)$ come from portions of $\Sigma \cap \overline{B}(R)$ and thus all of them are embedded. By Proposition 3.1.1, $2k+1$ must be one. This contradiction finishes Step 3.

As a direct consequence of the Claim in Step 3 it follows that M must be an annulus and, so, the Theorem is proved.

Remark 3.2.1. The hypotheses in Theorem 3.2.1 can be relaxed to impose that the force of any cycle in M which is nulhomologous in W is vertical. Moreover, the surface $\overline{M}$ needs not to be embedded but only to be the (piecewise smooth mean convex) boundary of an immersed compact 3-manifold W like in Figure 3.4, see Theorem 2 in [48].

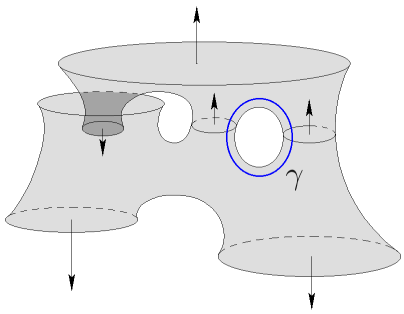

Fig. 3.4. The force along γ need not to be vertical.

Remark 3.2.2. The convexity of each boundary curve Γ_i guarantees that the corresponding curve $\psi_\lambda(\Gamma_i)$ remains embedded throughout the deformation. This hypothesis can be exchanged by a *capillarity condition:* for each i, M meets the plane containing Γ_i with constant angle. By Lemma 3.2.1, this alternative hypothesis implies that $\psi_\lambda(\Gamma_i)$ is homothetic to Γ_i and hence embedded for all $\lambda > 0$.

Theorem 3.2.1 has an interesting application to the *free boundary Plateau problem,* which we now describe. Suppose that Γ is a Jordan curve in the plane $\{x_3 = 1\}$ and Σ is an immersed compact minimal surface with boundary consisting of Γ together a nonvoid collection of immersed curves on a parallel plane to $\{x_3 = 1\}$, say $\Pi = \{x_3 = 0\}$. The surface Σ is called a *solution of the free boundary Plateau problem with data* $\{\Gamma, \Pi\}$ if Σ is orthogonal to Π along $\partial\Sigma \cap \Pi$. Schwarz reflection principle applies to any solution of the free boundary Plateau problem, giving rise to a minimal surface $M = \Sigma \cup \Sigma^*$ (the superindex * means the reflected image across Π). If we suppose additionally that Γ is convex, then Theorem 1 in Schoen [50] gives that Σ must be a graph over Π (thus embedded). Meeks and White [37], also assuming the convexity of Γ, proved that the free boundary Plateau problem with data $\{\Gamma, \Pi\}$ has at most two annular solutions. As the doubled surface $M = \Sigma \cup \Sigma^*$ of such an annular solution is a minimal annulus between two planes bounded by two convex curves, a beautiful Theorem of Schiffman [52]

gives that M is foliated by convex curves in horizontal planes, thus the same holds for Σ. Our next statement shows that the hypothesis on the annular topology can be removed.

Corollary 3.2.1. *Let Γ be a convex Jordan curve in the plane $\{x_3 = 1\}$. Then, the free boundary Plateau problem with data $\{\Gamma, x_3 = 0\}$ has at most two solutions, and any such solution is an embedded annulus foliated by convex curves in parallel planes.*

Proof. By the discussion before this Corollary, it suffices to check that any solution to the free boundary Plateau problem with data $\{\Gamma, \Pi\}$ is an annulus. Let Σ be such a solution. As Γ is convex, Σ must be a graph over Π. In particular Σ has genus zero and, therefore, its first homology group is generated by the components of $\Sigma \cap \Pi$. As Σ cuts Π orthogonally, its force along any component of $\Sigma \cap \Pi$ must be vertical. Now Theorem 3.2.1 joint with Remark 3.2.2 applies to Σ, concluding that it is an annulus.

3.3 Related Results.

The above arguments can also be adapted to the case of complete embedded minimal surfaces of finite total curvature and compact boundary. The main differences reside in dealing with noncompact flat 3-manifolds instead of compact ones. This difficulty can be overcome by taking into account that such 3-manifolds consist of a compact piece (where we argue as before) together with a finite number of ends bounded by one or two representatives of annular minimal ends of finite total curvature plus a compact surface, say a portion of a ball of sufficiently large radius. The controlled asymptotic geometry of complete embedded minimal ends of finite total curvature allows to modify successfully the ideas showed in the compact case. One key difference is that we can choose between gluing planar convex disks or the exterior of these disks in the planes containing the boundary curves, to find a properly immersed flat 3-manifold W with piecewise smooth mean convex boundary. We state without proof the following result

Theorem 3.3.1 ([48]). *Let M be a properly embedded nonflat minimal surface with finite total curvature and horizontal ends. Suppose that ∂M consists of a finite number of convex Jordan curves Γ_i in parallel planes Π_i, M being transversal to Π_i along Γ_i, $1 \leq i \leq k$. Let $\overline{M}$ be the piecewise smooth (immersed) surface obtained by gluing M, along each Γ_i, with the closure of one of the components of $\Pi_i - \Gamma_i$, for each $i = 1, \ldots, k$. Assume also that there exists a flat 3-manifold W with piecewise smooth mean convex boundary $\overline{M}$ and an isometric immersion $\phi : W \to \mathbb{R}^3$ extending the immersed surface $\overline{M}$ such that ϕ embeds properly a representative of each end of W. If any one-cycle in M which is nulhomologous in W has vertical force, then M is an annulus.*

As consequences of the preceeding Theorem, we point out the following statements.

Corollary 3.3.1. *There are no properly embedded minimal surfaces $M \subset \mathbb{R}^3$ with vertical forces satisfying*

1. *M is a global graph outside two disjoint convex disks in $\{x_3 = 0\}$, and*
2. *∂M consists of two closed convex curves in horizontal planes.*

Proof. Suppose M satisfies the conditions in the statement of the Corollary, with boundary components Γ_1, Γ_2 contained in horizontal planes Π_1, Π_2, respectively. The argument divides in three cases:

Case 1: *The end of M is of planar type.*
M is contained in the slab bounded by $\Pi_1 \cup \Pi_2$, by the maximum principle. Define $\overline{M}$ as the piecewise smooth embedded surface obtained by gluing M along its boundary with the disk enclosed by Γ_1 in Π_1 and with the noncompact component of $\Pi_2 - \Gamma_2$. Thus $\overline{M}$ bounds an embedded flat 3-manifold W with piecewise mean convex boundary, namely the region in the slab between M and Π_2, see Figure 3.5(a)). Now Theorem 3.3.1 applies, hence M must be an annulus, a contradiction.

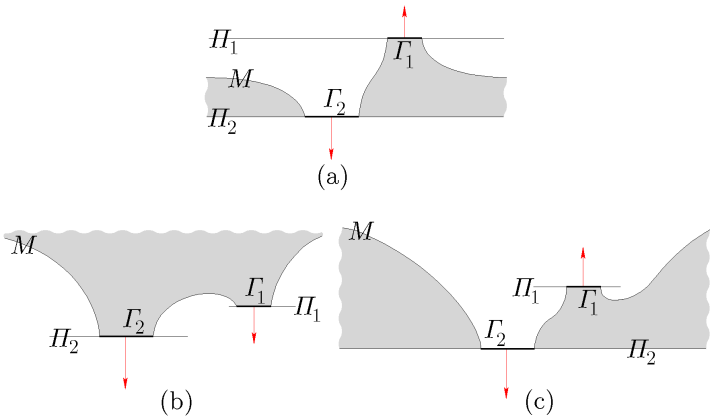

Fig. 3.5. The shaded zones represent the 3-manifold W enclosed by $\overline{M}$.

Case 2: *The end is of catenoid type and the forces along Γ_1, Γ_2 point to the same direction (say downward pointing).*
As the total force along M is zero, it follows that the logarithmic growth of the end must be positive. Taking $\overline{M}$ as the union of M with the two planar disks enclosed by Γ_1, Γ_2 and W as the component of $\mathbb{R}^3 - \overline{M}$ above $\overline{M}$ (see Figure 3.5(b)), we can repeat the argument before.

Case 3: *The end is of catenoid type and the forces along Γ_1, Γ_2 point to opposite directions.*

We can assume that the logarithmic growth of the end is again positive, and that the height of Π_1 is not less than the one of Π_2. From the maximum principle, $M \cap \Pi_2 = \Gamma_2$. Consider $\overline{M}$ as M joint with the planar disk enclosed by Γ_1 and with the exterior of Γ_2 in Π_2, and W as the region between M and Π_2 (Figure 3.5(c)), so we arrive to the same contradiction. This finishes the proof.

The last two statements we mention as consequences of Theorem 3.3.1 deal with minimal surfaces without boundary, symmetric respect to a plane, say $\{x_3 = 0\}$. In both cases, the portion of surface in one of the halfspaces determined by $\{x_3 = 0\}$ will satisfy the conditions in Theorem 3.3.1 (again exchanging the convexity of the boundary curves by the capillarity condition with angle $\pi/2$, see Remark 3.2.2). In the next Corollary, the verticality of the forces of $M^+ = M \cap \{x_3 > 0\}$ follows by imposing that M has genus one, thus M^+ has genus zero.

Corollary 3.3.2. *There are no properly embedded genus one minimal surfaces $M \subset \mathbb{R}^3$ with horizontal ends, symmetric with respect to $\{x_3 = 0\}$.*

In 1981, Costa [9] gave an example of a genus-one complete minimal surface with finite total curvature and three embedded ends. One year later, Hoffman and Meeks [16] proved that such surface is embedded by using that it is highly symmetric. Mathematical and computational analysis of this example allowed Hoffman and Meeks to construct, for any $k \geq 1$, a properly embedded minimal surface $M(k) \subset \mathbb{R}^3$ with finite total curvature, genus k and three ends [17], $M(1)$ being the Costa surface. Moreover, they characterized $M(k)$ by the order of its symmetry group (which is $4(k+1)$) among all surfaces with the same genus and number of ends. If one tries to extend this characterization fixing the genus but not the number of ends, then a careful analysis of the geometry of such a surface shows that we only have to discard the existence of a properly embedded minimal surface with finite total curvature, symmetric respect to $\{x_3 = 0\}$, such that $M^+ = M \cap \{x_3 > 0\}$ has genus zero and ∂M^+ consists of $k+1$ Jordan curves in $\{x_3 = 0\}$. In this setting, Theorem 3.3.1 gives the desired contradiction and we conclude the following

Corollary 3.3.3. *Let $M \subset \mathbb{R}^3$ be a properly embedded minimal surface with finite total curvature and genus $k > 0$. Then, the symmetry group of M, $\mathrm{Sym}(M)$, has at most $4(k+1)$ elements. Moreover, if $|\mathrm{Sym}(M)| = 4(k+1)$, then M is, up to homothety, the surface $M(k)$.*

4. Limits of Minimal Surfaces.

This Section is devoted to study under what conditions and in what sense we can take limits on a given sequence of minimal surfaces. This machinery is of fundamental importance in many situations as producing new examples, trapping surfaces in certain regions of space, or studying compactness questions of some moduli spaces of minimal surfaces. We develop different convergence results attending to the type of surfaces we deal with:

a) A sequence of minimal graphs,
b) A sequence of minimal surfaces with local uniform bounds for the area and for the Gaussian curvature,
c) A sequence of minimal surfaces with local uniform bounds for the Gaussian curvature (unbounded area),
d) A sequence of minimal surfaces in an open set with local uniform bounds for the area and for the total curvature,
e) A sequence of minimal surfaces in the whole $\mathbb{R}^3$ with uniformly bounded total curvature.

In a recent development, Colding and Minicozzi have described the structure of limits of minimal surfaces with bounded topology and no other restriction, see [7] and references therein.

4.1 Minimal Graphs.

Let $\Omega \subset \mathbb{R}^2$ be an open set and $u \in C^\infty(\Omega)$. Given a multi-index $\alpha = (a, b)$ with $a, b \in \mathbb{N} \cup \{0\}$, we denote the α-th partial derivative of u by $D_\alpha u = \frac{\partial^{|\alpha|} u}{\partial^a x \partial^b y}$, where $|\alpha| = a + b$ and $(x, y) \in \Omega$. Thus, $\nabla u = (D_1 u, D_2 u)$ is the gradient of u and $|\nabla^2 u|^2 = (D_{(1,1)} u)^2 + 2(D_{(1,2)} u)^2 + (D_{(2,2)} u)^2$ is the squared length of its Hessian.

If Ω' is a relatively compact open subset of Ω, we simply write $\Omega' \subset\subset \Omega$. We endow $C^\infty(\Omega)$ with the usual C^m-uniform topology on compacts subsets of Ω, for all $m \geq 0$.

Recall that the minimal surface equation is given by

$$(1 + u_y^2) u_{xx} - 2 u_x u_y u_{xy} + (1 + u_x^2) u_{yy} = 0. \tag{4.1}$$

The germ of all the results about convergence of minimal surfaces that we will see later on is the following statement for minimal graphs.

Theorem 4.1.1. *Consider a sequence $\{u_n\}_n \subset C^\infty(\Omega)$ of solutions of the minimal surface equation, satisfying*

1. *There exists $p \in \Omega$ such that $\{u_n(p)\}_n$ is bounded.*
2. *$\{|\nabla u_n|\}_n$ is uniformly bounded on compact subsets of Ω.*

Then, there exists a subsequence $\{u_k\}_k \subset \{u_n\}_n$ and a solution $u \in C^\infty(\Omega)$ of the minimal surface equation such that $\{u_k\}_k$ converges to u in the C^m-topology, for all m.

Proof. The result is consequence of Corollary 16.7 in [11], reasoning as follows. Take a subdomain $\Omega' \subset\subset \Omega$ such that $p \in \Omega'$. Hypotheses *1* and *2* together with the mean value Theorem give that $\{\sup_{\Omega'} |u_n|\}_n$ is bounded. Corollary 16.7 in [11] insures that for all multi-index α, the sequence of partial derivatives $\{D_\alpha u_n\}_n$ is uniformly bounded in Ω'. In this situation, Ascoli-Arzela's Theorem implies that a subsequence of $\{u_n\}_n$ converges to a function $u_\infty \in C^\infty(\Omega')$ in $C^m(\Omega')$, for all m. A standard diagonal process using an increasing exhaustive sequence of relatively compact domains gives a subsequence $\{u_k\}_k \subset \{u_n\}_n$ that converges to a function $u_\infty \in C^\infty(\Omega)$ in the C^m-topology in Ω, for all m. Clearly, u_∞ must also satisfy the minimal surface equation, which completes the proof.

As we are interested in taking limits in a sequence $\{M_n\}_n$ of minimal surfaces, in order to use Theorem 4.1.1 we need to control uniformly the relative size of the domain that expresses locally a minimal surface as a graph over the tangent plane. At this point, it is convenient to introduce some notation.

Let M be a surface in $\mathbb{R}^3$ with tangent plane TM_p, $p \in M$, Gauss map $N : M \to \mathbb{S}^2(1)$, shape operator A and Gaussian curvature K. Recall that for minimal surfaces we have $|A|^2 = -2K$. Given p in M and $r > 0$, we label by $D(p,r) = \{p + v \ / \ v \in TM_p, \ \ |v| < r\}$ the tangent disk of radius r. $W(p,r)$ stands for the infinite solid cylinder of radius r around the affine normal line at p,

$$W(p,r) = \{q + tN(q) \ / \ q \in D(p,r), \ \ t \in \mathbb{R}\}.$$

Inside $W(p,r)$ and for $\varepsilon > 0$, we have the compact slice

$$W(p,r,\varepsilon) = \{q + tN(q) \ / \ q \in D(p,r), \ \ |t| < \varepsilon\}.$$

Given an open set $O \subset \mathbb{R}^3$, we say that a minimal surface M immersed in O is *properly immersed* if for any relatively compact subdomain $O' \subset\subset O$ we have $M \cap O' \subset\subset M$.

If additionally M has no self-intersections and the topology of M is the induced by the one of O, we will say that M is *properly embedded in* O, and denote this fact simply by $M \overset{\text{p.e.}}{\subset} O$.

Lemma 4.1.1 (Uniform Graph Lemma). *Let M be a minimal surface properly immersed in O. Suppose that $|A| \leq c$ on M, for a given $c > 0$.*

1. *For all $p \in M$, consider $R = R(p)$ given by*

$$R = \min\{\frac{1}{4c}, \frac{1}{2}d(p, \partial O)\}. \tag{4.2}$$

   Then, the component of $W(p, R) \cap M$ through p is a graph over $D(p, R)$.
2. *If $u \in C^\infty(D(p, R))$ is the function which defines this graph, then we have the estimates $|u(q)| \leq 8c|p - q|^2$, $|\nabla u|(q) \leq 8c|p - q|$, and $|\nabla^2 u| \leq 16c$, for all $q \in D(p, R)$.*

Proof. Fix $p \in M$. Up to a rotation, we can assume $p = 0$, $TM_p = \{z = 0\}$ and $N(p) = (0, 0, 1)$. As M is locally a graph, there exists a radius $R > 0$ with the following properties:

i) M can be expressed as the graph of a function $u \in C^\infty(D(p, R))$. Hence $u(p) = 0$ and the map $\psi(x, y) = (x, y, u(x, y))$, $(x, y) \in D(p, R)$, is a parameterization of M with $\psi(p) = p$.
ii) The third component of the Gauss map $N_3 = \langle N, e_3\rangle = (1 + |\nabla u|^2)^{-1/2}$ satisfies $N_3 > \frac{1}{2}$ in $D(p, R)$ (note that $N_3(p) = 1$).

Then,

$$|(N_3)_x| = |\langle A\psi_x, e_3\rangle| \leq |A||\psi_x| \leq c\sqrt{1 + u_x^2} \leq c\sqrt{1 + |\nabla u|^2} < 2c,$$

and the same is true for the derivative of N_3 with respect to any unit vector. Thus $|\nabla N_3| \leq 2c$.

We assume that R is the maximal radius at p with the properties $i), ii)$ above. Note that if u were defined on $\partial D(p, R)$ and $N_3 > \frac{1}{2}$ were true along $\partial D(p, R)$, then u could be extended to a larger radius, which contradicts the maximality of R. Hence, one of the following possibilities hold:

a) The function u extends smoothly to a larger disk and there exists $q \in \partial D(p, R)$ such that $N_3(q) = \frac{1}{2}$.
b) There exists a sequence $\{q_n\}_n \subset D(p, R)$ with $d(\psi(q_n), \partial O) \to 0$.

In case *a)* we get $\frac{1}{2} = |N_3(p) - N_3(q)| \leq |\nabla N_3|(r)|p - q| \leq 2cR$, where r is some point in the segment $[p, q]$. If case *b)* holds, then $|p - \psi(q_n)| \leq \text{length}(\psi([p, q_n]))$, where $[p, q_n]$ is the segment in $D(p, R)$ joining p and q_n. We now estimate this length by

$$\text{length}(\psi([p, q_n])) = \int_0^{|q_n|} \sqrt{1 + |\nabla u|^2} ds_0 < \int_0^{|q_n|} 2\, ds_0 = 2|q_n| < 2R,$$

where ds_0 denotes the length element in the flat disk $D(p, R)$. So $d(p, \partial O) \leq |p - \psi(q_n)| + d(\psi(q_n), \partial O) < 2R + d(\psi(q_n), \partial O) \to 2R$. In summary, we obtain (4.2) which proves *i)*. Concerning *ii)*, firstly note that

$$\left|\frac{u_{xx}}{\sqrt{1+|\nabla u|^2}}\right| = |\langle N, \psi_{xx}\rangle| = |\langle N_x, \psi_x\rangle| \leq |A||\psi_x|^2 \leq c(1+u_x^2).$$

Therefore $|u_{xx}| \leq c(1+|\nabla u|^2)^{3/2} = cN_3^{-3} \leq 8c$. As the same holds for the other partial second derivatives, we have $|\nabla^2 u| \leq 16c$ in $D(p,R)$. Using the mean value theorem we see that $|u_x(q)| = |u_x(p) - u_x(q)| \leq 8c|p-q|$ from which one has $|\nabla u|(q) \leq 8c|p-q|$. Finally, $|u(q)| = |u(p) - u(q)| \leq 8c|p-q|^2$ and the proof is complete.

4.2 Sequences with Uniform Curvature Bounds.

4.2.1 Bounded Area.

Now we formulate the notion of convergence for minimal surfaces to be studied in this Section.

Definition 4.2.1. *Let $\{M_n \overset{\text{p.e.}}{\subset} O\}_n$ and $M \overset{\text{p.e.}}{\subset} O$ be minimal surfaces in an open set $O \subset \mathbb{R}^3$. We say that* $\{M_n\}_n$ converges to M in O with finite multiplicity, *if M is the accumulation set of $\{M_n\}_n$ and for all $p \in M$ there exist $r, \varepsilon > 0$ such that*

1. *$M \cap W(p,r,\varepsilon)$ can be expressed as the graph of a function $u : D(p,r) \to \mathbb{R}$.*
2. *For all n large enough, $M_n \cap W(p,r,\varepsilon)$ consists of a finite number (independent of n) of graphs over $D(p,r)$ which converge to u in the C^m-topology, for each $m \geq 0$.*

In the situation above, we define the *multiplicity* of a given $p \in M$ as the number of graphs in $M_n \cap W(p,r,\varepsilon)$, for n large enough. Clearly, this multiplicity remains constant on each connected component of M. Given a sequence of subsets $\{F_n\}_n$ in the open domain O, its *accumulation set* is defined by $\{p \in O \mid \exists\, p_n \in F_n \text{ with } p_n \to p\}$.

Given a minimal surface $M \overset{\text{p.e.}}{\subset} O$ and a 3-ball $B \subset\subset O$, we denote respectively by $A(M \cap B)$ and $K_{M\cap B}$ the area and the Gaussian curvature of the portion of M inside B. Next we state our first convergence result for minimal surfaces.

Theorem 4.2.1. *Let $\{M_n \overset{\text{p.e.}}{\subset} O\}_n$ be a sequence of minimal surfaces. Suppose that $\{M_n\}_n$ has an accumulation point and that for any 3-ball $B \subset\subset O$ there exist positive constants $c_i = c_i(B)$, $i = 1,2$, with $A(M_n \cap B) \leq c_1$ and $|K_{M_n \cap B}| \leq c_2$, $\forall n \in \mathbb{N}$. Then, there exists a subsequence $\{M_k\}_k \subset \{M_n\}_n$ and a minimal surface $M \overset{\text{p.e.}}{\subset} O$ such that $\{M_k\}_k$ converges to M in O with finite multiplicity.*

Proof. Fix an accumulation point p of the sequence $\{M_n\}_n$. Our curvature estimates assumption joint with Uniform Graph Lemma imply that there exist $R = R(p) > 0$ and disjoint graphs $U_n^i \subset \mathbb{R}^3$ of functions u_n^i defined over disks $B(p, 2R) \cap (p + \langle \nu_n^i \rangle^\perp)$, with $|\nu_n^i| = 1$ and $1 \leq i \leq s = s(p, n)$, such that

i) $M_n \cap B(p, R) = (U_n^1 \cup \ldots \cup U_n^s) \cap B(p, R)$.
ii) $|u_n^i|$, $|\nabla u_n^i|$, $|\nabla^2 u_n^i|$ are uniformly bounded in the corresponding disk of radius $2R$, for all $n \in \mathbb{N}$ and $i = 1 \ldots, s$.

As the area of M_n inside $B(p, 2R)$ is bounded by a constant $c_1 = c_1(p) > 0$, we deduce that the number s of such graphs is bounded above, independently of n. Taking a suitable subsequence we can assume that $s = s(p)$ does not depend of n and that $\{\nu_n^i\}_n$ converges to some unit vector ν^i. In fact we can assume that these sequences are constant, i. e. $\nu_n^i = \nu^i$, without destroying the derivative estimates in *ii)*. Using Theorem 4.1.1, there exist subsequences $\{U_k^i\}_k \subset \{U_n^i\}_n$ and minimal graphs U_∞^i over disks or radius $2R$ and center p in the planes $p + \langle \nu^i \rangle^\perp$ $(1 \leq i \leq s)$, such that each U_k^i converges to U^i. As the graphs U_k^i with k fixed are disjoint, maximum principle gives that each two limits graphs U^i, U^j must be disjoint or coincide when restricted to $B(p, 2R)$.

If $p \in O$ is not an accumulation point of $\{M_n\}$, then we can choose a subsequence $\{M_k\}_k$ and $R > 0$ such that $M_k \cap B(p, R) = \emptyset$ for all k.

Now take a countable dense set $\mathcal{A} = \{p_1, p_2, \ldots\} \subset O$. Applying the process above around p_1, we obtain a subsequence $\{M_{1,k}\}_k \subset \{M_n\}_n$ which converges in $B(p_1, R(p_1))$ to a disjoint union of at most s minimal graphs with finite multiplicity. Applying again the process to $\{M_{1,k}\}_k$ around p_2 we obtain another subsequence $\{M_{2,k}\}_k \subset \{M_{1,k}\}_k$ which converges in $B(p_1, R(p_1)) \cup B(p_2, R(p_2))$ to a minimal surface with finite multiplicity. Iterating the process and taking a diagonal subsequence, we obtain $\{M_k\}_k \subset \{M_n\}_n$ which converges in O to a minimal surface $M \overset{\text{p.e.}}{\subset} O$ with finite multiplicity, thereby proving the Theorem.

Later on, we will need to identify limits of sequences of properly embedded minimal surfaces provided that the multiplicity is greater than one.

Proposition 4.2.1. *Let $\{M_n \overset{\text{p.e.}}{\subset} O\}_n$ and $M \overset{\text{p.e.}}{\subset} O$ be minimal surfaces such that $\{M_n\}_n$ converges to M with finite multiplicity. If a connected component $M' \subset M$ is orientable and has multiplicity $m \geq 2$, then M' is stable.*

Proof. Fix a domain $\Omega \subset\subset M'$ with smooth boundary. As M' is orientable and embedded, Ω has an embedded regular neighborhood $\Omega(\varepsilon) = \{p + tN(p) : p \in \Omega, |t| < \varepsilon\}$ of positive radius ε with $\Omega(\varepsilon) \subset\subset O$, N being a unit normal vector field to M'. Denote by $\pi : \Omega(\varepsilon) \to \Omega$ and $d : \Omega(\varepsilon) \to \mathbb{R}$ the orthogonal projection of $\Omega(\varepsilon)$ onto Ω and the oriented distance to Ω, respectively. From convergence of M_n to M it follows that for n large enough, $\pi : M_n \cap \Omega(\varepsilon) \to \Omega$ is a m-sheeted covering map (m does not depend of n). As M_n is embedded, d must separate points at the fibers of this covering and thus $M_n \cap \Omega(\varepsilon)$

consists of m pairwise disjoint normal graphs $\Omega_{1,n}, \ldots, \Omega_{m,n}$ over Ω. These sheets are naturally ordered by d and each one of them converges to Ω.

If we consider two consecutive sheets $\Omega_{1,n}$, $\Omega_{2,n}$, we can construct a narrow, half-tubular shaped, compact surface $C_n \subset O$ with $\partial C_n = \partial\Omega_{1,n} \cup \partial\Omega_{2,n}$, in such a way that $\Omega_{1,n} \cup \Omega_{2,n} \cup C_n$ is a compact piecewise smooth embedded surface enclosing a 3-domain $W_n \subset\subset O$ with mean convex boundary. From Theorem 1.0.6, there exists a least-area surface $\Sigma'_n \subset W_n$ with $\partial\Sigma'_n = \partial\Omega_{1,n}$. Moreover Σ'_n is orientable. Varying n, the above procedure gives a sequence of minimal surfaces $\Sigma_n = \Sigma'_n \cap \Omega(\varepsilon)$ properly embedded in $\Omega(\varepsilon)$.

As ∂W_n collapses into Ω when $n \to \infty$ and Σ'_n meets (by topological reasons) the normal line $TM_p^{\perp}$ for any $p \in \Omega$, we deduce that the accumulation set of $\{\Sigma_n\}_n$ coincides with Ω. Moreover, the stability of Σ_n guarantees curvature estimates by Theorem 1.0.5. To see that $\{\Sigma_n\}_n$ has also local area bounds, consider a 3-ball $B \subset\subset \Omega(\varepsilon)$ such that $\Sigma_n \cap B \neq \emptyset$. Then, the area of $\Sigma_n \cap B$ is not greater than the one of any piecewise smooth surface $\Delta \subset W_n$ with $\partial\Delta = \partial(\Sigma_n \cap B)$. Note that we can construct such a surface Δ by considering suitable portions of ∂B and $\Omega_{i,n}$ $(i = 1, 2)$. As $\{\Omega_{i,n}\}_n$ has local area bounds because it converges to Ω, we conclude the desired area bounds for $\{\Sigma_n\}_n$. Now Theorem 4.2.1 implies that a subsequence $\{\Sigma_k\}_k \subset \{\Sigma_n\}_n$ converges to a minimal surface properly embedded in $\Omega(\varepsilon)$, with finite multiplicity. Clearly the limit surface must be Ω. As the area of Σ_k is not greater than the one of $\Omega_{1,k}$, we conclude that the multiplicity of $\Sigma_k \to \Omega$ is one. This implies easily that Ω is stable and concludes the proof of the Proposition.

4.2.2 Unbounded Area.

We will also need to construct limits of sequences of minimal surfaces under weaker conditions than in Theorem 4.2.1.

Theorem 4.2.2. *Let $\{M_n \overset{\text{p.e.}}{\subset} O\}_n$ be a sequence of minimal surfaces. Suppose that there exists a sequence $p_n \in M_n$ converging to a point $p \in O$ and that for any 3-ball $B \subset\subset O$ there exists a positive constant $c = c(B)$ with $|K_{M_n \cap B}| \leq c$, $\forall n \in \mathbb{N}$. Then, there exists a subsequence $\{M_k\}_k \subset \{M_n\}_n$ and a connected minimal surface M in O satisfying*

1. *M is contained in the accumulation set of $\{M_k\}_k$.*
2. *$p \in M$ and $K_M(p) = \lim_k K_{M_k}(p_k)$.*
3. *M is embedded in O (but not necessarily properly embedded).*
4. *Any divergent path in M either diverges in O or has infinite length.*

Proof. As the argument is similar to the one in Theorem 4.2.1, we only provide a sketch of proof. As $\{p_n\}_n$ accumulates at $p \in O$ and we have local uniform bounds for the curvature K_{M_n} around p, Uniform Graph Lemma gives $R = R(p) > 0$ such that the connected component of $M_n \cap B(p, 2R)$ passing through p_n contains a graph U_n over a planar disk of center p and radius R. Moreover the functions u_n which define the graphs satisfy that $|u_n|$, $|\nabla u_n|$

and $|\nabla^2 u_n|$ are uniformly bounded. By Theorem 4.2.1 or Theorem 4.1.1, there exists a subsequence $\{U_{k_1}\}_{k_1} \subset \{U_n\}_n$ converging to a minimal graph U over a disk of radius R with multiplicity one, and $p \in U$. An analytic prolongation argument allows us to construct a subsequence $\{M_k\}_k \subset \{M_{k_1}\}_{k_1}$ and a maximal sheet M in the accumulation set of $\{M_k\}_k$ which extends U. By construction, the minimal surface M satisfies items *1* and *2*. M must be embedded because transversal selfintersections of it would give rise to transversal selfintersections of M_n for n large, thus we have *3*. Finally, take a divergent path $\gamma : [0, \infty[\to M$ such that γ does not diverge in O (the existence of such a curve prevents M of being proper in O). Thus, there exists a compact set $C \subset O$ and a sequence of real numbers $\{t_i\}_i$ diverging to $+\infty$ such that $\gamma(t_i) \in C$ for all i. As $|K_{M \cap C}|$ is bounded, the uniform graph property implies that, up to a subsequence, there exists an interval I_i centered at t_i such that $I_i \cap I_{i+1} = \emptyset$ and $\text{length}(\gamma(I_i)) > \delta$ for a fixed $\delta > 0$. This gives that the length of γ is infinite.

4.3 Sequences with Total Curvature Bounds.

In this Section, we will exchange the local uniform curvature bounds of former results by total curvature bounds. Recall that given a minimal surface M in $\mathbb{R}^3$, its *total curvature* is defined by $C(M) = \int_M |K| \, dA$.

4.3.1 Limits in Open Domains.

We start by considering sequences of surfaces properly embedded in an open set $O \subset \mathbb{R}^3$, with local area and local total curvature bounds, see Choi and Schoen [6] and White [60].

Theorem 4.3.1. *Let $\{M_n \overset{\text{p.e.}}{\subset} O\}_n$ be a sequence of minimal surfaces. Suppose that $\{M_n\}_n$ has an accumulation point and that for any 3-ball $B \subset\subset O$ there exist positive constants $c_i = c_i(B) > 0$, $i = 1, 2$, with $A(M_n \cap B) \leq c_1$ and $C(M_n \cap B) \leq c_2$, $\forall n \in \mathbb{N}$. Then, there exists a subsequence $\{M_k\}_k \subset \{M_n\}_n$, a discrete set $X \subset O$ and a minimal surface $M \overset{\text{p.e.}}{\subset} O$ such that $\{M_k\}_k$ converges to M in $O - X$ with finite multiplicity.*

Moreover, given $p \in X$ and $R > 0$ it holds

$$\limsup_k C(M_k \cap B(p, R)) \geq 4\pi. \tag{4.3}$$

In what follows, we will call X the *singular set* of the sequence $\{M_k\}_k$.

Proof. Define $X = \{p \in O \ / \ \{|K_{M_n \cap B(p,r)}|\}_n \text{ is unbounded}, \forall r > 0\}$.

Fix a point $p \in X$ and take a radius $r > 0$ with $B(p, r) \subset\subset O$. Let p_n be a maximum of the function $|K_{M_n}(-)| d(-, \partial B(p, r))^2$ in the closure of

$M_n \cap B(p,r)$ (observe that this function is invariant under rescaling). Define the sequences $\lambda_n = \sqrt{|K_{M_n}(p_n)|}$ and $r_n = d(p_n, \partial B(p,r))$.

As $|K_{M_n \cap B(p,r/2)}|$ is unbounded, after passing to a subsequence we find points $q_n \in M_n \cap B(p,r/2)$ with $|K_{M_n}(q_n)| \to +\infty$. Note that $\lambda_n^2 r_n^2 \geq |K_{M_n}(q_n)| d(q_n, \partial B(p,r))^2 \geq |K_{M_n}(q_n)| \frac{r^2}{4}$, thus $\{\lambda_n r_n\}_n$ also diverges to ∞. Translate p_n to the origin and homothetically expand $M_n \cap B(p_n, r_n)$ by the factor λ_n, so we obtain new minimal surfaces $\widetilde{M}_n \overset{\text{p.e.}}{\subset} B(0, \lambda_n r_n)$ passing through the origin (see Figure 4.1), whose curvatures satisfy $|K_{\widetilde{M}_n}(0)| = 1$

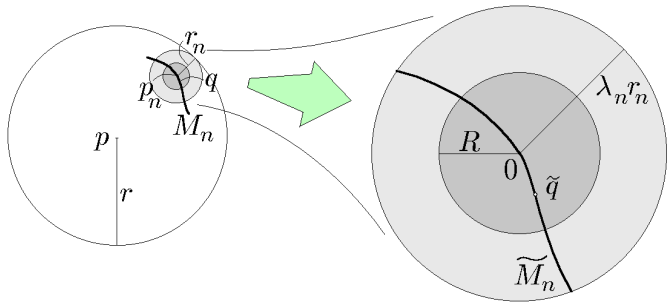

Fig. 4.1. Rescaling $M_n \cap B(p_n, r_n)$ by factor λ_n

for all n. Given $R > 0$ and $\widetilde{q} \in \widetilde{M}_n \cap B(0,R)$,

$$|K_{\widetilde{M}_n}(\widetilde{q})|(\lambda_n r_n - R)^2 \leq |K_{\widetilde{M}_n}(\widetilde{q})| d(\widetilde{q}, \partial B(0, \lambda_n r_n))^2 =$$

$$|K_{M_n}(q)| d(q, \partial B(p_n, r_n))^2 \leq |K_{M_n}(q)| d(q, \partial B(p,r))^2 \leq \lambda_n^2 r_n^2,$$

where $q \in M_n$ is the point which corresponds to $\widetilde{q} \in \widetilde{M}$ through the rescaling. Thus we get that $\{|K_{\widetilde{M}_n}|\}_n$ is uniformly bounded on compact subsets of $\mathbb{R}^3$. Note also that the invariance of the total curvature under rescaling shows that $C(\widetilde{M}_n)$ is bounded above by a constant that only depends on r. Therefore there exists a subsequence $\{\widetilde{M}_k\}_k \subset \{\widetilde{M}_n\}_n$ and a complete nonflat minimal surface $\widetilde{M} \subset \mathbb{R}^3$ such that $\widetilde{M}_k$ converges (in the sense of Theorem 4.2.2) to $\widetilde{M}$. Moreover, it is clear that $\widetilde{M}$ has finite total curvature. As $\widetilde{M}$ is nonflat, it must have total curvature at least 4π, by (1.3). Hence, coming back to the original scale, we deduce that $\limsup\ C(M_k \cap B(p,r)) \geq 4\pi$. This property, joint with the uniform control of the total curvature, imply that X is discrete.

By definition of X, $\{K_n\}_n$ is uniformly bounded on compact subsets of $O - X$. As we have local area bounds, Theorem 4.2.1 insures that a subsequence of $\{M_n\}_n$ converges to a minimal surface $M \overset{\text{p.e.}}{\subset} O - X$ with finite multiplicity. It only remains to prove that M can be extended to a properly embedded minimal surface in O. This fact will we a consequence of Lemma 4.3.1 below.

Remark 4.3.1. In the above proof we saw how to produce, around a singular point $p \in X$ and after rescaling, a nonflat minimal surface $\widetilde{M}$ with finite total curvature. As the extended Gauss of such a surface must be onto, we deduce that the Gauss map of the surfaces $M_k \cap B(p,R)$ cannot be contained in an open hemisphere.

Lemma 4.3.1. *Let $M \subset \overline{B}(0,1) - \{0\}$ be a properly embedded minimal surface with compact boundary contained in $\{|p| = 1\}$. If M has finite total curvature, then M extends through the origin giving rise to a properly embedded minimal surface in $\overline{B}(0,1)$.*

Proof. We first show that M is conformally a compact Riemann surface with boundary minus a finite number of points. Let $f : \mathbb{R}^3 - \{0\} \to \mathbb{R}^3 - \{0\}$ be the inversion given by $f(p) = \frac{p}{|p|^2}$, $p \in \mathbb{R}^3 - \{0\}$. As f is a conformal diffeomorphism, we have that $\widetilde{M} = f(M)$ is a properly embedded (nonminimal) surface in the exterior of the unit 3-ball, with boundary contained in $\{|p| = 1\}$. In particular, $\widetilde{M}$ is complete. The relationship between the induced metric ds^2 by the inner product in $\mathbb{R}^3$ and the pullback metric $d\widetilde{s}^2 = f^*\langle,\rangle$ is $d\widetilde{s}^2 = |p|^{-4}\, ds^2$, and the respective curvature elements are related by

$$\widetilde{K}\, d\widetilde{A} = \left(K + 4\frac{\langle p, N\rangle^2}{|p|^4} \right) dA,$$

N being the Gauss map of M. In particular,

$$\int_M \widetilde{K}^-\, d\widetilde{A} = -\int_{\{\widetilde{K}<0\}} \widetilde{K}\, d\widetilde{A} = -\int_{\{\widetilde{K}<0\}} \left(K + 4\frac{\langle p, N\rangle^2}{|p|^4} \right) dA$$

$$\leq -\int_{\{\widetilde{K}<0\}} K\, dA \leq -\int_M K\, dA = C(M) < +\infty.$$

Hence Huber's Theorem [19, 59] implies that M has the conformal structure of a compact Riemann surface with boundary with a finite number of points removed, $p_1, \ldots, p_k$. Thus for $r > 0$ small enough, $M(r) = M \cap \overline{B}(0,r)$ is a union of properly embedded minimal surfaces $D^*(p_1), \ldots, D^*(p_k) \subset \overline{B}(0,r) - \{0\}$ which are conformally equivalent to punctured disks and whose boundaries are contained in $\{|p| = r\}$. As the coordinate functions on these surfaces are harmonic and bounded by r, they can be extended through the punctures p_j so they produce minimal (possibly branched) disks $D(p_1), \ldots, D(p_k)$. As a minimal surface must have selfintersections in any neighborhood of a branch point, it follows that none of the $D(p_j)$ is branched. Finally, k must be one because otherwise we would have two minimal disks touching only at an interior point, in contradiction with the maximum principle. Now the Lemma is proved.

4.3.2 Limits in $\mathbb{R}^3$.

In the second part of this Section, we deal with sequences of surfaces properly embedded in the whole space. In this setting, to take limits we only require total curvature bounds.

Theorem 4.3.2 ([47]). *Let $\{M_n \overset{\text{p.e.}}{\subset} \mathbb{R}^3\}_n$ be a sequence of minimal surfaces with fixed finite total curvature $C(M_n) = c$ for all n. Then, there exists a subsequence $\{M_k\}_k \subset \{M_n\}_n$ such that one of the following possibilities hold:*

1. *$\{M_k\}_k$ has no accumulation points.*
2. *There exists a finite set $X \subset \mathbb{R}^3$ such that $\{M_k\}_k$ converges in $\mathbb{R}^3 - X$ to a finite union of parallel planes, with finite multiplicity. Moreover, equation (4.3) holds at any point of the singular set X.*
3. *There exists a minimal surface $M \overset{\text{p.e.}}{\subset} \mathbb{R}^3$ such that $C(M) \leq c$ and $\{M_k\}_k$ converges to M in $\mathbb{R}^3$ with multiplicity one.*

Proof. Assume that $\{M_n\}_n$ has an accumulation point. As the total curvature of the surfaces M_n is fixed, formula (1.3) implies that the number of ends of any M_n is bounded above by a fixed integer $r \geq 2$. Thus, Proposition 1.0.2 insures uniform local area bounds for the sequence $\{M_n\}_n$. Under these conditions, Theorem 4.3.1 says that there exist a subsequence $\{M_k\}_k \subset \{M_n\}_n$ and a minimal surface $M \overset{\text{p.e.}}{\subset} \mathbb{R}^3$ such that $\{M_k\}_k$ converges to M in $\mathbb{R}^3$ minus a discrete set X with finite multiplicity. As $C(M_k) = c$ for all k, it follows that $C(M)$ is finite and not bigger than c. Moreover, the inequality (4.3) for arbitrary $p \in X$ together with the hypothesis $C(M_k) = c$ insure that X is finite.

Note that as M is properly embedded in $\mathbb{R}^3$, it must be connected or a union of parallel planes by the strong halfspace Theorem in [18]. In this last case, the inequality $A(M_k \cap B(p,R)) \leq r\pi R^2$ of Proposition 1.0.2 implies that M is a union of at most, r parallel planes. It only remains to prove that if M is nonflat (hence connected), the multiplicity of the limit $\{M_k\}_k \to M$ is one and the singular set X is empty.

Reasoning by contradiction, suppose that $\{M_k\}_k \to M$ has multiplicity greater than one. Firstly note that as M is properly embedded in $\mathbb{R}^3$, it must be orientable and the same holds for $M - X$. As the multiplicity of the limit $\{M_k\}_k \to M - X$ is at least 2, Proposition 4.2.1 implies that $M - X$ is stable. But M extends smoothly through each $p \in X$, and a standard cutoff functions argument insure that the property of its Jacobi operator being positive semidefinite in a punctured surface extends to the whole surface. Thus M is a stable minimal surface properly embedded in $\mathbb{R}^3$. Then Theorem 1.0.4 implies that M is a plane. This contradiction shows that the multiplicity of the limit must be one.

Finally, take a singular point $p \in X$. Choose $r, \varepsilon > 0$ small such that $X \cap W(p,r,\varepsilon) = \{p\}$ and $M \cap W(p,r,\varepsilon)$ is a graph over the tangent disk

$D(p,r)$, say of a function u. As M_k is proper, $M_k \cap W(p,r,\varepsilon)$ must be compact for all k. Moreover, the convergence with multiplicity one of $\{M_k\}_k$ to M in $\mathbb{R}^3 - X$ insures that for k large enough, $M_k \cap \partial W(p,r,\varepsilon)$ is the graph of a function $v_k : \partial D(p,r) \to \mathbb{R}$ with $v_k \to u$ in $C^m(\partial D(p,r))$ for all $m \geq 0$. As $M_k \cap W(p,r,\varepsilon)$ is compact, Proposition 1.0.1 insures that this surface is indeed a graph over $D(p,R)$. In particular, the Gauss map of $M_k \cap W(p,r,\varepsilon)$ is contained in an open hemisphere, which contradicts that p is a singular point, see Remark 4.3.1. This finishes the proof of the Theorem.

5. Compactness of the Moduli Space of Minimal Surfaces.

Given integers $g \geq 0$ and $r \geq 1$, we will denote by $\mathcal{M}(g,r)$ the space of properly embedded minimal surfaces in $\mathbb{R}^3$ with genus g and r horizontal ends. As we saw in Subsection 1.0.2, $\mathcal{M}(g,1)$ is empty when $g \geq 1$, while $\mathcal{M}(0,1)$ is just the space of horizontal planes. Theorem 1.0.2 and Corollary 2.1.1 say that $\mathcal{M}(0,2)$ consists only of Catenoids, and that $\mathcal{M}(g,2)$, $\mathcal{M}(0,r)$ are empty for $g \geq 1$, $r \geq 3$.

Choi and Schoen [6] have proved that the space of embedded compact minimal surfaces of fixed genus in the standard unit 3-sphere $\mathbb{S}^3(1)$ is compact, in the sense that given any sequence in this space we can find a subsequence which converges smoothly to a minimal surface in $\mathbb{S}^3(1)$ with the same topology.

In our setting, a natural question is to decide whether the moduli space $\mathcal{M}(g,r)$ is compact, that is, if any sequence $\{M_n\}_n \subset \mathcal{M}(g,r)$ has a subsequence which converges (up to homotheties) to a minimal surface $M \in \mathcal{M}(g,r)$ with multiplicity one (in what follows, *homothety* stands for a homothety or a translation in $\mathbb{R}^3$). The spaces $\mathcal{M}(0,1)$ and $\mathcal{M}(0,2)$ are compact but $\mathcal{M}(1,3)$ is known to be noncompact. In fact, $\mathcal{M}(1,3)$ is the only nontrivial nonvoid moduli space which has been completely described: Costa proved in [10] that $\mathcal{M}(1,3)/\{\text{homotheties}\} = \mathbb{R}$. In this section we will see that for some prescribed topologies, the moduli space $\mathcal{M}(g,r)$ is compact (this holds, for instance, when $g = 1$ and $r = 5$). The results below were obtained by Ros in [47].

A central open problem in our setting is a conjecture by Hoffman and Meeks [17], which asserts that for each genus $g \geq 1$, there exists an integer $r(g)$ such that $\mathcal{M}(g,r)$ is empty for $r > r(g)$ (more precisely, they conjecture that $r(g) = g + 2$). The compactness result above may be viewed as a first step in the proof of the Hoffman and Meeks problem: what we expect is that these compact moduli spaces are in fact empty.

5.1 Weak Compactness.

In this Subsection we study convergence of sequences of surfaces in a fixed space $\mathcal{M}(g,r)$. Roughly speaking, we show that any sequence $\{M_n\}_n \subset \mathcal{M}(g,r)$ must have a partial that converges in an appropriate sense to a finite

collection of surfaces $M_{i,\infty} \in \mathcal{M}(g_i, r_i)$, $1 \leq i \leq k$, with $g_i \leq g$ and $r_i \leq r$. As we cannot insure the convergence with multiplicity one to a single surface in the original space $\mathcal{M}(g, r)$, we will use the expression *weak compactness* to refer to this property.

Recall that for $M \in \mathcal{M}(g, r)$, its total curvature depends only on g and r, see equation (1.3).

Theorem 5.1.1. *Fix integers $g \geq 0, r \geq 2$. Given a sequence of surfaces $\{M_n\}_n \subset \mathcal{M}(g, r)$, there exist a subsequence, denoted again by $\{M_n\}_n$, an integer $k > 0$, a collection of nonflat minimal surfaces $M_{i,\infty} \in \mathcal{M}(g_i, r_i)$ with $g_i \leq g$, $r_i \leq r$ for $i = 1, \ldots, k$, and k sequences of homotheties $\{h_{i,n}\}_n$ satisfying*

1. *$C(M_{i,\infty}) + \ldots + C(M_{k,\infty}) = C(M_n)$,*
2. *$\{h_{i,n}(M_n)\}_n$ converges to $M_{i,\infty}$ in $\mathbb{R}^3$ with multiplicity one, $1 \leq i \leq k$,*
3. *For any R, n large, there exist k disjoint balls $B_{1,n}, \ldots, B_{k,n} \subset \mathbb{R}^3$ with $h_{i,n}(B_{i,n}) = B(0, R)$ such that M_n decomposes as*

$$M_n = M_{1,n} \cup \ldots \cup M_{k,n} \cup \Omega_{1,n} \cup \ldots \cup \Omega_{r,n},$$

where $M_{i,n} = M_n \cap B_{i,n}$ (hence $h_{i,n}(M_{i,n})$ can be taken arbitrarily close to $M_{i,\infty} \cap B(0, R)$ for n large enough) and $\Omega_{j,n}$ is a graph over the exterior of some convex disks in $\{x_3 = 0\}$, containing exactly one end of M_n.

In this setting, we will call $M_{1,n}, \ldots, M_{k,n}$ the *bounded domains*, $\Omega_{1,n}, \ldots, \Omega_{r,n}$ the *unbounded domains* of the surface M_n (see Figure 5.1) and $M_{1,\infty}, \ldots, M_{k,\infty}$ the *weak limit of the subsequence* $\{M_n\}_n$.

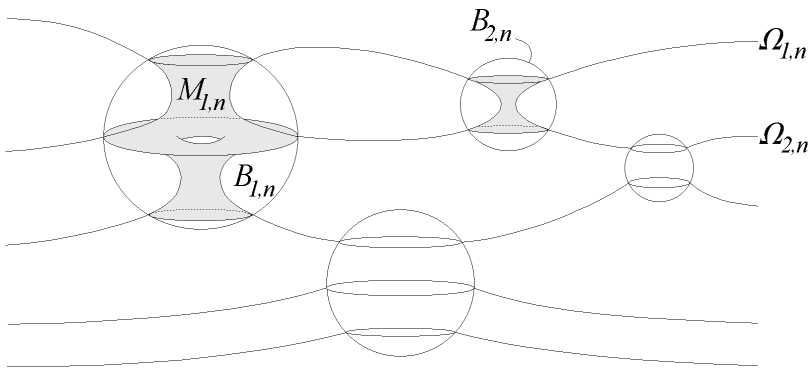

Fig. 5.1. A surface with five ends decomposed in bounded and unbounded domains.

Proof. Equation (1.3) gives that the total curvature of all the M_n is fixed, say $C(M_n) = c$. Given $n \in \mathbb{N}$, consider balls $B \subset \mathbb{R}^3$ with $C(M_n \cap B) = 2\pi$.

As $r \geq 2$, we have that M_n is nonflat and so, $c \geq 4\pi$. In particular, the above family of balls is nonvoid. Clearly if the center of a ball in the family goes to infinity, then its radius must also diverge to infinity. Thus we can find a ball $B'_{1,n}$ in this family with minimum radius. Let $h_{1,n}$ be the homothety that transforms $B'_{1,n}$ into $B(0,1)$. All the rescaled surfaces $\{h_{1,n}(M_n)\}_n$ have total curvature c. By Theorem 4.3.2, there exists a subsequence, again denoted by $\{M_n\}_n$, such that one of the following possibilities hold:

a) there exists a finite set $X_1 \subset \mathbb{R}^3$ such that $\{h_{1,n}(M_n)\}_n$ converges in $\mathbb{R}^3 - X_1$ to a finite union of parallel planes with finite multiplicity, or

b) there exists a minimal surface $M_{1,\infty} \overset{\text{p.e.}}{\subset} \mathbb{R}^3$ such that $C(M_{1,\infty}) \leq c$ and $\{h_{1,n}(M_n)\}_n$ converges to $M_{1,\infty}$ in $\mathbb{R}^3$ with multiplicity one.

Let us see that case *a)* is impossible. Reasoning by contradiction, take a point p in the singular set X_1. Equation (4.3) at p implies that we can find a ball $B(p,R)$ of arbitrarily small radius such that $C(h_{1,n}(M_n) \cap B(p,R)) \geq 3\pi$. As $h_{1,n}(B'_{1,n}) = B(0,1)$, the existence of $B(p,R)$ contradicts the minimality of the radius of $B'_{1,n}$. As consequence, only case *b)* can hold. In particular, $C(M_{1,\infty} \cap B(0,1))$ must be 2π and so, $M_{1,\infty}$ is nonflat. Denote by g_1, r_1 the genus and number of ends of $M_{1,\infty}$, respectively. From the convergence $\{h_{1,n}(M_n)\}_n \to M_{1,\infty}$ it follows that $g_1 \leq g$. Finally, consider a large positive number ρ_1 such that $|C(M_{1,\infty}) - C(h_{1,n}(M_n) \cap B(0,\rho_1))| < \varepsilon$, with $\varepsilon > 0$ small. Note that ρ_1 can be chosen so that $h_{1,n}(M_n) \cap \partial B(0,\rho_1)$ consists of r_1 Jordan curves projecting bijectively onto convex curves in the limit tangent plane to $M_{1,\infty}$ (which at this moment need not to be horizontal, although this will certainly be the case). Denote by $B_{1,n} = h_{1,n}^{-1}(B(0,\rho_1))$. This finishes the first step in our construction of the weak limit.

Assuming $C(M_{1,\infty}) < c$, we will construct the second partial limit of our sequence. As both $C(M_{1,\infty})$ and c are integer multiples of 4π, the family of balls $B \subset \mathbb{R}^3$ such that $C([M_n - B_{1,n}] \cap B) = 2\pi$ is nonvoid. As before, we can choose a ball $B'_{2,n}$ in this family with minimum radius. Clearly the radius of $B'_{2,n}$ cannot be smaller than the radius of $B'_{1,n}$. We label as $h_{2,n}$ the homothety such that $h_{2,n}(B'_{2,n}) = B(0,1)$. Thus, the radius of $h_{2,n}(B'_{1,n})$ is at most one, and the one of $h_{2,n}(B_{1,n})$ is bounded above by ρ_1.

We claim that if $\{h_{2,n}(B_{1,n})\}_n$ has a limit, then it must be necessarily a single point: by contradiction, assume that $\{h_{2,n}(B_{1,n})\}_n$ converges to a ball of positive radius. Then, the surfaces $h_{1,n}(M_n)$ and $h_{2,n}(M_n)$ differ in a homothety whose center and ratio are controlled independently of n. As $\{h_{1,n}(M_n)\}_n$ converges in $\mathbb{R}^3$ to $M_{1,\infty}$ with multiplicity one, we deduce that $C(h_{2,n}(M_n) \cap [B(0,1) - h_{2,n}(B_{1,n})])$ can be made arbitrarily small, which contradicts our choice of $B'_{2,n}$. Hence our claim holds.

Using again Theorem 4.3.2 and after passing to a subsequence, we have two possibilities: either $\{h_{2,n}(M_n)\}_n$ converges in $\mathbb{R}^3$ minus a finite subset X_2 to a finite union of parallel planes with finite multiplicity, or $\{h_{2,n}(M_n)\}_n$ converges in $\mathbb{R}^3$ to a properly embedded minimal surface $M_{2,\infty}$ with multiplicity one, $C(M_{2,\infty})$ being less that or equal to c. Our next goal is showing that the

singular set X_2 must be empty and only the second possibility can occur. On the contrary, if $X_2 \neq \emptyset$ then the minimizing property of $B'_{2,n}$ implies that $X_2 = \{p\} = \lim_n h_{2,n}(B_{1,n})$. Moreover, $p \in \overline{B}(0,1)$ (otherwise for n large we would have $h_{2,n}(B_{1,n}) \cap B(0,1) = \emptyset$ thus $\{h_{2,n}(M_n) \cap B(0,1)\}_n$ would converge in $B(0,1)$ to a finite union of parallel disks with finite multiplicity, which contradicts that $C(h_{2,n}(M_n) \cap B(0,1)) = 2\pi$ for all n).

Fix $\varepsilon > 0$ small. Taking n large enough, we can suppose $h_{2,n}(B_{1,n}) \subset\subset B(p,\varepsilon)$. Then

$$2\pi = C\left([h_{2,n}(M_n) \cap B(0,1)] - h_{2,n}(B_{1,n})\right)$$

$$= C\left([h_{2,n}(M_n) \cap B(0,1)] - B(p,\varepsilon)\right) + C\left(h_{2,n}(M_n - B_{1,n}) \cap B(p,\varepsilon)\right). \quad (5.1)$$

As $\{h_{2,n}(M_n)\}_n$ converges in $\mathbb{R}^3 - \{p\}$ to a finite union of parallel planes with finite multiplicity, the first summand in (5.1) goes to zero as $n \to \infty$. Take a component S of $h_{2,n}(M_n - B_{1,n}) \cap B(p,\varepsilon)$. The components of the boundary of S are divided in two kinds: the ones which lie on $\partial B(p,\varepsilon)$, where the Gauss map of S converges to a constant value of the sphere $\mathbb{S}^2(1)$ because outside p the limit of $\{h_{2,n}(M_n)\}_n$ is flat, and those lying on $\partial h_{2,n}(B_{1,n})$, where the Gauss map of S is again almost constant because of the existence of the limit surface $M_{1,\infty}$. As the Gauss map of a nonflat minimal surface is an open map, all these constants in $\mathbb{S}^2(1)$ must be the same (otherwise the Gauss map image of S would cover almost all $\mathbb{S}^2(1)$, hence $C(S)$ would be close to a positive multiple of 4π, a contradiction with the definition of $B'_{2,n}$). Equivalently, the Gauss map image of S is contained in a small neighborhood of some vector $a \in \mathbb{S}^2(1)$. Moreover, $\pm a$ must coincide with the limit normal vectors at the ends of $M_{1,\infty}$ and with the normal vectors to the flat limit of $\{h_{2,n}(M_n)\}_n$ outside p. Clearly, $C(S)$ can be taken arbitrarily small by choosing n large enough. Note also that the number of such components S is bounded above independently of n, because the number of boundary components of $h_{2,n}(M_n - B_{1,n}) \cap B(p,\varepsilon)$ is controlled by the (finite) number of planes in the flat limit of $\{h_{2,n}(M_n)\}_n$ and the number r_1 of ends of $M_{1,\infty}$. As consequence, the second summand in (5.1) will be also arbitrarily small for n large, which is the desired contradiction.

So we have proved that $X_2 = \emptyset$ and, therefore, $\{h_{2,n}(M_n)\}_n$ converges in $\mathbb{R}^3$ to a properly embedded minimal surface $M_{2,\infty}$ with multiplicity one. As before, $C(M_{2,\infty} \cap B(0,1)) = 2\pi$ hence $M_{2,\infty}$ is nonflat. Calling g_2, r_2 to its genus and number of ends, it follows that $g_2 \leq g$. Take $\rho_2 > 0$ large such that $|C(M_{2,\infty}) - C(h_{2,n}(M_n) \cap B(0,\rho_2))| < \varepsilon$ and $h_{2,n}(M_n) \cap \partial B(0,\rho_2)$ consists of r_2 Jordan curves projecting bijectively onto convex curves in the limit tangent plane to $M_{2,\infty}$. Finally, denote by $B_{2,n} = h_{2,n}^{-1}(B(0,\rho_2))$. Recall that the radius of $h_{2,n}(B_{1,n})$ is at most ρ_1. This inequality together with the fact that $X_2 = \emptyset$, imply that the sequence of balls $\{h_{2,n}(B_{1,n})\}_n$ must diverge in $\mathbb{R}^3$. In particular, we can assume that $h_{2,n}(B_{1,n}) \cap B(0,\rho_2) = \emptyset$ and so, $B_{1,n}$, $B_{2,n}$ are disjoint. Now our second step is finished.

Clearly, $C(M_{1,\infty}) + C(M_{2,\infty}) \leq c$. If the equality does not hold, then we repeat the arguments above and in a finite number of steps, say k, we reach the equality. To finish the proof, we must prove that each limit surface $M_{i,\infty}$ has $r_i \leq r$ horizontal ends and M_n decomposes as in item *3* of the statement. With this aim, let Ω_n be the closure of a component of $M_n - (B_{1,n} \cup \ldots \cup B_{k,n})$. Ω_n is a properly embedded minimal surface in $\mathbb{R}^3 - (B_{1,n} \cup \ldots \cup B_{k,n})$ whose boundary consists of a finite number of Jordan curves in the boundaries of some of the balls $B_{i,n}$. As the homothetical expansion of $M_n \cap B_{i,n}$ by $h_{i,n}$ is arbitrarily close to the intersection of a big ball with the properly embedded nonflat minimal surface of finite total curvature $M_{i,\infty}$, it follows that each component Γ_n of $\partial\Omega_n$, say in $\partial B_{i,n}$, is close to a round circle for n large and the Gauss map of M_n along Γ_n is uniformly close to a constant value, namely the limit normal vector of the corresponding end of $M_{i,\infty}$. In particular, the Gauss map of Ω_n applies $\partial\Omega_n$ into curves contained in small neighborhoods of some constants values of $\mathbb{S}^2(1)$. As such Gauss map is an open map and the total curvature of Ω_n is small, it follows that all these constants are the same. In other words, the Gauss map image of Ω_n is contained in a small neighborhood of a vector $a \in \mathbb{S}^2(1)$. Along each component $\Gamma_n \subset \partial B_{i,n}$ of $\partial\Omega_n$, glue Ω_n smoothly with a compact surface $D_{i,n} \subset B_{i,n}$, $D_{i,n}$ being a graph over the orthogonal plane $\langle a \rangle^\perp$ to a (we can take such $D_{i,n}$ close to a planar disk parallel to $\langle a \rangle^\perp$). After these gluing processes, we obtain a properly embedded (nonminimal) surface Ω'_n without boundary, whose Gauss map image is contained in a small neighborhood of a. The projection of Ω'_n over $\langle a \rangle^\perp$ is a proper local diffeomorphism, thus a covering map and then necessarily a global diffeomorphism. In particular, Ω_n is graph over a noncompact region in $\langle a \rangle^\perp$ bounded by a finite number of disjoint convex curves. This implies that Ω_n contains exactly one end of M_n, and that a must be the value of the Gauss map of M_n at this end, $a = \pm(0,0,1)$, so item *3* of the statement is true. Note also that all the limit surfaces $M_{i,\infty}$ corresponding to the balls $B_{i,n}$ joined to a given unbounded domain Ω_n along components of $\partial\Omega_n$ must have horizontal limit tangent plane. As all the limit surfaces $M_{i,\infty}$ will appear when considering all the unbounded domains, we conclude that $M_{i,\infty}$ has horizontal ends for all i.

To finish the proof, we check that the number of ends r_i of $M_{i,\infty}$ is less than or equal to r: note that the boundary components of $M_n \cap B_{i,n}$ correspond bijectively with the ends of $M_{i,\infty}$. Moreover, $M_n \cap B_{i,n}$ is joined along each one of these components to a certain unbounded domain, which contains exactly one end of M_n. So we have an injective map from the ends of $M_{i,\infty}$ into the ends of M_n. Then $r_i \leq r$ and we have proved the Theorem.

Remark 5.1.1. Item *1* in Theorem 5.1.1 joint with equation (1.3) lead us to

$$\sum_{i=1}^{k} g_i + \sum_{i=1}^{k} r_i - k = g + r - 1. \tag{5.2}$$

5.2 Strong Compactness.

Next we prove the compactness results stated at the beginning of this Section. Firstly we study more carefully the unbounded domains appearing in Theorem 5.1.1. Following the above notation, take a sequence $\{M_n\}_n \subset \mathcal{M}(g,r)$ weakly convergent to $M_{1,\infty},\ldots,M_{k,\infty}$. Decompose each M_n with n large in bounded and unbounded domains $M_n = M_{1,n}\cup\ldots\cup M_{k,n}\cup\Omega_{1,n}\cup\ldots\cup\Omega_{r,n}$. Both the set $\mathbf{P} = \{\Omega_{1,n},\ldots,\Omega_{r,n}\}$ of unbounded domains and the set of boundary components of bounded domains are naturally ordered by their heights with respect to the vertical direction. Fix $\Omega\in\mathbf{P}$ (note that $\partial\Omega\neq\emptyset$). A component Γ of $\partial\Omega$ is said to be a *top boundary component* if Γ is the top boundary component of the bounded domain $M_{i,n}$ which contains Γ. *Bottom boundary components* of Ω are defined similarly. Clearly, the top (resp. bottom) unbounded domain only contains top (resp. bottom) boundary components. We decompose $\mathbf{P}$ as the disjoint union of the following three sets:

$\mathbf{A} = \{$the top and bottom unbounded domains$\}$,
$\mathbf{B} = \{\Omega\in\mathbf{P}-\mathbf{A}\ /\ \Omega$ has only top or bottom boundary components$\}$,
$\mathbf{C} = \mathbf{P}-(\mathbf{A}\cup\mathbf{B})$.

Proposition 5.2.1. *If $\Omega\in\mathbf{B}$, then Ω has at least three boundary components.*

Proof. Fix $\Omega\in\mathbf{B}$. Firstly suppose that $\partial\Omega$ is connected. As $\Omega\in\mathbf{B}$, $\partial\Omega$ must be the top or boundary component of a bounded domain $M_{i,n}\subset B_{i,n}$ joined with Ω along its boundary. Hence we can find an open disk $D\subset B_{i,n}$ with $\partial D=\partial\Omega$ and $M_{i,n}\cap D=\emptyset$. Note that $\Omega\cup D$ is a properly embedded topological plane, so it separates $\mathbb{R}^3$ in two components. As M_n is disjoint with D, it follows that M_n is contained in the closure of one the of the components of $\mathbb{R}^3-(\Omega\cup D)$, thus Ω is the top or the bottom unbounded domain, a contradiction. So, Ω must have least two boundary components.

Assume now that Ω has exactly two boundary components Γ_1,Γ_2. For $i=1,2$, Ω is joined along Γ_i to a bounded domain $M_{i,n}$, Γ_i being a top or bottom boundary component of $M_{i,n}$. As $n\to\infty$, a suitable rescaling of $M_{i,n}$ converges by Theorem 5.1.1 to the intersection with a big ball of one of the surfaces $M_{i,\infty}$ in the weak limit. As the top and bottom ends of each $M_{i,\infty}$ are of Catenoid type, we can suppose by taking n large enough that near each Γ_i, Ω looks like a neighborhood of a horizontal section in a vertical halfcatenoid. Cutting transversally Ω with suitable horizontal planes, we obtain a proper subdomain $\Omega'\subset\Omega$ whose boundary consists of two convex Jordan curves Γ_1',Γ_2' in horizontal planes. Moreover Ω' is close along Γ_i' to the intersection of a vertical halfcatenoid with a horizontal slab, $i=1,2$. If we prove that the force of Ω' along each Γ_i' is vertical we will contradict Corollary 3.3.1, thereby finishing the proof of the Proposition. To prove that the forces along Γ_1',Γ_2' are vertical, it suffices to check that each one of these curves disconnects M_n

(in such case each Γ_i' would be homologous to a sum of curves around the ends of M_n, whose forces are vertical).

Let D_i be the planar convex open disk bounded by Γ_i'. Thus $M_n \cap D_i = \emptyset$ for n large, and $\Omega' \cup D_1 \cup D_2$ is again a properly embedded topological plane hence it separates $\mathbb{R}^3$ in two components N_1, N_2. As Ω is neither the top or the bottom unbounded domain of M_n, we deduce that M_n meets N_1 and N_2. If both Γ_1', Γ_2' bounded top ends of $M_{1,\infty}, M_{2,\infty}$ (i.e. if the corresponding curves Γ_1, Γ_2 were top boundary components of $M_{1,\infty}$ and $M_{2,\infty}$), then $M_n - \Omega'$ would be below $\Omega' \cup D_1 \cup D_2$ in a neighborhood of $\Gamma_1' \cup \Gamma_2'$, hence $M_n - \Omega'$ would be entirely contained in the component N_j below $\Omega' \cup D_1 \cup D_2$, which is impossible. Similarly, both Γ_1', Γ_2' cannot bound bottom ends of $M_{1,\infty}, M_{2,\infty}$, thus one bounds a top and the other bounds a bottom end. This implies that one component of $\partial\Omega'$, say Γ_1', is the topological boundary of $M_n \cap N_1$ and Γ_2' is the one of $M_n \cap N_2$. In particular, each one of these curves disconnects M_n, which finishes the proof.

Let $\#(\mathbf{P})$ and $\#(\mathbf{P}_\partial)$ be respectively the number of elements in $\mathbf{P}$ and in

$$\mathbf{P}_\partial = \{\text{boundary components of elements in } \mathbf{P}\}.$$

We use similar notations with $\mathbf{A}, \mathbf{B}, \mathbf{C}$ instead of $\mathbf{P}$. Clearly $\#(\mathbf{P}) = r$, $\#(\mathbf{P}_\partial) = \sum_i r_i$, $\#(\mathbf{A}) = 2$ and $\#(\mathbf{A}_\partial) \geq 2$. As $\mathbf{A}, \mathbf{B}, \mathbf{C}$ form a partition of $\mathbf{P}$, we get

$$r = 2 + \#(\mathbf{B}) + \#(\mathbf{C}). \tag{5.3}$$

On the other hand, each $\Omega \in \mathbf{C}$ has at least a boundary component which is neither a top nor a bottom boundary component. As each one of the k surfaces $M_{i,\infty}$ in the weak limit has $r_i - 2$ middle ends, it holds $\#(\mathbf{C}) \leq \sum_{i=1}^k r_i - 2k$, and thus,

$$r \leq 2 + \#(\mathbf{B}) + \sum_{i=1}^k r_i - 2k. \tag{5.4}$$

Proposition 5.2.1 insures that $3\#(\mathbf{B}) \leq \#(\mathbf{B}_\partial)$. Note that the total number of top and bottom ends in all surfaces $M_{i,\infty}$ is $2k$, and at least two of them correspond to the elements of $\mathbf{A}$ (because the top and bottom unbounded domains only contain top and bottom boundary components, at least one each). Thus $\#(\mathbf{B}_\partial) \leq 2k - 2$ and

$$\#(\mathbf{B}) \leq \frac{2k-2}{3}. \tag{5.5}$$

To finish these general counting arguments, from $\#(\mathbf{P}_\partial) = \#(\mathbf{A}_\partial) + \#(\mathbf{B}_\partial) + \#(\mathbf{C}_\partial)$ and $3\#(\mathbf{B}) \leq \#(\mathbf{B}_\partial)$ one has

$$\sum_{i=1}^k r_i \geq 2 + 3\#(\mathbf{B}) + \#(\mathbf{C}_\partial). \tag{5.6}$$

Next result, which deals with the case where the weak limit has the simplest topology, is related with the Hoffman-Meeks conjecture.

Theorem 5.2.1. *Fix integers $g \geq 0, r \geq 2$. If a sequence $\{M_n\}_n \subset \mathcal{M}(g,r)$ is weakly convergent to $M_{1,\infty}, \ldots, M_{k,\infty}$ and each $M_{i,\infty}$ is a Catenoid, then $r \leq 2g+2$.*

Proof. We continue with the same notation as before. As each $M_{i,\infty}$ is a Catenoid we have $g_i = 0, r_i = 2$ $(1 \leq i \leq k)$ and all the boundary components of any unbounded domain are top or bottom boundary components, which implies $\mathbf{C} = \emptyset$. Thus equations (5.3) and (5.6) transform respectively into $r = 2 + \#(\mathbf{B})$ and $2k \geq 2 + 3\#(\mathbf{B})$. These two relations give $2k \geq 3r - 4$. On the other hand, (5.2) gives $k = g + r - 1$, from where $r \leq 2g + 2$ follows directly.

If in the statement of Theorem 5.1.1 we have $k = 1$, then $\{M_n\}_n$ converges, up to homotheties, to a properly embedded minimal surface M_∞ with multiplicity one, without loss of total curvature or topology. The following result gives a condition to insure such kind of strong convergence.

Theorem 5.2.2. *Fix integers $g \geq 0, r \geq 2$. If a sequence $\{M_n\}_n \subset \mathcal{M}(g,r)$ is weakly convergent to $M_{1,\infty}, \ldots, M_{k,\infty}$ and $M_{1,\infty}$ has genus g, then $k = 1$.*

Proof. As $g_1 + \ldots + g_k \leq g$ and $g_1 = g$, for $i \geq 2$ we get $g_i = 0$ hence $r_i = 2$ by Corollary 2.1.1. To compute r_1 we use (5.2), which now gives $r = r_1 + k - 1$. Hence (5.4) writes as $r_1 + k - 1 \leq 2 + \#(\mathbf{B}) + \sum_{i=1}^{k} r_i - 2k = \#(\mathbf{B}) + r_1$, i.e. $\#(\mathbf{B}) \geq k - 1$. This inequality joint with (5.5) force k to be one.

If any sequence in $\mathcal{M}(g,r)$ has a subsequence converging to a weak limit with $k = 1$, we will say that $\mathcal{M}(g,r)$ is *compact (up to homotheties)*. Finally we prove two results about this property.

Corollary 5.2.1. *For any $r \geq 5$, the space $\mathcal{M}(1,r)$ is compact (up to homotheties).*

Proof. Fix $r \geq 5$ and take a sequence $\{M_n\}_n \subset \mathcal{M}(g,r)$ weakly convergent to $M_{1,\infty}, \ldots, M_{k,\infty}$. The Corollary will be proved if we check that $k = 1$. On the contrary, if $k \geq 2$ then Theorem 5.2.2 insures that all the $M_{i,\infty}$ have genus zero. Using Theorem 5.2.1, we must have $r \leq 4$, a contradiction.

As a generalization of Corollary 5.2.1, we have

Corollary 5.2.2. *Fix integers $g \geq 1$, $r > 2g + 2$. If $\mathcal{M}(g',r') = \emptyset$ for each pair (g',r') with $0 \leq g' < g$ and $r' > 2g' + 2$, then $\mathcal{M}(g,r)$ is compact (up to homotheties).*

Proof. Let $\{M_n\}_n \subset \mathcal{M}(g,r)$ be a sequence which converges weakly to $M_{1,\infty}, \ldots, M_{k,\infty}$. Reasoning by contradiction, assume $k \geq 2$. By Theorem 5.2.2, the genus g_i of each $M_{i,\infty}$ must be strictly less than g, hence our hypotheses imply $r_i \leq 2g_i + 2$, r_i being the number of ends of $M_{i,\infty}$. Plugging this inequality and $r > 2g+2$ into (5.2) we get $3(\sum_i r_i - r) < 4(k-1)$, which joint with (5.4) gives $2k - 2 < 3\#(\mathbf{B})$, in contradiction with (5.5).

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