Uniqueness of the Riemann Minimal Examples

William H. Meeks III*
Mathematics Department
University of Massachusetts
Amherst, MA 01003, U.S.A.

Joaquín Pérez†
Departamento de Geometría y Topología
Universidad de Granada
Fuentenueva 18071, Granada, Spain - FAX: 34 58 243281

Antonio Ros†
Departamento de Geometría y Topología
Universidad de Granada
Fuentenueva 18071, Granada, Spain

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ABSTRACT.-We prove that a properly embedded minimal surface in $\mathbb{R}^3$ of genus zero with infinite symmetry group is a plane, a catenoid, a helicoid or a Riemann minimal example. We introduce the language of Hurwitz schemes to understand the underlying moduli space of surfaces in our setting.

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1 Introduction.

In a posthumously published paper, B. Riemann [20] (see also [4], pages 461-464) used elliptic functions to classify all minimal surfaces in $\mathbb{R}^3$ that are foliated by circles and straight lines in horizontal planes. He showed that these examples are the plane, the helicoid, the catenoid and a one-parameter family $\{R_t\}_{t>0}$ with infinite topology. The new surfaces R_t, called *Riemann minimal examples*, intersect horizontal planes in lines at precisely integer heights, with the radius of the circles going to infinity near the lines. Each surface R_t is also invariant under reflection in the (x_1, x_3)-plane and by the translation by $(t, 0, 2)$. Furthermore, in the complement of a solid open cylinder in the direction of $(t, 0, 2)$, the surface R_t consists topologically of disjoint punctured disks, each one close and asymptotic to a plane at integer height.

Each R_t is naturally conformally diffeomorphic to a vertical cylinder in $\mathbb{R}^3$, punctured at integer heights corresponding to the ends. In particular, R_t is conformally equivalent to $\mathbb{C}^* = \mathbb{C} - \{0\}$ punctured in a discrete set of points with 0 and ∞ being limit points of the planar ends. In topological terms, this means that R_t is a properly embedded periodic minimal surface in $\mathbb{R}^3$ of genus zero, with exactly two limit ends.

Our main theorem is that the Riemann examples are unique in this setting:

Theorem 1 *If M is a properly embedded periodic minimal planar domain in $\mathbb{R}^3$ with two limit ends, then M is one of the Riemann examples.*

The above theorem is equivalent, after taking quotients by a translation or screw motion symmetry, to proving that a genus one properly embedded minimal surface with planar ends in a nonsimply connected flat three-manifold is the quotient of some R_t. We will demonstrate Theorem 1 by proving the uniqueness of Riemann examples in the quotient space. This result follows from an exhaustive study of the global and local properties of the *Flux map*, which associates to each surface in our setting its flux along a horizontal section. In our analysis, we introduce *Hurwitz schemes*, which seem to be the ideal setting for analyzing the type of moduli space and/or uniqueness questions that often arise in the global theory of minimal surfaces.

Recent work of Frohman, Kusner, Meeks and Rosenberg [6] shows that an embedded periodic genus zero minimal surface that is not the plane or the helicoid must have exactly two limit ends. Their result, together with our uniqueness theorem, shows that the following more general classification holds:

Theorem 2 *Suppose that M is a properly embedded minimal surface in $\mathbb{R}^3$ with genus zero. If the symmetry group of M is infinite, then M is a plane, a catenoid, a* *helicoid or a Riemann example. In particular, a properly embedded minimal surface with genus zero has infinite symmetry group if and only if it is foliated by circles and/or lines in parallel planes.*

For previous partial results on Theorem 1, see [5, 10, 12, 13, 17, 18, 19, 22]. See also [13] for an up to date complete discussion and further references to the history behind our Theorem 1. More results about singly-periodic minimal surfaces with planar ends can be found in [1, 2].

2 Proof of the Main Theorem.

We now prove Theorem 1, assuming the results in later sections. Let M be a periodic properly embedded minimal surface in $\mathbb{R}^3$ with genus zero and two limit ends. Since every periodic minimal surface with more than one end has a top and a bottom limit end, it follows that all the middle ends are simple ends, which in the case of finite genus means annular ends (see [7] for a complete discussion of the ordering of the ends of a properly embedded minimal surface). The structure theorem in Callahan, Hoffman and Meeks [3] implies that there exists a nontrivial screw motion or a translation in $\mathbb{R}^3$ which preserves the surface and such that the quotient surface M/Λ, Λ being the cyclic group generated by the above rigid motion, has genus one and finitely many planar parallel ends. In particular, M/Λ has finite total curvature. For the remainder of the proof, we will suppose that the ends of M/Λ are horizontal. By a result of Pérez and Ros [19], we have that Λ must be generated by a translation. Hence, Theorem 1 follows from the next one:

Theorem 3 *Let M be a properly embedded minimal surface in $\mathbb{R}^3/T$ of genus one and a finite number of planar ends, T being a nontrivial translation. Then, M is a quotient of a Riemann example.*

We will let $\mathcal{S}$ denote the space of properly embedded minimal (oriented) tori in a quotient of $\mathbb{R}^3$ by a translation T, that depends on the surface, with $2n$ horizontal planar (ordered) ends. By the maximum principle at infinity, see [11] and also [16], the ends are separated by a positive distance. Moreover, embeddedness insures that consecutive ends have reversed (vertical) limit normal vectors. The allowed orders for the ends with normal vector $(0,0,-1)$, $p_1,\ldots,p_n$, and for the ends with normal vector $(0,0,1)$, $q_1,\ldots,q_n$, will be those in which the list $(p_1,q_1,\ldots,p_n,q_n)$

corresponds to consecutive ends in the quotient space. We will identify in our space $\mathcal{S}$ two surfaces which differ by a translation that preserves both orientation and the order of the above list of ends, or equivalently, that maps the first end in the list of one of the surfaces into the first end of the list of the second surface.

A surface $M \in \mathcal{S}$ cuts transversally any horizontal plane nonasymptotic to its ends in a compact Jordan curve γ. We will orient γ in such a way that the flux vector v along γ has positive third coordinate. As the flux is a homological invariant and vanishes around a planar end, it follows that v does not depend on the height of the intersecting plane. From the results in [19] we know that v cannot be vertical. We will rescale our surfaces so that this flux has always third coordinate equal to one. We define the *Flux map*

$$F : \mathcal{S} \longrightarrow \mathbb{R}^2 - \{0\}$$

by letting $F(M)$ be the horizontal part of v.

Denote by $\mathcal{R}$ the subset of $\mathcal{S}$ consisting of the Riemann examples and their rotations around the vertical x_3-axis, see more details in Section 3. We claim $\mathcal{R}$ is open and closed in $\mathcal{S}$: it is proved in Pérez [18] that any small deformation in $\mathcal{S}$ of a Riemann example must be another Riemann example. This gives the openness of $\mathcal{R}$. For closedness, observe that Riemann examples are characterized by the fact of being foliated by circles or lines in horizontal planes, see [20]. So, if a sequence of Riemann examples converges to a surface $M \in \mathcal{S}$ in the uniform topology on compact subsets, then horizontal sections on M will be circles or lines and, thus, $M \in \mathcal{R}$.

Proof of Theorem 3. The result follows from a combination of the following three statements:

- The Flux map is proper (which will be proved in Theorem 5).
- The Flux map is open (see Theorem 6).
- There exists a positive number ε such that if $M \in \mathcal{S}$ satisfies $|F(M)| < \varepsilon$, then $M \in \mathcal{R}$ (Theorem 7).

Indeed, suppose $\mathcal{S}' := \mathcal{S} - \mathcal{R} \neq \emptyset$. As $\mathcal{S}'$ is open and closed in $\mathcal{S}$ and the Flux map $F : \mathcal{S} \longrightarrow \mathbb{R}^2 - \{0\}$ is proper (Theorem 5) and open (Theorem 6), it follows that its restriction to $\mathcal{S}'$ is also a proper and open map. In particular, $F(\mathcal{S}') = \mathbb{R}^2 - \{0\}$. This contradicts Theorem 7, which insures that $F(\mathcal{S}')$ omits a punctured neighborhood of zero in $\mathbb{R}^2$. Now the proof is complete. $\square$

3 Background.

We now recall some general properties to be satisfied by any surface in the setting of Theorem 1. With this aim, take an orientable surface $M \subset \mathbb{R}^3/T$ in the space $\mathcal{S}$ of properly embedded minimal tori in quotients of $\mathbb{R}^3$ by nontrivial translations, with $2n$ horizontal planar ends. Note that M has finite total curvature and well-defined Gauss map, that we will identify with the meromorphic function g obtained after stereographic projection from the North Pole of the sphere $\mathbb{S}^2$. We will also denote by $\phi = \frac{\partial x_3}{\partial z} dz$ the height differential, where z denotes a holomorphic coordinate. Both g, ϕ extend meromorphically to the conformal compactification Σ of M through the $2n$ punctures corresponding to the ends. As the ends are planar and horizontal, ϕ has neither poles or zeroes on Σ, g has double zeroes at n of the ends, $p_1, \ldots, p_n$, double poles at the remaining n ends, $q_1, \ldots, q_n$, and M has no other points with vertical normal vector. In particular, the minimal surface intersects transversally in a compact Jordan curve each horizontal plane in $\mathbb{R}^3/T$ whose height does not coincide with the height of an end. Denote by γ one of these horizontal compact sections. Then, from Weierstrass representation one easily deduces that the condition to close the real period of the surface along γ is equivalent to

$$\int_\gamma \frac{1}{g}\phi = \overline{\int_\gamma g\phi}. \tag{1}$$

and in this case, the horizontal flux along γ is given by

$$F(M, \gamma) = i \int_\gamma g\phi = i \overline{\int_\gamma \frac{1}{g}\phi}.$$

At an end p_i with $g(p_i) = 0^{(2)}$, both ϕ and $g\phi$ have no singularities, and $\frac{1}{g}\phi$ has a double pole without residue. Similarly, the period problem at an end q_i with $g(q_i) = \infty^{(2)}$ can be solved if and only if the residue of $g\phi$ at q_i vanishes. Finally, note that as the sum of the residues of $g\phi$ and $\frac{1}{g}\phi$ on Σ is zero, to solve the period problem at the ends it suffices to insure that

$$\mathrm{Res}_{p_i}\left(\frac{1}{g}\phi\right) = \mathrm{Res}_{q_i}(g\phi) = 0, \qquad 1 \leq i \leq n-1. \tag{2}$$

Reciprocally, consider a genus one surface Σ and a pair $(g, [\gamma])$ where g is a meromorphic function with degree $2n$ and divisor $(g) = p_1^2 \ldots p_n^2 q_1^{-2} \ldots q_n^{-2}$, for distinct points $p_i, q_j \in \Sigma$, and $[\gamma]$ is a nonzero homology class in $H_1(\Sigma, \mathbb{Z})$. Denote by ϕ

the unique meromorphic differential on Σ satisfying $\int_\gamma \phi = 2\pi i$. Then, (g, ϕ) defines a complete immersed minimal surface in a quotient of $\mathbb{R}^3$ by a translation if and only if (1) and (2) hold simultaneously. Furthermore, when it exists, the surface above has $2n$ embedded planar horizontal ends, although the surface itself may not be embedded.

We now describe briefly the Riemann minimal examples, which will be characterized in this paper. They form a one-parameter family $\{R_\lambda\}_{\lambda>0}$ of properly embedded singly periodic minimal surfaces, each one foliated by circles and lines in parallel planes, which we will think about as horizontal. In fact, a surface satisfying such hypotheses must be a Riemann example. Each surface R_λ is invariant by a cyclic group of translations, such that the quotient with a smaller number of ends is a twice punctured torus that can be conformally parametrized by $\{(z, w) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}} \mid w^2 = z(z-\lambda)(\lambda z+1)\} - \{(0,0), (\infty, \infty)\}$, with Weierstrass data

$$g(z, w) = z, \qquad \phi = A_\lambda \frac{dz}{w},$$

where A_λ is a nonzero complex number satisfying $A_\lambda^2 \in \mathbb{R}$. R_λ has a reflection symmetry in a vertical plane (which in the parametrization above is the (x_1, x_3)-plane), and a 180 degree rotation around straight lines orthogonal to the symmetry plane, that are situated at half distance between each pair of consecutive ends and cut the surface orthogonally. If we consider any multiple of the generator of the translation group leaving R_λ invariant, we will find a quotient surface with the topology of a torus minus $2n$ points, $n \in \mathbb{N}$ (here we have a different parametrization of the Riemann minimal examples from the one we used in the introduction).

We will use some standard results in several complex variables theory. For convenience of the reader, we list here these results as stated in the book of P. Griffiths and J. Harris [9]. A subset V of a complex manifold M is called an *analytic subvariety* if for any $p \in M$, there exists a neighborhood U of p in M and a finite collection of holomorphic functions $f_1, \ldots, f_k$ on U such that $U \cap V = \{q \in U \mid f_1(q) = \ldots = f_k(q) = 0\}$, see [9], p. 20. In particular, V is a closed subset. Analytic subvarieties of $\mathbb{C}$ are $\mathbb{C}$ itself and its discrete subsets. It follows directly from the definition that if V is an analytic subvatiety of M and V' is an open and closed subset of V, then also V' is an analytic subvariety of M.

Now we state some important theorems concerning analytic subvarieties.

Riemann's extension theorem ([9] p. 9) *Let V be an analytic subvariety of an open connected subset U of $\mathbb{C}^n$, $V \neq U$, and $f : U - V \longrightarrow \mathbb{C}$ a bounded holomorphic function. Then, f extends holomorphically to U.*

Proper mapping theorem ([9] p. 395) *Let $f : M \longrightarrow N$ be a holomorphic map between complex manifolds and $V \subset M$ an analytic subvariety. If $f|_V$ is proper, then $f(V)$ is an analytic subvariety of N.*

We will apply the result above only in when V is compact. In this case, properness is clear. If moreover $N = \mathbb{C}$, then we conclude that $f(V)$ is a finite set of points.

In general, nonconstant holomorphic maps between manifolds with the same dimension are not open maps. However, openness holds for the subclass of *finite maps*, i.e. maps such that the inverse image of any point is finite. In fact, we have the following theorem:

Openness theorem ([9] p. 667) *Let U be an open subset of $\mathbb{C}^n$ with $0 \in U$ and $f : U \longrightarrow \mathbb{C}^n$ a holomorphic map such that $f^{-1}(\{0\}) = \{0\}$. Then, there exists a neighborhood W of 0 in U such that $f|_W$ is an open map.*

4 The Properness of the Flux Map.

In this section we will prove the properness of the Flux map $F : \mathcal{S} \longrightarrow \mathbb{R}^2 - \{0\}$ on the space of properly embedded minimal genus one examples in $\mathbb{R}^3/T$ for varying T. This is equivalent to proving that if $\{M(i) \mid i \in \mathbb{N}\}$ is a sequence of examples whose horizontal fluxes (i.e., the two first components of the flux vectors) in norm lie in some interval $[\frac{1}{\varepsilon}, \varepsilon]$, $\varepsilon > 0$, then a subsequence of the $\{M(i)\}_i$ converges smoothly to another such example. Also, some of the results proved here will be useful in understanding the limit of a sequence $\{M(i)\}_i$ whose horizontal fluxes converge to zero.

Lemma 1 *Suppose that $\Sigma \subset \mathbb{R}^3/T$ is a properly embedded nonflat orientable minimal surface with finite topology and horizontal planar ends. Also suppose that the absolute value of the Gaussian curvature of Σ is bounded from above by some positive constant c. Let $\Sigma(t)$ be a parallel surface at distance t from Σ, $0 \leq |t| < \sqrt{c}$. Then,*

$\Sigma(t)$ is embedded and has nonnegative mean curvature. Furthermore, the family $\{\Sigma(t) \mid 0 \leq |t| < \sqrt{c}\}$ gives rise to a foliation of a regular neighborhood of Σ. In particular, the distance between the ends of Σ at infinity is greater than or equal to $2\sqrt{c}$ and the vertical component of T is greater than or equal to $4n\sqrt{c}$, where $2n$ is the number of ends of Σ.

Proof. The proof of this assertion is standard in the case of a closed orientable minimal surface in a flat three dimensional torus, by a simple application of the maximum principle for surfaces of nonnegative mean curvature. In the present noncompact setting with the ends of Σ asymptotic to planes, a simple application of the maximum principle for oppositly oriented graphs of nonnegative mean curvature shows that the minimal distance between two disjoint surfaces of nonpositive curvature (and opposite unit normals), whose ends are asymptotic to planes, cannot go to zero at infinity (see [16] for similar arguments). With this in mind, the proof of the Lemma is straightforward and details are left to the reader (see also Lemma 4 in [21] for a detailed proof of a similar assertion in the case of an embedded minimal surface of finite total curvature in $\mathbb{R}^3$). $\square$

Lemma 2 *For any positive number k, there exists a positive constant $\varepsilon(k)$ such that the following statement holds. Suppose M is a complete minimal surface and P is a horizontal plane that intersects M orthogonally at a point x_0, and $\alpha = M \cap P$ is the intersection curve. If the vertical flux of α is less than k, then in the intrinsic neighborhood of distance k from x_0, there is a point of M where the absolute Gaussian curvature is greater than $\varepsilon(k)$.*

Proof. Suppose that the lemma fails for some positive number k. In this case, there exists a complete minimal surface $M(i) \subset \mathbb{R}^3$, passing through the origin, with the (x_1, x_2)-plane P orthogonal to $M(i)$ at the origin, and such that the intrinsic neighborhood of the origin, $N(i)$, of radius k, has maximum absolute curvature less than $\frac{1}{i}$. Since the $N(i)$ converge to vertical planar disks as $i \to \infty$, the vertical flux of $N(i) \cap P$ is close to $2k$ for i large. This implies that the flux of the $M(i)$ is greater than k, which is a contradiction. This finishes the proof. $\square$

Proposition 1 *Suppose that $\{\widetilde{M}(i) \mid i \in \mathbb{N}\}$ is a sequence of embedded Riemann-type minimal surfaces in $\mathbb{R}^3$ invariant under translation by $T(i)$, and such that*

$\widetilde{M}_i/T(i) \subset \mathbb{R}^3/T(i)$ has $2n$ horizontal planar ends. Suppose further that the point of maximal absolute Gaussian curvature of each $\widetilde{M}(i)$ occurs at the origin and equals to one. Then, after choosing a subsequence, we may assume that the $\widetilde{M}(i)$ converge to a properly embedded minimal surface $\widetilde{M}_\infty$, which must be one of the following examples.

i) $\widetilde{M}_\infty$ is a vertical catenoid. In this case, the flux vectors of the $\widetilde{M}(i)$ converge to $(0,0,2\pi)$ and $\{\|T(i)\|\}_i$ is unbounded.

ii) $\widetilde{M}_\infty$ is a vertical helicoid. In this case, we may assume after choosing a subsequence that $\{T(i)\}_i \to (0,0,2\pi k)$ for some fixed positive integer k. Furthermore, the vertical components of the fluxes of the $\widetilde{M}(i)$ go to infinity as $i \to \infty$.

iii) $\widetilde{M}_\infty$ is another Riemann-type example. In this case, we may assume after choosing a subsequence that $T(i) \to T \in \mathbb{R}^3$, $\widetilde{M}_\infty/T \subset \mathbb{R}^3/T$ has $2n$ horizontal ends and the fluxes of $\widetilde{M}(i)$ converge to the flux of $\widetilde{M}_\infty$.

Proof. Lemma 1 implies that there are uniform local area bounds for the family $\{\widetilde{M}(i)\}_i$. As curvature is also uniformly bounded, it follows, after choosing a subsequence, that $\{\widetilde{M}(i)\}_i$ converges smoothly on compact subsets of $\mathbb{R}^3$ to a properly embedded minimal surface $\widetilde{M}_\infty$ with absolute Gaussian curvature 1 at the origin.

By Lemma 1, the vertical component of $T(i)$ is bounded from below by $4n$. Suppose that $\{\|T(i)\|\}_i$ is not bounded from above. Then, $\widetilde{M}_\infty$ is a limit of surfaces contained in fundamental regions of $\widetilde{M}(i)$, which would imply that $\widetilde{M}_\infty$ has finite total curvature. Since $\widetilde{M}(i)$ has genus zero, the same holds for $\widetilde{M}_\infty$, and by the uniqueness theorem of López and Ros [14], $\widetilde{M}_\infty$ is a catenoid. As the $\widetilde{M}(i)$ have no vertical normals, $\widetilde{M}_\infty$ has the same property and thus it is a vertical catenoid. The maximum absolute Gaussian curvature of $\widetilde{M}_\infty$ equals one, so its flux vector equals $(0,0,2\pi)$. Since $\{\widetilde{M}(i)\}_i$ converges smoothly to $\widetilde{M}_\infty$, there must be simple closed curves of $\widetilde{M}(i)$ whose fluxes converge to $(0,0,2\pi)$ as $i \to \infty$. Thus, in the case of $\{\|T(i)\|\}_i$ is not uniformly bounded, case *i)* of the Proposition must hold.

Suppose now that $\widetilde{M}_\infty$ is not as in description *i)* above. After passing to a subsequence, we will assume that $\widetilde{M}(i)$ converges to $\widetilde{M}_\infty$. Since $\{\|T(i)\|\}_i$ is bounded and $\|T(i)\| \geq 4n$, a subsequence of the $T(i)$ converges to some nontrivial vector $T \in \mathbb{R}^3$. Thus $\widetilde{M}_\infty$ is invariant under translation by T. Suppose that $\widetilde{M}_\infty$ is simply connected. Since $\widetilde{M}_\infty$ is periodic and nonplanar, Theorem 2 in [15] states that it must be a helicoid. Since the normal vectors on $\widetilde{M}(i)$ are never vertical, the same holds for $\widetilde{M}_\infty$. This is sufficient to show that $\widetilde{M}_\infty$ is a vertical helicoid and T is vertical. Since the maximum absolute Gaussian curvature of $\widetilde{M}_\infty$ is one, $T = (0, 0, 2\pi k)$ for some positive integer k. Since the $\widetilde{M}(i)$ converge uniformly on compact subsets of $\mathbb{R}^3$ to $\widetilde{M}_\infty$ and the vertical helicoid has infinite vertical flux, the last statement in *ii)* holds.

Suppose that $T(i) \to T$ as $i \to \infty$ and $\widetilde{M}_\infty$ is not simply connected. We claim that $\widetilde{M}_\infty$ is another Riemann-type example. First note that in a fundamental slab region, $\widetilde{M}_\infty$ has finite absolute total curvature at most $8\pi n$. The reason for this bound on total curvature is that such a fundamental slab region for $\widetilde{M}_\infty$ is obtained as limit of parallel slab fundamental regions for the $\widetilde{M}(i)$, and in each one of these regions the corresponding surface has absolute total curvature $8\pi n$. This implies that the quotient surface $M_\infty = \widetilde{M}_\infty/T \subset \mathbb{R}^3/T$ has absolute total curvature at most $8n\pi$. Hence, M_∞ has finite topology and its ends are annular. By Theorem 1 in [3], the ends of M_∞ are planar. Since every such surface has Gauss map that misses only possibly two values corresponding to the normals of the ends and the Gauss map of M_∞ misses the values $\{0, \infty\}$, then the ends of M_∞ are horizontal. We now show that M_∞ has exactly $2n$ ends, which are the limits of the $2n$ ends of the $M(i) = \widetilde{M}(i)/T(i)$. By Lemma 1, the spacing between the ends of the $\widetilde{M}(i)$ is bounded away from zero and hence, the relative heights of the ends of the $M(i)$ converge to distinct heights in $\mathbb{R}^3/T$. It is easy to check that these limit heights in $\mathbb{R}^3/T$ correspond to ends of M_∞, since the level sets of these heights are noncompact (because they are smooth limits of noncompact curves on the $M(i)$). As the finite total curvature of M_∞ cannot be greater than the total curvature of the $M(i)$, it follows that M_∞ cannot have more than $2n$ ends. This finishes the proof of the proposition. $\square$

We now prove the main technical result of this section.

Theorem 4 *Let $\{M(i) \mid i \in \mathbb{N}\} \subset \mathcal{S}$ be a sequence of examples with $2n$ ends in $\mathbb{R}^3/T(i)$, normalized so that the vertical component of the flux of $M(i)$ is one. Suppose that the horizontal fluxes of the family $\{M(i)\}_i$ are bounded. Then, the Gaussian curvature of the family $\{M(i)\}_i$ is uniformly bounded.*

Proof. Let $\lambda_i := \max_{M(i)} \sqrt{|K|}$ (here K denotes Gaussian curvature), and $\Sigma(i) := \lambda_i M(i) \subset \mathbb{R}^3/\lambda_i T(i)$. We will denote by $\widetilde{\Sigma}(i)$ the lifted surface of $\Sigma(i)$ to $\mathbb{R}^3$ (and similarly with $\widetilde{M}(i)$ for $M(i)$). *Assume that the theorem fails.* We will derive a contradiction by proving the existence of almost helicoids in $M(i)$ for i large and then, use them for calculating the horizontal part of the flux, which will turn out to be unbounded.

By replacing $M(i)$ by a subsequence, we can suppose that λ_i is strictly increasing. After a translation, we will also assume that a point of maximal absolute Gaussian curvature of $\widetilde{\Sigma}(i)$ occurs at the origin. By Proposition 1, a subsequence of the $\{\widetilde{\Sigma}(i)\}_i$ converges smoothly on compact sets of $\mathbb{R}^3$ to a properly embedded minimal surface H_1, which must be a vertical catenoid, a vertical helicoid or another Riemann-type example. Since the vertical fluxes of the $\widetilde{\Sigma}(i)$ go to infinity as $i \to \infty$, Proposition 1 also implies that H_1 is a vertical helicoid with vertical translation vector T, and a subsequence of the $\lambda_i T(i)$ converge to T. Again after replacing the $M(i)$ by this subsequence, we may assume that the surfaces $\widetilde{\Sigma}(i)$ converge to H_1.

Now let $\pi_i : \mathbb{R}^3/\lambda_i T(i) \longrightarrow \mathbb{R}^2$ denote the linear projection along the direction of $\lambda_i T(i)$ onto the (x_1, x_2)-plane $\mathbb{R}^2 \subset \mathbb{R}^3/\lambda_i T(i)$. From now on, we will only consider very large values of i, so that inside the solid cylinder $\pi^{-1}(D)$, $D = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| < 1\}$, the surface $\Sigma(i)$ is connected and extremely close to a piece of a vertical helicoid containing its axis and $\lambda_i T(i)$ is almost vertical. On the compactified genus one surface $\overline{\Sigma}(i)$ obtained by attaching to $\Sigma(i)$ its $2n$ ends, $H_1(i) := \Sigma(i) \cap \pi_i^{-1}(D)$ is a open annulus that does not separate $\overline{\Sigma}(i)$, hence its complement is a compact annulus $F(i)$ that contains the $2n$ ends of $\Sigma(i)$. If the Gauss map g_i of $F(i)$ were never horizontal, then the extended projection $\pi_i : F(i) \longrightarrow (\mathbb{R}^2 \cup \{\infty\}) - D$, sending the end points to ∞, would be a proper submersion and hence a covering map (note that $F(i)$ has no points above D by definition of $H_1(i)$). This is impossible since $(\mathbb{R}^2 \cup \{\infty\}) - D$ is simply connected but $F(i)$ is not. Denote by $\mathbb{S}^1 = \{z \in \mathbb{S}^2 = \overline{\mathbb{C}} \mid \|z\| = 1\}$. Since $H_1(i)$ is almost a vertical helicoid inside of $\pi_i^{-1}(D)$, $\left(g_i|_{H_1(i)}\right)^{-1}(\mathbb{S}^1)$ is one simple closed curve on $H_1(i)$ that covers evenly (with finite multiplicity) $\mathbb{S}^1$ through g_i. Since the Gauss map is horizontal somewhere on $F(i)$, the multiplicity of the above covering is less that the degree of g_i. Now it follows that for every $\theta \in \mathbb{S}^1$ there exists at least one point in $F(i)$ with normal vector θ.

Recall that the maps $g_i : \Sigma(i) \longrightarrow \mathbb{C}$ have at most $2n$ distinct branch values. Hence, there is a fixed small $\varepsilon > 0$ such that for every i, there exists a $\theta(i) \in \mathbb{S}^1 \subset \mathbb{S}^2$ with the disk of radius ε in $\mathbb{S}^2$ around $\theta(i)$ disjoint from the branch locus of g_i. Renormalize $\widetilde{\Sigma}(i)$ by translating a point $p(i)$ with normal vector $\theta(i)$ to the origin. As before, after choosing a subsequence we may assume that the $\Sigma(i)$ converge to a properly embedded minimal surface H_2 in $\mathbb{R}^3/T$. If the curvature of H_2 is not identically zero, then our previous arguments show that H_2 must be a vertical helicoid.

We now show that the curvature of H_2 is not identically zero, which will imply that H_2 is a vertical helicoid. Consider the neighborhood $N(i) = \{x \in \Sigma(i) \mid \text{dist}(x, p(i)) < \frac{3}{2}\|T\|\}$ of $p(i)$. If the curvature of H_2 were identically zero, then H_2 would contain a component that is a flat vertical annulus in $\mathbb{R}^3/T$ passing through the origin. This means that for i large, $N(i)$ intersects the (x_1, x_2)-plane in $\mathbb{R}^3/\lambda_i T(i)$ in either $1, 2$ or 3 arcs near the origin, which are C^1-close (and converging as i gets large) to straight line segments. The first part of Lemma 1 shows that there can be just one arc for i large. The one arc intersection implies that for i large, $N(i)$ contains a loop α_i of $\Sigma(i)$ giving rise to the period vector $\lambda_i T(i)$. However, by construction $g_i(\alpha_i)$ lies in the disk around $\theta(i)$ with radius going to zero as i goes to ∞. In particular, $g_i(\alpha_i)$ eventually lies in the disk on $\mathbb{S}^2$ of radius ε around $\theta(i)$. Since this disk is disjoint from the branch values of g_i, the inverse image of this disk via g_i consists of disks on $\Sigma(i)$, one of which contains α_i. But this contradicts the fact that α_i is homotopically nontrivial on $\Sigma(i)$. This contradiction completes the proof that H_2 is a vertical helicoid in $\mathbb{R}^3/T$.

So far we have shown that for i large, the surfaces $\Sigma(i)$ have two large regions in which $\Sigma(i)$ is close to two helicoids, that can occur in the limit. In $\mathbb{R}^2 \subset \mathbb{R}^3/\lambda_i T(i)$, choose disjoint open round disks $D_1(i), D_2(i)$ of the same radius such that the solid cylinders $\pi_i^{-1}(D_j(i))$ capture the forming helicoids H_1, H_2 minus neighborhoods of their ends. Furthermore, these disks can be chosen such that $H_j(i) := \pi_i^{-1}(D_j(i)) \cap \Sigma(i)$ is a connected annulus "close to" H_j, $j = 1, 2$. After replacing $\Sigma(i)$ by a subsequence, we may assume the following properties:

1. The radius $r(i)$ of $D_1(i)$ goes to infinity as i goes to ∞.

2. $\lim\limits_{i\to\infty} \dfrac{r(i)}{\lambda_i} = 0.$

3. The normal line on the boundary "helices" of $H_j(i)$ makes an angle of less than $\frac{1}{i}$ with $(0, 0, 1)$.

We claim that the extended Gauss map $\overline{g}_i$ of $\overline{\Sigma}(i) - (H_1(i) \cap H_2(i))$ is contained in the two disks of radius $\frac{1}{i}$ around the North and the South Poles in $\mathbb{S}^2$. By the open mapping property of $\overline{g}_i$ on the compactification $\overline{\Sigma}(i)$, this is equivalent to proving that g_i has no horizontal normal vectors on $\Sigma(i) - (H_1(i) \cap H_2(i))$. If g_i has such a horizontal vector, our previous argument used in the construction of H_2 proves that for large i we have another vertical helicoid forming. Repeating this argument a finite number of times yields a sequence $H_1(i), H_2(i), \ldots, H_k(i)$ of almost helicoids on

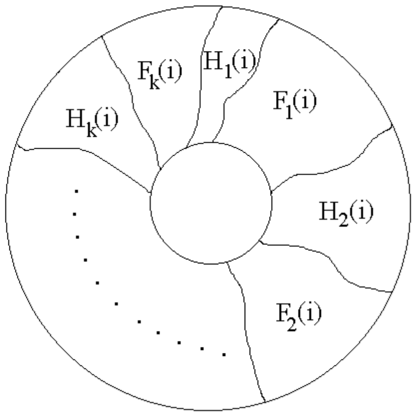

Figure 1:

$\Sigma(i)$ for i large. Since the absolute total curvature of $\Sigma(i)$ is $8n\pi$ and each $H_j(i)$ has approximately some positive multiple of 4π for absolute total curvature, this process of construction cannot produce more that $2n$ such helicoids. The complements of the k helicoids in $\overline{\Sigma}(i)$ gives k closed annular strips $F_1(i), \ldots, F_k(i)$ (see Figure 1). This means that one eventually arrives at a sequence $H_1(i), H_2(i), \ldots, H_k(i)$ and $g_i\left(\Sigma(i) - \cup_{j=1}^{k} H_j(i)\right)$ is contained in some small neighborhood of $\{0, \infty\}$ in $\mathbb{S}^2$. Consider the projection $\pi_i : F_1(i) \longrightarrow (\mathbb{R}^2 \cup \{\infty\}) - \cup_{j=1}^{k} D_j(i)$. This projection is a proper submersion and therefore a covering space. Since $\partial F_1(i)$ has two components, the boundary of the base space can only have two components, which implies $k = 2$.

Now consider the implications that these results, already proven for $\Sigma(i)$, have for the original surfaces $M(i)$ in $\mathbb{R}^3/T(i)$. In this case we will use some of the previous notation from the case of $\Sigma(i)$. Let $\pi_i : \mathbb{R}^3/T(i) \longrightarrow \mathbb{R}^2$ denote the projection on the (x_1, x_2)-plane in the direction of $T(i)$, and $\frac{1}{\lambda_i}H_1(i), \frac{1}{\lambda_i}H_2(i)$ the forming helicoids, contracted by λ_i. The cylindrical regions containing the contracted almost helicoids project onto the disks $\frac{1}{\lambda_i}D_1(i), \frac{1}{\lambda_i}D_2(i)$, both with the same radius going to zero as i goes to infinity. Since a homothety does not change the Gauss map, the Gauss map $g_i : M(i) \longrightarrow \mathbb{S}^2$ (we are working with the same notation as in the case of $\Sigma(i)$) restricted to $M(i) - \left(\frac{1}{\lambda_i}H_1(i) \cup \frac{1}{\lambda_i}H_2(i)\right)$ is contained in the disks of radius $\frac{1}{i}$ around the points $0, \infty \in \mathbb{S}^2$.

Assume that $F_1(i), F_2(i)$ are labeled so that the normal vectors of $M(i)$ are upward pointing on $F_1(i)$ and downward pointing on $F_2(i)$. Let $L(i)$ denote the

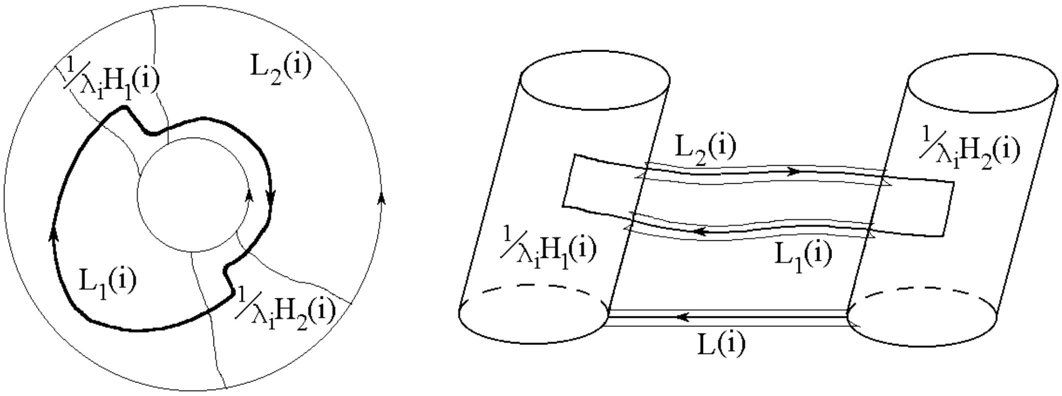

Figure 2:

line segment joining $\partial D_1(i)$ to $\partial D_2(i)$ that minimizes the distance. Recall that the compactified torus $\overline{M}(i)$ is divided into four annuli, two of them corresponding to $\frac{1}{\lambda_i}H_1(i), \frac{1}{\lambda_i}H_2(i)$ and the other two annuli, that we denote by $F_1(i), F_2(i)$, corresponding to the two connected components of the covering (see Figure 1 and Figure 2)

$$\pi_i : \overline{M}(i) - \left(\frac{1}{\lambda_i}H_1(i) \cup \frac{1}{\lambda_i}H_2(i)\right) \longrightarrow \left(\mathbb{R}^2 \cup \{\infty\}\right) - \left(\frac{1}{\lambda_i}D_1(i) \cup \frac{1}{\lambda_i}D_2(i)\right).$$

Assume that $F_1(i), F_2(i)$ are labeled so that the normal vectors of $M(i)$ are upward pointing on $F_1(i)$ and downward pointing on $F_2(i)$. Let $L_1(i)$ denote a lift of $L(i)$ to the component $F_1(i)$ of the above covering. Since $M(i)$ is embedded, the lifts of $L(i)$ to $F_1(i) \cup F_2(i)$ are cyclicly ordered by height, so we can chose another lift, $L_2(i)$, to lie on the sheet directly above $L_1(i)$. Note that $L_2(i) \subset F_2(i)$. We now join the end point of $L_1(i)$ in $\frac{1}{\lambda_i}H_1(i)$ to the end point of $L_2(i)$ in $\frac{1}{\lambda_i}H_1(i)$ by a curve contained in $\frac{1}{\lambda_i}H_1(i)$, and similarly join up the corresponding end points on $\frac{1}{\lambda_i}H_2(i)$ (see Figure 2).

Note that the joined up curve γ_i represents in homology the curve for which one calculates fluxes for $M(i)$. We choose the orientation of γ_i so that the vertical component of the flux along γ_i is one. The orientation of γ_i induces an orientation on $L_1(i)$ and $L_2(i)$. We orient $L(i)$ with the orientation of $L_1(i)$.

We now wish to show that the length $\|L(i)\|$ of $L(i)$ goes to infinity as $i \to \infty$. Initially, we can assume that $\|\gamma_i \cap \left(\frac{1}{\lambda_i}H_1(i) \cup \frac{1}{\lambda_i}H_2(i)\right)\| \to 0$ as $i \to \infty$. We now give an estimate for the vertical flux of γ_i. Since the normal vector to $M(i)$ along

$\gamma_i \cap (F_1(i) \cup F_2(i))$ makes an angle of less that $\frac{1}{i}$ with the vertical, the vertical flux along $L_j(i)$ is less than or equal to $\sin\left(\frac{1}{i}\right) \|L(i)\|$. Hence the total vertical flux, which is one, is bounded from above by

$$\begin{aligned} & \|\gamma_i \cap \left(\frac{1}{\lambda_i} H_1(i) \cup \frac{1}{\lambda_i} H_2(i)\right) \| + 2 \sin\left(\frac{1}{i}\right) \|L(i)\| \\ \leq \; & \|\gamma_i \cap \left(\frac{1}{\lambda_i} H_1(i) \cup \frac{1}{\lambda_i} H_2(i)\right) \| + \frac{2}{i} \|L(i)\|. \end{aligned}$$

Thus $\|L(i)\| > \frac{i}{3}$ for i large.

We now estimate the horizontal flux of γ_i. First note that while the orientation of $\pi_i(L_1(i))$ and $L(i)$ agree, the orientation of $\pi_i(L_2(i))$ and $L(i)$ are opposite. Since $\pi_i|_{F_1}$ preserves orientation but $\pi_i|_{F_2}$ reverses orientation, the projection of the conormal field along $L_1(i)$ and $L_2(i)$ projects to a positive function times the conormal field along $L(i)$. Since the lengths of the part of γ_i inside $\frac{1}{\lambda_i} H_1(i) \cup \frac{1}{\lambda_i} H_2(i)$ converge to zero as $i \to \infty$, we know that the horizontal flux of the limit can be calculated as the limit of the horizontal fluxes along $L_1(i)$ and $L_2(i)$. Recall that the sheets in $F_1(i) \cup F_2(i)$ converge to horizontal sheets as $i \to \infty$, and so the horizontal flux of $L_1(i) \cup L_2(i)$ divided by the length of $L(i)$ converges to 2 as $i \to \infty$. As the length of $L(i)$ is greater than $\frac{i}{3}$, it follows that the horizontal fluxes of the $M(i)$ are unbounded, contrary to our hypotheses. This contradicts our earlier assumption that the curvature of the family $\{M(i)\}_i$ is unbounded, and completes the proof of the theorem. $\square$

We now prove the properness of the Flux map $F : \mathcal{S} \longrightarrow \mathbb{R}^2 - \{0\}$.

Theorem 5 *The Flux map is proper.*

Proof. Consider a sequence $\{M(i) \mid i \in \mathbb{N}\}$ of examples in $\mathcal{S}$ whose normed flux, $\|F(M(i))\|$, lies in $[\varepsilon, \frac{1}{\varepsilon}]$ for some $\varepsilon > 0$. It is sufficient to show that a subsequence of the $\{M(i)\}_i$ converges to another surface in $\mathcal{S}$ with vertical flux one and horizontal flux in $[\varepsilon, \frac{1}{\varepsilon}]$.

By Theorem 4, there is an upper bound on the Gaussian curvatures of the family $\{M(i)\}_i$. For simplicity, we consider the lifts $\widetilde{M}(i) \subset \mathbb{R}^3$ of $M(i)$, invariant under translation by $T(i)$. Translate each $\widetilde{M}(i)$ so that a point of maximal absolute Gaussian curvature of $\widetilde{M}(i)$ occurs at the origin. As before we may assume after choosing a subsequence that $\{\widetilde{M}(i)\}_i$ converges to a surface $\widetilde{M}_\infty$. Since $\widetilde{M}(i)$ has points with horizontal normal vector, Lemma 2 implies that the maximums of the absolute Gaussian curvatures of the family $\{\widetilde{M}(i)\}$ are bounded from below by a positive constant. Hence, the maximal absolute value of Gaussian of $\widetilde{M}_\infty$, which occurs at the origin, is some positive number c^2.

We now apply Proposition 1 to conclude that $\widetilde{M}_\infty$ is another example in $\mathcal{S}$. First, replace the surfaces $\widetilde{M}(i)$ by $\frac{1}{c(i)}\widetilde{M}(i)$, where $c(i)$ is the maximal Gaussian curvature of $M(i)$. Since $c(i) \to c$, the surfaces $\frac{1}{c(i)}\widetilde{M}(i)$ converge to $\frac{1}{c}\widetilde{M}_\infty$. Since the vertical fluxes of these surfaces are bounded, $\frac{1}{c}\widetilde{M}_\infty$ is not a helicoid. As the horizontal components of the fluxes are uniformly bounded away from zero, $\frac{1}{c}\widetilde{M}_\infty$ cannot be a catenoid. Hence $\frac{1}{c}\widetilde{M}_\infty$ is a Riemann-type example with the fluxes of the $\frac{1}{c(i)}\widetilde{M}(i)$ converging to the flux of $\frac{1}{c}\widetilde{M}_\infty$. It follows that the quotient surface M_∞ obtained from $\widetilde{M}_\infty$ is an element of $\mathcal{S}$ and the horizontal flux of M_∞ lies in $[\varepsilon, \frac{1}{\varepsilon}]$. This completes the proof of properness. $\square$

The last result of this section will be useful for proving the uniqueness of the Riemann examples among surfaces $M \in \mathcal{S}$ with very small horizontal flux, which will be demonstrated in Section 6.

Lemma 3 *Suppose $\{M(i) \mid i \in \mathbb{N}\} \subset \mathcal{S}$ is a sequence of minimal surfaces such that the norm of the horizontal fluxes of $M(i)$ goes to zero as i goes to infinity. Then, for i large, $M(i)$ is close to n translates of larger and larger compact regions of the vertical catenoid with vertical flux one, together with very flat regions that are graphs and contain the planar ends of $M(i)$.*

Proof. Since the sequence in the Lemma is arbitrary, it suffices to prove there is some subsequence that satisfies the conclusions in the statement of the Lemma. We now prove this assertion.

For each i, choose an angle $\theta(i) \in \mathbb{S}^1 \subset \overline{\mathbb{C}}$ which is a regular value of the Gauss map $g_i : M(i) \longrightarrow \overline{\mathbb{C}}$. Let $p(i,1), \ldots, p(i,2n)$ denote the $2n$ distinct preimages of $\theta(i)$ on $M(i)$. Let $\widetilde{M}(i)$ denote the lift of $M(i)$ to $\mathbb{R}^3$ and $\widetilde{p}(i,j)$, $1 \le j \le 2n$, the lifts of the $p(i,j)$ to a fundamental region of $\widetilde{M}(i)$ in $\mathbb{R}^3$. Let $\widetilde{M}(i,j)$ be the translate of $\widetilde{M}(i)$ which has $\widetilde{p}(i,j)$ at the origin. By Lemma 1 and Theorem 4, there are uniform curvature estimates and local area bounds for this family. Therefore, up to passing to a subsequence, $\{\widetilde{M}(i,1)\}_i$ converges to a properly embedded surface $\widetilde{M}(\infty,1)$.

By Lemma 2, $\widetilde{M}(\infty,1)$ is not a plane. Proposition 1 implies that $\widetilde{M}(\infty,1)$ is a vertical catenoid, a vertical helicoid or corresponds to the lift of another Riemann-type example (actually to apply Proposition 1 directly one needs to normalize the curvature as was done in the proof of the previous Theorem). Since the vertical fluxes of the $\widetilde{M}(i,1)$ are one, Proposition 1 shows that $\widetilde{M}(\infty,1)$ is not a helicoid. Since a Riemann example has nonzero horizontal flux, Proposition 1 shows that $\widetilde{M}(\infty,1)$ must be a vertical catenoid with vertical flux one.

For i large, $\widetilde{M}(i,1)$ has a large compact region close to a catenoid, so this region can be chosen with injective Gauss map. It follows that the points $p(i,j)$, $j \geq 2$, are far from the origin on the surface. The previous argument shows that a subsequence of $\{\widetilde{M}(i,2)\}_i$ converges to a vertical catenoid with vertical flux one, passing through the origin at $\widetilde{p}(i,2)$. Thus, for each large first index and passing again to a subsequence, each surface has two regions which are close to translates of a fixed large compact region of a vertical catenoid with vertical flux one. Continuing in this manner one can show that passing to a subsequence, around each $\widetilde{p}(i,j)$, the surface $\widetilde{M}(i)$ is close to a translation of larger and larger compact pieces of a fixed vertical catenoid with vertical flux one. In particular, the vertical components of the translation vectors of the $\widetilde{M}(i)$ must be going to infinity, since the large compact regions of the $2n$ catenoids embed simultaneously in the quotient spaces. As the level sets of any Riemann-type example are connected, these compact regions in the quotient space can be chosen as the intersection of $M(i)$ with horizontal slabs whose widths go to infinity as i increases.

We now claim that in each one of the complementary slabs found above, the surface $M(i)$ contains one of its planar ends and is a flat graph over a horizontal plane minus two disjoint disks. Let $A(i)$ denote one of these complementary slabs. $M(i) \cap A(i)$ has two boundary components, each one close to a large circle and with almost vertical normal vector. Since the total curvature of the complement of the $2n$ catenoidal regions is arbitrarily small, a simple application of the openness property of the Gauss map shows that the Gauss map of $M(i) \cap A(i)$ is contained in a small neighborhood of one vertical direction, hence the projection on the horizontal plane is a proper submersion that is one-to-one on each boundary component. Since the surface $M(i) \cap A(i)$ is connected, elementary topology arguments imply that the projection is one-to-one. Now we prove that $M(i) \cap A(i)$ is diffeomorphic to $\mathbb{R}^2$ minus two open disks. First, note that the projection is not a compact annulus, since $M(i) \cap A(i)$ lies locally on the outside of each boundary curve. Since $M(i)$ intersects each of the $2n$ complementary slabs in noncompact sets, and there are $2n$ ends on the surface, each such intersection must be diffeomorphic to $\mathbb{R}^2$ minus two open disks. This proves that $M(i) \cap A(i)$ is a flat graph over the horizontal plane minus two large disks, as desired. Now the proof of the Lemma is complete. $\square$

Remark 1 Suppose that $\{M(i)\}_i \subset \mathcal{S}$ is a sequence whose horizontal fluxes converge to zero. By Lemma 3, for i large, the branch points of the Gauss map of the $M(i)$ lie in the very flat regions which are graphs and contain the ends. Hence, the branch values of the Gauss maps of these surfaces converge to 0 and ∞ in $\overline{\mathbb{C}}$. By carefully choosing the flat regions, each flat region together with the point at infinity is a degree two branched cover of a disk neighbourhood of 0 or ∞. The Riemann-Hurwitz formula implies that there are exactly two branch points, one of them being a finite point in the surface. This remark will be used in the proof of Theorem 7.

5 The Openness of the Flux Map.

Let n be a positive integer. We will denote by $\mathcal{W}$ the space of all lists $(\Sigma, g, p_1, \ldots, p_n, q_1, \ldots, q_n, [\gamma])$ where Σ is a closed genus one Riemann surface and $g : \Sigma \longrightarrow \overline{\mathbb{C}}$ is a meromorphic function of degree $2n$, having n distinct double zeroes $p_1, \ldots, p_n$ and n distinct double poles $q_1, \ldots, q_n$ on Σ. So, the divisor of g is given by $(g) = p_1^2 \ldots p_n^2 q_1^{-2} \ldots q_n^{-2}$. Note that we are assuming an ordering on the points p_i and q_i, and we will think of different orderings as different elements of $\mathcal{W}$. The last component $[\gamma]$ in the list above is a homology class in $H_1(\Sigma - \{p_1, \ldots, p_n, q_1, \ldots, q_n\}, \mathbb{Z})$ which induces a nontrivial homology class in the torus Σ, $[\gamma] \in H_1(\Sigma, \mathbb{Z}) - \{0\}$. Elements of $\mathcal{W}$ will be called *marked meromorphic maps*, and will be denoted simply by g. Riemann-Hurwitz formula gives that each $g \in \mathcal{W}$ has exactly $2n$ branch points, not necessarily distinct, other than its zeroes and poles. The branch values corresponding to these points will be called $\mathbb{C}^*$-branch values.

Natural families of meromorphic functions with prescribed degree and genus can be endowed with structures of finite-dimensional complex manifolds, called *Hurwitz schemes*, see for instance [8]. We will explain some aspects of this construction in our particular case.

Let $g \in \mathcal{W}$ with $\mathbb{C}^*$-branch values $a_1, \ldots, a_k$, $k \leq 2n$, without counting multiplicity. A neighborhood, $U(g)$, of g in $\mathcal{W}$ can be described as follows. Take $k+2$ small pairwise disjoint disks D_i in $\overline{\mathbb{C}}$, centered at the points a_i, $1 \leq i \leq k$, D_{k+1} and D_{k+2} centered at 0 and ∞, respectively. Denote by $\Omega = \overline{\mathbb{C}} - \bigcup D_i$, the complementary domain. The restriction $g : g^{-1}(\Omega) \longrightarrow \Omega$ is an unbranched $2n$-sheeted covering map with connected total space. Each component of $g^{-1}(D_i)$, $i = 1, \ldots, k+2$, is a disk and contains at most one branch point of g, possibly of high multiplicity. When $i = k+1$ or $k+2$, $g^{-1}(D_i)$ has just n components and g has just one simple branch point, a double zero or a double pole, in each of these components. A marked meromorphic map $f \in \mathcal{W}$ lies in $U(g)$ if it satisfies the following conditions:

(i) *f has no branch values on Ω and the covering maps $f : f^{-1}(\Omega) \longrightarrow \Omega$ and $g : g^{-1}(\Omega) \longrightarrow \Omega$ are isomorphic.* Thus we can identify, from the conformal point of view, $f^{-1}(\Omega)$ and $g^{-1}(\Omega)$.

(ii) *Each component of $f^{-1}(D_i)$ is a disk in $f^{-1}(\overline{\mathbb{C}})$, $i = 1, \ldots, k+2$.* The identification in *(i)* determines a bijective correspondence between the components of $f^{-1}(D_i)$ and those of $g^{-1}(D_i)$ and the total branch order of g and f at the corresponding components necessarily coincides. In particular, if g has a simple branch point in a component, the same holds for f at the corresponding component.

(iii) The correspondence in *(ii)* determines, when $i = k+1$ and $k+2$, a bijection between the zeroes (resp. poles) of g and those of f. *We impose that the ordering in the set of zeroes and poles of f coincides with the one in the zero and pole set of g through this bijection.* We will denote the zeroes and poles of f again by $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$, respectively.

(iv) The homology class of g, $[\gamma] \in H_1\left(g^{-1}(\overline{\mathbb{C}}) - \{p_1, \ldots, p_n, q_1, \ldots, q_n\}, \mathbb{Z}\right)$ can be represented by a closed curve γ immersed in $g^{-1}(\Omega)$, which by means of the identification in *(i)* induces a homology class in the punctured surface determinated by the meromorphic map f, also denoted by $[\gamma] \in H_1\left(f^{-1}(\overline{\mathbb{C}}) - \{p_1, \ldots p_n, q_1, \ldots, q_n\}, \mathbb{Z}\right)$. Using *(i)* and *(ii)* we can check that $[\gamma] \neq 0$ in $H_1\left(f^{-1}(\overline{\mathbb{C}}), \mathbb{Z}\right)$. *We will impose on the marked meromorphic map f that its associated homology class coincides with $[\gamma]$.*

If an element $g \in \mathcal{W}$ has $2n$ distinct $\mathbb{C}^*$-branch values, then the same holds for any $f \in \mathcal{W}$ near g. Furthermore, the map which applied to f yields its list of $\mathbb{C}^*$-branch values is a local chart for $\mathcal{W}$ around g. In particular, $\dim \mathcal{W} = 2n$. If g has multiple branch points or different branch points with the same $\mathbb{C}^*$-branch value, coordinates around g are a bit more complicated to describe, see [8], but they follow the essential idea that elements in $\mathcal{W}$ are locally parametrized by the list of their branch values in $\mathbb{C}^*$.

On $\mathcal{W}$ we have some globally defined holomorphic functions $\sigma_i : \mathcal{W} \longrightarrow \mathbb{C}$, $1 \leq i \leq 2n$. For any $g \in \mathcal{W}$, $\sigma_i(g)$ is the symmetric elementary polynomial of degree i in the variables $a_1, \ldots, a_{2n}$, where $\{a_1, \ldots, a_{2n}\}$ is the set of $\mathbb{C}^*$-branching values of g, each one repeated as many times as its multiplicity indicates. Note that the a_i themselves are not globally well-defined functions because $\{a_1, \ldots, a_{2n}\}$ is an unordered set. Recall that the numbers $\sigma_1(g), \ldots, \sigma_{2n}(g)$ are the coefficients of the polynomial $\prod_i (z - a_i)$, and so, they determine the unordered set $\{a_1, \ldots, a_{2n}\}$. There are only finitely many (unmarked) meromorphic maps g with the same $\mathbb{C}^*$-branch values or, equivalently, with the same image under all the functions σ_i, $i = 1, \ldots, 2n$. As a consequence of the description above of the topology in $\mathcal{W}$, we have that given $\alpha_1, \ldots, \alpha_{2n} \in \mathbb{C}$, then the set $\{g \in \mathcal{W} \mid \sigma_1(g) = \alpha_1, \ldots, \sigma_{2n}(g) = \alpha_{2n}\}$ is a discrete subset of $\mathcal{W}$. Note that when nonvoid, the above subset is infinite because of the infinitely many possible choices for $[\gamma]$ in $H_1\left(g^{-1}(\overline{\mathbb{C}}) - \{p_1, \ldots, p_n, q_1, \ldots, q_n\}, \mathbb{Z}\right)$.

For any $g \in \mathcal{W}$, we have an unique holomorphic differential $\phi = \phi(g)$ on the complex torus $g^{-1}(\overline{\mathbb{C}})$ satisfying

$$\int_\gamma \phi = 2\pi i. \tag{3}$$

Moreover, ϕ depends holomorphically on g in the sense that the map from $U(g) \times g^{-1}(\Omega)$ into $\mathbb{C}^*$ given by

$$(f, z) \longmapsto \frac{\phi(f)}{\phi(g)}(z) \tag{4}$$

is holomorphic. As (g, ϕ) are potential Weierstrass data for surfaces in the setting of our theorem, we will call $\mathcal{W}$ the space of *Weierstrass representations.*

Later, we will need the following property for the analytic subvarieties of $\mathcal{W}$.

Lemma 4 *The only compact analytic subvarieties of $\mathcal{W}$ are the finite subsets.*

Proof. Let $\mathcal{V}$ be a compact analytic subvariety of $\mathcal{W}$. From the proper mapping theorem (see Section 3), it follows that $\sigma_i(\mathcal{V})$ is a compact analytic subvariety of $\mathbb{C}$. This implies that $\sigma_i(\mathcal{V})$ is a finite subset, for $1 \le i \le 2n$ and, thus, $\mathcal{V}$ itself is a discrete subset of $\mathcal{W}$. From this we obtain directly the finiteness of $\mathcal{V}$. $\square$

We consider the *Period map* $P : \mathcal{W} \longrightarrow \mathbb{C}^{2n}$ that associates to each $g = (g^{-1}(\overline{\mathbb{C}}), g, p_1, \ldots, p_n, q_1, \ldots, q_n, [\gamma])$ of $\mathcal{W}$ the $2n$-tuple

$$P(g) =$$

$$\left(\int_\gamma \frac{1}{g}\phi, \int_\gamma g\phi,\ \mathrm{Res}_{p_1}\left(\frac{1}{g}\phi\right), \ldots,\ \mathrm{Res}_{p_{n-1}}\left(\frac{1}{g}\phi\right),\ \mathrm{Res}_{q_1}(g\phi), \ldots,\ \mathrm{Res}_{q_{n-1}}(g\phi)\right),$$

where Res_p means the residue of the corresponding differential at the point p. We claim that P is a holomorphic map. Take f in a neighborhood $U(g)$ of g in $\mathcal{W}$ as above. Holomorphicity is clear for the first two coordinates of P because of the analytic dependence of (4). For the remaining coordinates, note that the residues appearing in $P(f)$ can be computed as integrals along curves contained in the boundary of Ω.

We consider the subset $\mathcal{M}$ of $\mathcal{W}$ given by the marked meromorphic maps g such that $P(g) = (z, \overline{z}, 0, \ldots, 0)$, for some $z \in \mathbb{C}$. As we pointed out in Section 3, for each $g \in \mathcal{M}$ the pair $(g, \phi(g))$ is the Weierstrass representation of a complete minimal surface M immersed in $\mathbb{R}^3/T$, for some nonhorizontal vector $T \in \mathbb{R}^3$, with $2n$ planar ends at the points $p_1, \ldots, p_n, q_1, \ldots, q_n$. Moreover, each compact horizontal section of this surface consists of exactly one Jordan curve in M. Note that a geometric minimal surface produces infinitely many elements when considered in $\mathcal{M}$, by taking all the possible orderings of its ends and all the different choices of the homology class $[\gamma]$ in the punctured torus that give the same class in the compactified surface. Nevertheless, all those infinitely many elements of $\mathcal{M}$ form a discrete subset in $\mathcal{W}$. We will refer to $\mathcal{M}$ as the space of immersed minimal surfaces.

Let be $\mathcal{S}$ denote the space of properly embedded minimal (oriented) tori in a quotient of $\mathbb{R}^3$ by a translation T, that depends on the surface, with $2n$ horizontal planar (ordered) ends. Recall (see Section 2) that surfaces in $\mathcal{S}$ are defined up to translations. We embed $\mathcal{S}$ in $\mathcal{M}$ as follows. Each $M \in \mathcal{S}$ determines a marked meromorphic map $g = (g^{-1}(\overline{\mathbb{C}}), g, p_1, \ldots, p_n, q_1, \ldots, q_n, [\gamma])$, where g is the (extended) Gauss map of M, $(p_1, q_1 \ldots, p_n, q_n)$ is the list of its ends and γ is the compact horizontal section of M by a plane which lies between the asymptotic planes at the ends p_1 and q_1, oriented in such a way that the third coordinate of the conormal vector is always positive.

The following assertion can be demonstrated by using the arguments that appear in the proof of Lemma 6.

Lemma 5 *On $\mathcal{S}$, both topologies, the smooth convergence of the lifted surfaces on compact subsets of $\mathbb{R}^3$ and the one induced by the inclusion $\mathcal{S} \subset \mathcal{W}$, coincide.*

Remark 2 The Flux map $F : \mathcal{S} \longrightarrow \mathbb{R}^2 - \{0\}$ defined in Section 2 applied to $M \in \mathcal{S}$ is the horizontal part $z \in \mathbb{R}^2 - \{0\} = \mathbb{C} - \{0\}$ of the flux of M along γ. The map P applied to M is $(i\overline{z}, -iz, 0, \ldots, 0) \in \mathbb{C}^{2n}$. Thus, the Period map restricted to $\mathcal{S}$

identifies with the Flux map. We will use this identification in the proof of the next Lemma.

Lemma 6 *$\mathcal{S}$ is an open and closed subset in $\mathcal{M}$. In particular, for any $z \in \mathbb{R}^2 - \{0\}$, the space $\mathcal{S}(z) = \{M \in \mathcal{S} \mid F(M) = z\}$ is an analytic subvariety of $\mathcal{W}$.*

Proof. If $\{M(i) \mid i \in \mathbb{N}\} \subset \mathcal{S}$ is a sequence, thought of as maps, which converges to a surface $M \in \mathcal{M}$, then it follows from Lemma 1 that the mapping M is an embedding. So, $M \in \mathcal{S}$ and the closedness is proved.

To prove the openness part, take a sequence $\{M(j)\}_j \subset \mathcal{M}$ which converges to $M \in \mathcal{S}$ in the topology of $\mathcal{W}$. If we denote by g the (extended) Gauss map of M and by f_j the one of $M(j)$, then, with the notation in the beginning of this Section, it follows that $f_j \in U(g)$ for j large, and large compact pieces of $M(j)$ and M are parametrized by maps $\psi_j : g^{-1}(\Omega) \longrightarrow \mathbb{R}^3/T_j$ and $\psi : g^{-1}(\Omega) \longrightarrow \mathbb{R}^3/T$ such that the ψ_j converge to ψ. In particular, ψ_j is an embedding for j large. Moreover, we can assume, by taking the neighborhood $U(g)$ small enough, that the connected components of $M - g^{-1}(\Omega)$ consist either of small graphs $\Delta_1, \ldots, \Delta_r$ over convex planar domains, or bounded graphs $\Delta_1^*, \ldots, \Delta_{2n}^*$ over the complements of large convex domains in a horizontal plane.

If $\Delta_k(j)$ is the component of $M(j) - g^{-1}(\Omega)$ that corresponds to Δ_k, then $\Delta_k(j)$ is a disk whose boundary values are arbitrarily close, depending on j, to those of Δ_k, and whose Gauss map image (that coincides with the one of Δ_k) is small. In this situation, it is a standard fact that $\Delta_k(j)$ is a graph which converges to Δ_k. Therefore, for j large, the domain $[M(j) - g^{-1}(\Omega)] \cup \Delta_1(j) \cup \ldots \cup \Delta_r(j)$ is embedded in $\mathbb{R}^3/T_j$.

Consider now the component $\Delta_k^*(j)$ in $M(j) - g^{-1}(\Omega)$ that corresponds to Δ_k^*. It is a representative of a planar end of $M(j)$, with almost vertical Gauss map and whose boundary values converge to those of Δ_k^*. So, for j large enough, each $\Delta_k^*(j)$ is a graph over the exterior of a Jordan curve in the (x_1, x_2)-plane, because its horizontal projection is a proper local diffeomorphism that maps injectively its boundary onto a convex horizontal curve. From the maximum principle at infinity [11], the distance between distinct ends of M, $\Delta_1^*, \ldots, \Delta_{2n}^*$ is positive and, so, we can assume that each two of these ends are strictly separated by a horizontal plane. For j large, this plane will also separate the closed curves $\partial\Delta_k^*(j)$, $k = 1, \ldots, s$. As the x_3-coordinate on $\Delta_k^*(j)$ is a harmonic function which extends to the puncture, it follows from the maximum principle that the planes above separate the entire ends

$\Delta_1^*(j), \ldots, \Delta_{2n}^*(j)$. Therefore, $M(j)$ is embedded and we conclude that $\mathcal{S}$ is open in $\mathcal{M}$.

For any $z \in \mathbb{C}$, the subset $\mathcal{M}(z) = \{g \in \mathcal{W} \mid P(g) = (i\overline{z}, -iz, 0, \ldots, 0)\} \subset \mathcal{M}$ is clearly an analytic subvariety of $\mathcal{W}$. Therefore, $\mathcal{S}(z) = \mathcal{M}(z) \cap \mathcal{S}$, which is open and closed in $\mathcal{M}(z)$, is also an analytic subvariety of $\mathcal{W}$. $\square$

Theorem 6 *The Flux map $F : \mathcal{S} \longrightarrow \mathbb{R}^2 - \{0\}$ is open.*

Proof. We show that F is open in a neighborhood of any surface M in the space $\mathcal{S}$. Let $z = F(M) \in \mathbb{R}^2 - \{0\}$. Lemma 6 and the properness of F (Theorem 5) imply that $\mathcal{S}(z) = F^{-1}(z)$ is a compact analytic subvariety of $\mathcal{W}$. From Lemma 4, we obtain that $\mathcal{S}(z)$ is finite. Therefore, using the openness Theorem in Section 3, we get that there exists a neighborhood $\mathcal{U}$ of M in $\mathcal{W}$ such that $P|_{\mathcal{U}}$ is open. Using Lemma 6, we can assume that $\mathcal{U} \cap \mathcal{M} = \mathcal{U} \cap \mathcal{S}$. This gives directly the openness of $F : \mathcal{U} \cap \mathcal{S} \longrightarrow \mathbb{R}^2 - \{0\}$, as we claimed. $\square$

6 Surfaces with Almost Vertical Flux.

Consider $\varepsilon \in (0, 1)$, and denote by $D(\varepsilon) = \{z \in \mathbb{C} \mid \|z\| < \varepsilon\}$ and $D^*(\varepsilon) = D(\varepsilon) - \{0\}$. Given $\mathbf{a} = (a_1, \ldots, a_{2n}) \in [D^*(\varepsilon)]^{2n}$, a marked meromorphic map associated to $\mathbf{a}$ can be constructed as follows:

Consider $2n$ copies of $\overline{\mathbb{C}}$, labelled by $\overline{\mathbb{C}}_1, \ldots, \overline{\mathbb{C}}_{2n}$. For i odd, cut $\overline{\mathbb{C}}_i$ along two arcs α_i, α_{i+1} joining 0 with a_i and ∞ with $\frac{1}{a_{i+1}}$, respectively. When i is even, cut $\overline{\mathbb{C}}_i$ along two arcs α_i, α_{i+1} joining ∞ with $\frac{1}{a_i}$ and 0 with a_{i+1}, respectively. Choose these arcs in such a way that consecutive copies $\overline{\mathbb{C}}_i$, $\overline{\mathbb{C}}_{i+1}$ share the arc α_{i+1}, so we can glue together $\overline{\mathbb{C}}_i$ with $\overline{\mathbb{C}}_{i+1}$ (in the same way, glue $\overline{\mathbb{C}}_{2n}$ with $\overline{\mathbb{C}}_1$ along the common arc α_1), forming a genus one surface Σ with the $2n$ copies of $\overline{\mathbb{C}}$. The z map on each copy $\overline{\mathbb{C}}_i$ is now well-defined on Σ as a meromorphic map g with degree $2n$, which has a double zero (resp. double pole) at 0 in each copy $\overline{\mathbb{C}}_i$ (resp. at ∞ in each $\overline{\mathbb{C}}_i$) and $\mathbb{C}^*$-branch values $a_1, \frac{1}{a_2}, \ldots, a_{2n-1}, \frac{1}{a_{2n}}$, each one being shared by two of the copies of $\overline{\mathbb{C}}$. The ordered list of zeroes and poles associated to our marked meromorphic map is defined to be $(0_1, \infty_1, \ldots, 0_n, \infty_n)$, where $A_1 = \{0_1, \infty_1\}$, $A_2 = \{\infty_1, 0_2\}$, $A_3 = \{0_2, \infty_2\}, \ldots, A_{2n} = \{\infty_n, 0_1\}$ are the sets of zeroes and poles of the z-map on the copies $\overline{\mathbb{C}}_1, \ldots, \overline{\mathbb{C}}_{2n}$, respectively. Finally, the nontrivial homology class

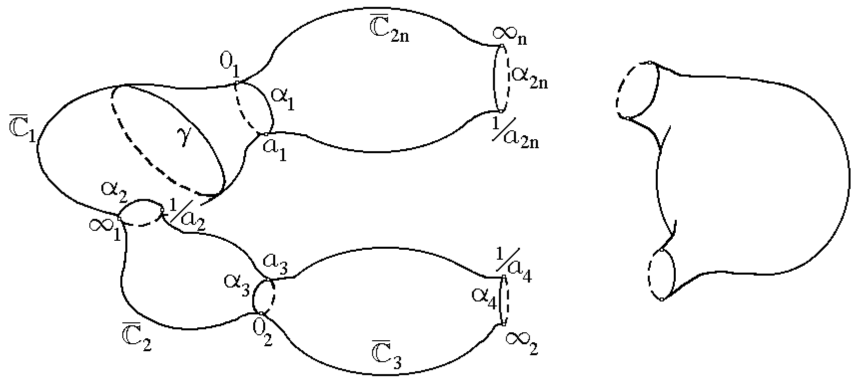

Figure 3:

$[\gamma] \in H_1\left(\Sigma - \{0_1, \infty_1, \ldots, 0_n, \infty_n\}, \mathbb{Z}\right)$ is defined to be the class induced by the loop $\{\|z\| = 1\}$ in the first copy $\overline{\mathbb{C}}_1$, see Figure 3.

In fact, this correspondence between lists $(a_1, \ldots, a_{2n}) \in [D^*(\varepsilon)]^{2n}$ and marked meromorphic maps is a local parametrization in the complex manifold $\mathcal{W}$. Also note that by choosing different orderings for the zero and pole set and different homology classes in the construction above, we construct an infinite number of other elements in $\mathcal{W}$, with the same underlying meromorphic function. However, these elements are far away, in the topology of $\mathcal{W}$, from the one constructed above.

If we consider the restriction of the Period map P to the open subset of $\mathcal{W}$ where the above local chart is defined, then P can be considered as a holomorphic map from $[D^*(\varepsilon)]^{2n}$ into $\mathbb{C}^{2n}$. Next we will prove that P extends holomorphically to $[D(\varepsilon)]^{2n}$. With this aim, we need the following result.

Lemma 7 *Let S be a Riemann surface and $\{f_j : S \longrightarrow \mathbb{C}^*\}_j$ a sequence of injective holomorphic functions. Assume that there exists a closed curve $\Gamma \subset S$ such that $f_j(\Gamma)$ is not nulhomotopic in $\mathbb{C}^*$, for any $j \in \mathbb{N}$. Then, there exists a subsequence of $\{f_j\}_j$, denoted in the same way, and a sequence $\{\lambda_j\}_j \subset \mathbb{C}^*$ such that $\{\lambda_j f_j\}_j$ converges on compact subsets of S to a injective holomorphic function $f : S \longrightarrow \mathbb{C}^*$.*

Proof. Take a point $p \in S$ and consider the functions $\widetilde{f_j} = \frac{1}{f_j(p)} f_j$. Viewed as functions on $S - \{p\}$, the family $\{\widetilde{f_j}\}_j$ omits the values $0, 1, \infty$ in $\overline{\mathbb{C}}$. Hence Montel-Caratheodory Theorem says that a $\{\widetilde{f_j}\}_j$ is a normal family and thus, it has a subsequence, denoted in the same way, that converges to a meromorphic function $f : S - \{p\} \longrightarrow \overline{\mathbb{C}}$.

Take a complex coordinate z, $\|z\| < 2\varepsilon$, centered at p. From the mean value property of holomorphic maps we have that

$$\frac{1}{2\pi\varepsilon}\int_{\{\|z\|=\varepsilon\}} \widetilde{f_j}(z)|dz| = 1.$$

As the $\widetilde{f_j}$ converge on $\{\|z\| = \varepsilon\}$, we conclude that, if f were constant, then it would be identically one. In particular, $\widetilde{f_j}(\Gamma)$ would be contained in a small neighborhood of 1 in $\mathbb{C}^*$ for j large enough, which is contrary to the assumption on $f_j(\Gamma)$. Therefore, f is nonconstant. The maximum modulus principle implies that the $\widetilde{f_j}$ converge to f on the whole surface S. In this situation, a well-known theorem of Hurwitz implies that f is injective and has no zeroes or poles on S. $\square$

Lemma 8 *The Period map P extends holomorphically to $[D(\varepsilon)]^{2n} \subset \mathbb{C}^n$.*

Proof. Following the notation at the beginning of this Section, note that given $(a_1, \ldots, a_{2n}) \in [D^*(\varepsilon)]^{2n}$ with associated marked meromorphic map $g \in \mathcal{W}$, we can rewrite the Period map P in a more homogeneus way as follows. Consider the $2n$ closed curves $\gamma = \gamma_1, \gamma_2, \ldots, \gamma_{2n}$ corresponding to the loops $\{\|z\| = 1\}$ on $\overline{\mathbb{C}}_1, \ldots, \overline{\mathbb{C}}_{2n}$, respectively. Note that all the γ_i are homologous in Σ and each residue at a pole (resp. a zero) shared by consecutive copies $\overline{\mathbb{C}}_j$, $\overline{\mathbb{C}}_{j+1}$ of the meromorphic differential $g\phi$ (resp. of $\frac{1}{g}\phi$) can be computed as the expression $\frac{1}{2\pi i}\left(\int_{\gamma_{j+1}} g\phi - \int_{\gamma_j} g\phi\right)$ (resp. $\frac{1}{2\pi i}\left(\int_{\gamma_{j+1}} \frac{1}{g}\phi - \int_{\gamma_j} \frac{1}{g}\phi\right)$). Also note that if the copies $\overline{\mathbb{C}}_j, \overline{\mathbb{C}}_{j+1}$ have a zero along the common arc α_{j+1}, then $\int_{\gamma_j} g\phi = \int_{\gamma_{j+1}} g\phi$, and the same holds for $\frac{1}{g}\phi$ if α_{j+1} joins a pole with $\frac{1}{a_{j+1}}$. Now an elementary exercise in linear algebra shows that the Period map can be expressed as the composition of a regular linear transformation in $\mathbb{C}^{2n}$ with the holomorphic map $G : [D^*(\varepsilon)]^{2n} \longrightarrow \mathbb{C}^{2n}$ defined by

$$G(a_1, \ldots, a_{2n}) = G(g) = \left(\int_{\gamma_1} g\phi, \int_{\gamma_2} \frac{1}{g}\phi, \int_{\gamma_3} g\phi, \int_{\gamma_4} \frac{1}{g}\phi, \ldots, \int_{\gamma_{2n-1}} g\phi, \int_{\gamma_{2n}} \frac{1}{g}\phi\right).$$

As consequence, it suffices to prove that G extends holomorphically to $[D(\varepsilon)]^{2n}$.

As $[D(\varepsilon)]^{2n} - [D^*(\varepsilon)]^{2n}$ is an analytic subvariety of $[D(\varepsilon)]^{2n}$, by the Riemann extension theorem (see Section 3) G will extend holomorphically provided that it is bounded. With this aim, consider a sequence $\{\mathbf{a}(j) = (a_1(j), \dots, a_{2n}(j))\}_j \subset [D^*(\varepsilon)]^{2n}$ that converges to $\mathbf{a} = (a_1, \dots, a_{2n}) \in [D(\varepsilon)]^{2n}$, such that at least one of the components of $\mathbf{a}$ is zero. By the construction at the beginning of this Section, each list $\mathbf{a}(j)$ determines an element $g(j) = (\Sigma(j), g(j), p_1(j), \dots, p_n(j), q_1(j), \dots, q_n(j), [\gamma(j)])$ in $\mathcal{W}$ and a holomorphic one-form $\phi(j)$ such that $\int_{\gamma(j)} \phi(j) = 2\pi i$. Denote by $b_1, \dots, b_\tau$ the distinct elements in the set of limit branch values $\{a_1, \frac{1}{a_2}, \dots, a_{2n-1}, \frac{1}{a_{2n}}, 0, \infty\}$. For $\alpha = 1, \dots, \tau$, take pairwise disjoint small disks D_α centered at the b_α, and let $\Omega = \overline{\mathbb{C}} - \cup_{\alpha=1}^{\tau} D_\alpha$. For j large enough, $g(j) : g(j)^{-1}(\Omega) \longrightarrow \Omega$ is an unbranched $2n$-sheeted covering, any two of these being isomorphic. This isomophism allows us to identify each component $B_1(j), \dots, B_k(j)$ of $g(j)^{-1}(\Omega)$ with open subsets $B_1, \dots, B_k$ of a collection of genus zero surfaces $S_1, \dots, S_k$, in such a way that $g(j)$ restricted to $B_m(j)$ identifies with g_m restricted to B_m, for certain nonconstant meromorphic maps $g_m : S_m \longrightarrow \overline{\mathbb{C}}$, $m = 1 \dots, k$ satisfying that the branch values of g_m are contained in $\{b_1, \dots, b_\tau\}$ and $\deg(g_1) + \dots + \deg(g_k) = 2n$. When we shrink the disks D_α, the domains B_m increase until in the limit they fill the whole sphere S_m minus a finite subset Z_m. Moreover, when B_m is viewed as a subdomain of the torus $\Sigma(j)$, each component of ∂B_m bounds a disk in $\Sigma(j)$ except in two of them, which are homologous to the curve $\gamma(j)$. These two boundary components correspond to two elements in the list $\mathbf{a}(j)$ that converge to zero, and determine two distinct points $P_m, Q_m \in Z_m$, where g_m takes the value 0 and/or ∞. On the other hand, the curves $\gamma_1, \dots, \gamma_{2n}$ in $\Sigma(j)$ induce $2n$ curves (denoted again by $\gamma_1, \dots, \gamma_{2n}$), distributed in $B_1, \dots, B_k$ such that each one of them, when viewed inside the corresponding annulus $S_m - \{P_m, Q_m\}$, generates its homology group.

Fix $m \in \{1, \dots, k\}$ and consider the inclusion map from B_m into $\Sigma(j)$, for j large. Let Γ be one of the curves in $\{\gamma_1, \dots, \gamma_{2n}\}$, which lies in B_m. Thus, Γ is a generator of the first homology group of $S_m - \{P_m, Q_m\}$. Consider the cyclic covering Π of $\Sigma(j)$ with image subgroup the cyclic group generated by the loop Γ in $\pi_1(\Sigma(j))$, where Γ is viewed as a nontrivial homotopy class in $\Sigma(j)$. The total space of this covering can be conformally parametrized by $\mathbb{C}^*$ so that Γ lifts to the homotopy class of $\{\|z\| = 1\}$ in $\mathbb{C}^*$. Hence the inclusion of B_m into $\Sigma(j)$ lifts to a holomorphic injective function $f_j : B_m \longrightarrow \mathbb{C}^*$. Lemma 7 implies that, after passing to a subsequence, $\{\lambda_j f_j\}_j$ converges on compact subsets of B_m to a injective holomorphic function $f : B_m \longrightarrow \mathbb{C}^*$, where $\{\lambda_j\}_j$ is a sequence of nonzero complex numbers. The domain B_m can be thought as an element in a increasing exhaustion of $S_m - Z_m$, so we can assume, again after choosing a subsequence, that neither the sequence $\{\lambda_j\}_j$ nor the limit function f depend on B_m, hence $f : S_m - Z_m \longrightarrow \mathbb{C}^*$

is a injective holomorphic function and $\{\lambda_j f_j\}_j$ converges uniformly on compact subsets of $S_m - Z_m$ to f. Therefore, f extends to a biholomorphism between S_m and $\overline{\mathbb{C}}$. On the other hand, our normalization on $\phi(j)$ implies that its lifting to $\mathbb{C}^*$ is $\Pi^*\phi(j) = \frac{dz}{z}$. Thus, $\phi(j)$ restricted to B_m coincides with $f_j^* \frac{dz}{z} = \frac{df_j}{f_j} = \frac{d(\lambda_j f_j)}{\lambda_j f_j}$ and, so, when j goes to infinity, $\phi(j)$ converges to $\phi = \frac{df}{f}$ on $S_m - Z_m$. Observe that ϕ is a meromorphic differential on S_m with exactly two simple poles. As the integral of $\phi(j)$ along the boundary components of $B_m \subset S_m$ that enclose the points P_m, Q_m is $\pm 2\pi i$, we conclude that the same holds for the one-form ϕ, or equivalently, that the poles of ϕ occur at P_m and Q_m.

In summary, we have shown that after passing to a subsequence, when $j \to \infty$ the torus $\Sigma(j)$ disconnects into k genus zero surfaces $S_1, \ldots, S_k$, the meromorphic maps $g(j)$ converge uniformly on compact subsets of each S_m minus a finite subset Z_m to a nonconstant meromorphic map $g_m : S_m \longrightarrow \overline{\mathbb{C}}$, $1 \leq m \leq k$, in such a way that the total degree of $g_1, \ldots, g_k$ is equal to $2n$. Moreover, the holomorphic differentials $\phi(j)$ converge uniformly on compact subsets of $S_m - Z_m$ to the (unique) meromorphic differential ϕ on S_m which has just two simple poles at certain points $P_m, Q_m \in Z_m$ with residues $1, -1$ at these points.

Now we conclude easily that $\{G(\mathbf{a}(j))\}_j$ is bounded in $\mathbb{C}^{2n}$, because each component of $G(\mathbf{a}(j))$ is written as an integral along a fixed closed curve contained in $S_m - Z_m$ for some m, of certain holomorphic differentials that converge uniformly on the closed curve to another holomorphic differential. $\Box$

Remark 3 From the proof above, it follows that if $\mathbf{a} \in [D(\varepsilon)]^{2n} - [D^*(\varepsilon)]^{2n}$ determines the spheres $S_1, \ldots, S_k$, then the value of G at $\mathbf{a}$ is given by

$$G(\mathbf{a}) = \left(\int_{\gamma_1} g\phi, \int_{\gamma_2} \frac{1}{g}\phi, \int_{\gamma_3} g\phi, \int_{\gamma_4} \frac{1}{g}\phi, \ldots, \int_{\gamma_{2n-1}} g\phi, \int_{\gamma_{2n}} \frac{1}{g}\phi \right).$$

where g is taken as g_m in the r-th component, provided that the curve γ_r lies in the sphere S_m, and ϕ is the holomorphic differential on the annulus $S_m - \{P_m, Q_m\}$ that has simple poles at P_m and Q_m, and period $2\pi i$ along the generator of its homology.

Lemma 9 *There exists a neighborhood $\mathcal{U}$ of $0 \in \mathbb{C}^{2n}$ such that $P|_{\mathcal{U}}$ is biholomorphic.*

Proof. Using the arguments in the proof of Lemma 8, it suffices to check that the differential at zero of $G : [D(\varepsilon)]^{2n} \longrightarrow \mathbb{C}^{2n}$ is regular.

Fix $j \in \{1, \ldots, 2n\}$ and compute $\frac{\partial G}{\partial a_j}(0)$. As we showed in the proof of Lemma 8, the $2n$-tuple $(0, \ldots, a_j, \ldots, 0)$, $a_j \in D^*(\varepsilon)$, determines 2n-1 genus zero surfaces $S_1, \ldots, S_{j-1}, S_{j+1}, \ldots, S_{2n}$ and meromorphic maps $g_1, \ldots, g_{j-1}, g_{j+1}, \ldots, g_{2n}$ on these surfaces, satisfying

1. For $m = 1, \ldots, j-2, j+1, \ldots, 2n$, $g_m : S_m \longrightarrow \overline{\mathbb{C}}$ has degree one. If we parametrize S_m conformally by $\overline{\mathbb{C}}$, then $g_m(z) = z$, and the curve γ_m is homologous to $\{\|z\| = 1\}$ in $\mathbb{C}^*$.

2. $g_{j-1} : S_{j-1} \longrightarrow \overline{\mathbb{C}}$ has degree two. When j is odd, g_{j-1} has a double zero and a_j is a $\mathbb{C}^*$-branch value with ramification order one. If j is even, its branch values are ∞ and $\frac{1}{a_j}$. Moreover, S_{j-1} contains the curves γ_{j-1}, γ_j that appear in the definition of G.

Take $m \neq j-1, j$. The m-th component of G is either $\int_{\gamma_m} g\phi$ or $\int_{\gamma_m} \frac{1}{g}\phi$, depending on the parity of m, where γ_m lies in one of the surfaces in case 1. Then, as our normalization of ϕ on $S_m = \overline{\mathbb{C}}$ gives $\phi = \frac{dz}{z}$, it follows that

$$\int_{\gamma_m} g\phi = \int_{\|z\|=1} dz = 0, \qquad \int_{\gamma_m} \frac{1}{g}\phi = \int_{\|z\|=1} \frac{dz}{z^2} = 0.$$

This implies that the m-th component of G is identically zero along the curve $(0, \ldots, a_j, 0, \ldots, 0)$, hence the same holds for its derivative.

We now consider the $(j-1)$-th, j-th components of G. Assume firstly that j is odd. If we parametrize S_{j-1} by $S = \{(z, w) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}} \mid w^2 = z(z - a_j)\}$, then $g_{j-1}(z, w) = z$, and the curves γ_{j-1}, γ_j are homologous in $S - \{(0, 0), (a_j, 0), (\infty, \infty)_1, (\infty, \infty)_2\}$ to the two connected lifts of $\{\|z\| = 1\}$ to S by the z-projection. To compute $G(0, \ldots, 0, a_j, 0, \ldots, 0)$, according to Remark 3, we need to determine a meromorphic differential ϕ on S by the conditions

(1.a) ϕ has no zeroes or poles in $S - \{(\infty, \infty)_1, (\infty, \infty)_2\}$, and

(1.b) $\int_{\|z\|=1} \phi = 2\pi i$ (here $\{\|z\| = 1\}$ denotes either of the two liftings of the unit circle to S by the z-projection).

Conditions *(1.a)* and *(1.b)* imply that $\phi = c(a_j)\frac{dz}{w}$, where $c(a_j) \in \mathbb{C}^*$ satisfies the equation

$$c(a_j) \int_{\|z\|=1} \frac{dz}{\sqrt{z(z-a_j)}} = 2\pi i.$$

Hence $c(a_j)$ is a holomorphic function of a_j that extends to zero as $c(0) = 1$. Now by direct computation one has

$$\left.\frac{d}{da_j}\right|_0 \int_{\|z\|=1} \frac{1}{g}\phi = \left.\frac{d}{da_j}\right|_0 c(a_j) \int_{\|z\|=1} \frac{dz}{z\sqrt{z(z-a_j)}} = c'(0) \int_{\|z\|=1} \frac{dz}{z^2} + \frac{1}{2} \int_{\|z\|=1} \frac{dz}{z^3} = 0.$$

$$\left.\frac{d}{da_j}\right|_0 \int_{\|z\|=1} g\phi = \left.\frac{d}{da_j}\right|_0 c(a_j) \int_{\|z\|=1} \frac{zdz}{\sqrt{z(z-a_j)}} = c'(0) \int_{\|z\|=1} dz + \frac{1}{2} \int_{\|z\|=1} \frac{dz}{z} = \pi i.$$

Similarly, if j is even, we can write $S_{j-1} = S = \{(z,w) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}} \mid w^2 = a_j z - 1\}$, $g(z,w) = z$, and the curves γ_{j-1}, γ_j are the liftings by g of $\{\|z\| = 1\}$. Therefore, $\phi = \widehat{c}(a_j)\frac{dz}{zw}$, where $\widehat{c}(a_j)$ is holomorphic in a_j with $\widehat{c}(0) = i$, and we obtain

$$\left.\frac{d}{da_j}\right|_0 \int_{\|z\|=1} \frac{1}{g}\phi = -\pi i, \qquad \left.\frac{d}{da_j}\right|_0 \int_{\|z\|=1} g\phi = 0.$$

In conclusion, the partial derivative of G at the origin is

$$\frac{\partial G}{\partial a_j}(0) = \left(0, \ldots, 0, (-1)^{j+1}\pi i, 0 \ldots, 0\right),$$

and the differential of G at the origin is bijective, as desired. □

Now we can state the main result in this Section.

Theorem 7 *There exists $\varepsilon > 0$ such that if $M \in \mathcal{S}$ satisfies $|F(M)| < \varepsilon$, then M is a Riemann minimal example.*

Proof. Consider the neighborhood $\mathcal{U}$ of $0 \in \mathbb{C}^{2n}$ where $P|_{\mathcal{U}}$ is bijective given by Lemma 9, and note that P extends by $P(0) = 0$.

By applying the properness Theorem 5 and the openness Theorem 6 to any connected component $\mathcal{C}$ in $\mathcal{W}$ corresponding to the space of Riemann examples (which is open and closed in $\mathcal{S}$), we deduce that in $\mathcal{C}$ we can find Riemann examples with almost vertical flux. By Remark 1, these surfaces lie in $\mathcal{U}$ if the horizontal part of their fluxes are small enough. On the other hand, if $M \in \mathcal{S}$ lies in $\mathcal{U}$, then its horizontal flux coincides with the horizontal flux of a Riemann example R, suitably rotated around the vertical axis. Thus P takes the same value at M and at R, which implies these surfaces coincide.

Now the theorem follows from the relation between the two first components of $P|_{\mathcal{S}}$ and the Flux map F (see Remark 2 in Section 5). $\square$

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William H. Meeks, III
bill@gang.umass.edu
Mathematics Department University of Massachusetts
Amherst, MA 01003, U.S.A.

Joaquin Perez
jperez@goliat.ugr.es
Departamento de Geometria y Topologia
Facultad de Ciencias - Universidad de Granada
Fuentenueva, 18071 - Granada - Spain

Antonio Ros
aros@goliat.ugr.es
Departamento de Geometria y Topologia
Facultad de Ciencias - Universidad de Granada
Fuentenueva, 18071 - Granada - Spain