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Sinh-Gordon type equations for CMC surfaces

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Resumen We review how the classical Sinh-Gordon equation appears in the theory of constant mean curvature surfaces in space forms and minimal surfaces in product spaces, and introduce a new equation of this type occurring for constant mean curvature surfaces in homogeneous three-manifolds with isometry group of dimension four.

1. Introduction

The Sinh-Gordon equation is an elliptic PDE which appears naturally in surface theory, among others in the following two different contexts: for constant mean curvature (CMC) surfaces in space forms and for minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$, where $\mathbb{S}^2$ denotes the 2-sphere with its usual metric. In both situations, the Sinh-Gordon equation has been shown to be a useful tool in obtaining both existence and uniqueness results (see e.g. Pinkall and Sterling [7], Ritoré and Ros [8], Hauswirth, Sa Earp and Toubiana [5], Kilian and Schmidt [6] and references therein). In this note we will review how to find the Sinh-Gordon equation in these known settings, and provide a new related extension which allows to find a geometrical Sinh-Gordon (or Sin-Gordon) type equation valid for CMC surfaces in more general ambient geometries, namely in homogeneous three-manifolds N with isometry group $\mathrm{Iso}(N)$ of dimension four. Since our result is purely local, we can assume that N is simply-connected. It is well-known that every simply-connected, homogeneous three-manifold is of one of the following three types: If $\dim \mathrm{Iso}(N) = 6$, then N is a space form $\mathbb{R}^3, \mathbb{S}^3$ or $\mathbb{H}^3$; if $\dim \mathrm{Iso}(N) = 4$, then N admits a Riemannian submersion $\Pi\colon N \to \mathbb{M}^2(\kappa)$ over a surface of constant curvature $\kappa \in \mathbb{R}$; finally, if $\dim \mathrm{Iso}(N) = 3$, then N is a Lie group endowed with a left invariant metric. In this note we will focus on case 2 above, although for the sake of completeness, we will also discuss to some extent the already known situation in case 1.

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2. CMC surfaces in $\mathbb{M}^3(c)$.

Along this section, $\mathbb{M}^3(c)$ will denote the three-dimensional space form of constant sectional curvature $c \in \mathbb{R}$. Let $\Sigma \looparrowright \mathbb{M}^3(c)$ be an immersed surface of CMC $H \in \mathbb{R}$ in $\mathbb{M}^3(c)$. Let $ds^2 = \lambda |dz|^2$ be the induced metric on Σ, where $z = x + iy$ is a local holomorphic coordinate and λ a smooth positive function. If σ stands for the second fundamental form of Σ, then after complexifying the tangent bundle and taking the $(2,0)$-part we find the well-known *Hopf differential* of the immersion,

$$\sigma^{2,0} = p\,(dz)^2, \qquad p = 4\sigma(\partial_z, \partial_z), \tag{1}$$

which is a globally defined holomorphic quadratic differential on Σ (holomorphicity comes from the Codazzi equation). The zeros of $\sigma^{2,0}$ are the umbilical points of Σ, hence $\mathfrak{U} = \{\sigma^{2,0} = 0\}$ is either a discrete set of points or Σ is totally umbilical. We will assume in the sequel that Σ is not umbilical, and work away from $\mathfrak{U}$. We will also suppose that $H^2 + c > 0$ (note that this condition is meaningless in $\mathbb{S}^3(c)$ and excludes minimal surfaces in $\mathbb{R}^3$). Consider the metric $ds_0 = 2\sqrt{H^2 + c}\,|\sigma^{2,0}|$ on $\Sigma - \mathfrak{U}$, which is flat and conformal to ds^2. Thus, there exists $\psi \in C^\infty(\Sigma - \mathfrak{U})$ such that

$$ds^2 = \frac{e^{2\psi}}{4(c + H^2)}\, ds_0^2. \tag{2}$$

The following result is well-known.

Proposition 2.1 *ψ satisfies the Sinh-Gordon equation $\Delta_0\psi + \frac{1}{2}\sinh(2\psi) = 0$ in $\Sigma - \mathfrak{U}$, where Δ_0 is computed with respect to ds_0^2.*

Proof. By the Gauss equation,

$$c = K - \det S, \tag{3}$$

where K is the Gauss curvature of ds^2 and S the shape operator of Σ. A direct computation (valid in every ambient space) gives

$$\frac{|p|^2}{\lambda^2} = |\sigma|^2 = 4\left(H^2 - \det S\right). \tag{4}$$

Plugging (4) in (3) and using the classical relation between the Gaussian curvatures of conformally related metrics, we have

$$c + H^2 = K + \frac{|p|^2}{4\lambda^2} = -\frac{4(c + H^2)}{e^{2\psi}}\Delta_0\psi + \frac{|p|^2}{4\lambda^2}. \tag{5}$$

On the other hand,

$$\lambda\,|dz|^2 = ds^2 = \frac{e^{2\psi}}{4(c + H^2)}\, ds_0^2 = \frac{e^{2\psi}}{2\sqrt{c + H^2}}|\sigma^{2,0}| = \frac{e^{2\psi}}{2\sqrt{c + H^2}}|p|\,|dz|^2,$$

hence $\frac{|p|}{\lambda} = \frac{2\sqrt{c+H^2}}{e^{2\psi}}$ and (5) gives $c+H^2 = -4(c+H^2)e^{-2\psi}\Delta_0\psi + (c+H^2)e^{-4\psi}$. Therefore, $0 = \Delta_0\psi + \frac{e^{2\psi}}{4} - \frac{1}{4e^{2\psi}} = \Delta_0\psi + \frac{1}{2}\sinh(2\psi)$. □

Reciprocally, every solution ψ of $\Delta_0\psi + \frac{1}{2}\sinh(2\psi) = 0$ defined in a simply-connected domain of $\Sigma \subset \mathbb{C}$ produces a metric ds^2 by equation (2) (with ds_0^2 being the usual metric on $\mathbb{C}$, which is equivalent to taking $p = \frac{e^{i\theta}}{2\sqrt{H^2+c}}$ for certain $\theta \in [0, 2\pi)$). Now $(ds^2, \sigma^{2,0}, H)$ satisfy the Gauss and Codazzi equations and thus define an immersed H-surface $\Sigma \looparrowright \mathbb{M}^3(c)$ (up to congruences) with this fundamental data. Roughly speaking, we can say that CMC surfaces in $\mathbb{M}^3(c)$ are described by solutions of the Sinh-Gordon equation. We remark that moving $\theta \in [0, 2\pi)$ we get the well-known family of *associated surfaces,* all of them isometric and with the same CMC H.

3. Minimal surfaces in $M \times \mathbb{R}$.

Let (M, g_M) be a Riemannian surface with metric $g_M = \mu(w)|dw|^2$, $\mu \in C^\infty(M)$, $\mu > 0$. We endow the product space $M \times \mathbb{R}$ with the product metric $\mu|dw|^2 + dt^2$, where t parameterizes the vectical factor. Consider an immersion $X = (h, f)\colon \Sigma \looparrowright M \times \mathbb{R}$. As before, call $ds^2 = \lambda^2|dz|^2$ to the induced metric on Σ, $z = x+iy$ being a conformal coordinate on Σ. The tangent bundle of the immersion is generated by $X_x = (h_x, f_x)$ and $X_y = (h_y, f_y)$. Then,

$$\langle X_x, X_y\rangle = g_M(h_x, h_y) + f_x f_y = (\mu\circ h)^2 \Re(h_x\overline{h_y}) + f_x f_y,$$

where $\Re(\cdot)$ stands for real part. Analogously, $\|X_x\|^2 = (\mu\circ h)^2|h_x|^2 + f_x^2$ and $\|X_y\|^2 = (\mu\circ h)^2|h_y|^2 + f_y^2$. Using the classical operators $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, we can easily get

$$\left(\|X_x\|^2 - \|X_y\|^2\right) - 2i\langle X_x, X_y\rangle = 4(\mu\circ h)^2 h_z\overline{h}_z + 4f_z^2,$$

from where we have that

$$X \text{ is conformal if and only if } (f_z)^2 = -(\mu\circ h)^2 h_z\overline{h}_z = -g_M(h_z, h_z). \qquad (6)$$

Assume from now on that X is conformal. Then, the conformal factor of the induced metric ds^2 on Σ is given by

$$\begin{aligned}\lambda^2 &= \tfrac{1}{2}(\|X_x\|^2 + \|X_y\|^2) = \tfrac{1}{2}\left[2(\mu\circ h)^2(|h_z|^2 + |h_{\overline{z}}|^2) + 4|f_w|^2\right]\\ &\overset{(6)}{=} (\mu\circ h)^2(|h_z|^2 + |h_{\overline{z}}|)^2.\end{aligned}$$

Consider the smooth map $g\colon \Sigma \to \mathbb{C}$ given by

$$g = \frac{f_z h_{\overline{z}} - f_{\overline{z}} h_z}{\lambda|h_{\overline{z}}|}.$$

Note that the gradient of h has no zeros (otherwise we contradict that X is an immersion), hence the denominator of the last formula cannot vanish and, thus, g is smooth. The geometric meaning of the map g is stated in the following lemma, whose proof can be found in Sa Earp and Toubiana [9], Proposition 4:

Lemma 3.1 *Let* $X = (h, f) \colon (\Sigma, \lambda^2|dz|^2) \looparrowright (M \times \mathbb{R}, \mu|dw|^2 + dt^2)$ *be a conformal immersion. Then, the following formula defines a unit normal vector field along* X*:*

$$N = \frac{1}{|g|^2+1}\left(\frac{2\Re(g)}{\mu\circ h}, \frac{2\Im(g)}{\mu\circ h}, |g|^2-1\right), \tag{7}$$

where $\Im(\cdot)$ *stands for imaginary part. Furthermore,*

$$g^2 = -\frac{h_z}{\overline{h}_z}. \tag{8}$$

We remark that in the case $M = \mathbb{R}^2$ with its usual metric, then $\mu = 1$ and (7) says that g is the stereographic projection of the unit normal vector field from the north pole of the sphere.

Coming back to the general case, we now state a formula for the mean curvature of the immersion X, whose proof can be found in [9], Proposition 7. Note that if $w = a+ib$ is a local conformal coordinate in M, then $\partial_a, \partial_b, \partial_t$ give a (local) trivialization of the tangent bundle of $M \times \mathbb{R}$.

Lemma 3.2 *Let* $X = (h, f) \colon (\Sigma, \lambda^2|dz|^2) \looparrowright (M \times \mathbb{R}, \mu|dw|^2 + dt^2)$ *be a conformal immersion. Then, the mean curvature vector is given by*

$$2\lambda^2\vec{H} = 4\Re(\Theta)\,\partial_a + 4\Im(\Theta)\,\partial_b + \Delta_0 f\,\partial_t \tag{9}$$

where $\Theta = \Theta(z) = h_{z\overline{z}} + 2\left(\log\mu\right)_w h_z h_{\overline{z}}$ *and* Δ_0 *is the laplacian with respect to the flat metric* $|dz|^2$.

An immediate consequence of (9) is that X is a conformal and minimal if and only if both h and f are harmonic. In the sequel, we will assume this is the case. The fact that h is harmonic with values on the Riemann surface M allows us to associate to it an holomorphic object, namely the *Hopf differential* of h:

$$Q_h = \phi\,(dz)^2, \quad \text{where } \phi = g_M(h_z, h_z) = (\mu\circ h)^2 h_z \overline{h}_z. \tag{10}$$

By equation (6), $Q_h = -(f_z)^2\,(dz)^2 = -(\partial f)^2$, so Q_h is the square of a holomorphic differential on Σ. It is worth mentioning that the holomorphic differential Q_h also coincides up to a multiplicative constant with the celebrated *Abresch-Rosenberg* differential Q_{AR}, although these two authors defined Q_{AR} for any CMC surface in a three-dimensional homogeneous space with isometry group of dimension 4, see [1, 2]. We will not make use of this remarkable property.

With the above ingredients, we next explain how the Sinh-Gordon equation appears in this setting. A straightforward computation (see for instance page 8 of Schoen and Yau [10]) gives that

$$0 \overset{(A)}{\leq} \left|\frac{h_{\overline{z}}}{h_z}\right| = \frac{|dh|^2 - 2\,\mathrm{Jac}(h)}{|dh|^2 + 2\mathrm{Jac}(h)} \overset{(B)}{\leq} 1,$$

where dh is the differential of h and $\mathrm{Jac}(h)$ its Jacobian, and equality holds in (A) if and only if $h_{\overline{z}} = 0$ while equality holds in (B) if and only if $\mathrm{Jac}(h) = 0$. Since the gradient of h has no zeros, then equality in (A) cannot hold. Comparing with (8) we deduce that $1 \leq |g| < \infty$ in Σ, and thus the function $\psi\colon \Sigma \to \mathbb{R}$ given by

$$\psi = \log|g| \tag{11}$$

is smooth and non-negative. Furthermore, the zeros of ψ coincide with the zeros of $\mathrm{Jac}(h)$, that is to say, with the points where the tangent plane to the immersion is vertical. We next compute the laplacian of ψ with respect to the flat metric $|dz|^2$:

$$\Delta_{|dz|^2}\psi = \frac{1}{2}\Delta_{|dz|^2}\log|g|^2 = \frac{1}{2}\Delta_{|dz|^2}\log\frac{|h_z|}{|h_{\overline{z}}|} = -2K_M\,\mathrm{Jac}_{|dz|^2}(h),$$

where K_M is the Gaussian curvature of $(M, \mu|dw|^2)$, $\mathrm{Jac}_{|dz|^2}(h)$ is the Jacobian of h with respect to the metrics $|dz|^2$ on Σ and $\mu|dw|^2$ on M, and in the last equality we have used the Bochner formula for harmonic maps between surfaces (see [10] page 10). On the other hand,

$$\sinh(2\psi) = \frac{1}{2}\left(e^{2\psi} - e^{-2\psi}\right) \overset{(11)}{=} \frac{1}{2}\left(|g|^2 - |g|^{-2}\right) \overset{(8)}{=} \frac{1}{2}\left(\frac{|h_z|}{|h_{\overline{z}}|} - \frac{|h_{\overline{z}}|}{|h_z|}\right)$$

$$= \frac{1}{2}\frac{|h_z|^2 - |h_{\overline{z}}|^2}{|h_z|\,|h_{\overline{z}}|} = \frac{1}{2}\frac{|\partial h|^2 - |\overline{\partial} h|^2}{|\partial h|\,|\overline{\partial} h|} = \frac{1}{2}\frac{\mathrm{Jac}_{|dz|^2}(h)}{(\mu\circ h)^2|h_z|\,|h_{\overline{z}}|} \overset{(10)}{=} \frac{1}{2}\frac{\mathrm{Jac}_{|dz|^2}(h)}{|\phi|},$$

from where we obtain

$$\Delta_{|dz|^2}\psi + 4|\phi|K_M\sinh(2\psi) = 0.$$

We can absorb the multiplicative factor $|\phi|$ in the last equation by considering the metric $ds_0^2 = 8|\phi|\,|dz|^2 = 8|Q_h|^2$ on Σ, which is unbranched since the gradient of h does not vanish, and flat since ϕ is holomorphic. In this way we conclude the following well-known result (see Proposition 1 in [5]):

Proposition 3.3 *Let $X = (h, f)\colon (\Sigma, \lambda^2|dz|^2) \looparrowright (M\times\mathbb{R}, \mu|dw|^2 + dt^2)$ be a conformal minimal immersion. Then, the function ψ given by (11) satisfies the PDE*

$$\Delta_0\psi + \frac{1}{2}K_M\sinh(2\psi) = 0 \qquad \text{on } \Sigma, \tag{12}$$

where the laplacian is computed with respect to the flat metric $ds_0 = 8|Q_h|$ and Q_h is the Hopf differential associated to the harmonic function h.

In the case K_M is constant (i.e. $M = \mathbb{M}^2(\kappa)$, $\kappa \in \mathbb{R}$), then (12) is a genuine Sinh-Gordon equation, at least in the case $\kappa > 0$. Proposition 4.2 admits a converse, which allows to construct conformal minimal immersions in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ by prescribing the induced metric and a holomorphic quadratic differential $Q = Q_h$. For details, see Theorem 7 in [5].

4. New Sinh-Gordon and Sin-Gordon type equations for CMC surfaces in spaces $\mathbb{E}(\kappa, \tau)$.

In the sequel, we will study CMC surfaces in a simply-connected homogeneous three-manifold whose isometry group has dimension 4. These homogeneous spaces are classified in terms of two real numbers κ, τ with $\kappa \neq 4\tau^2$, and are usually denoted by $\mathbb{E}(\kappa, \tau)$. The space $\mathbb{E}(\kappa, \tau)$ admits a Riemannian fibration Π over the complete simply-connected surface $\mathbb{M}^2(\kappa)$ of constant curvature κ (the sphere $\mathbb{S}^2(\kappa)$ when $\kappa > 0$, the Euclidean plane $\mathbb{R}^2$ when $\kappa = 0$ and the hyperbolic plane $\mathbb{H}^2(\kappa)$ when $\kappa < 0$); here τ is the bundle curvature. The fibers of $\Pi\colon \mathbb{E}(\kappa, \tau) \to \mathbb{M}^2(\kappa)$ are geodesics, and translations along these fibers generate a unit Killing vector field E_3, called the *vertical vector field.* If $\tau = 0$, then $\mathbb{E}(\kappa, \tau)$ is the product space $\mathbb{M}^2(\kappa) \times \mathbb{R}$. In the case $\tau \neq 0$, we get three types of manifolds depending on the sign of κ: if $\kappa > 0$ we have the *Berger spheres,* if $\kappa = 0$ we obtain the *Heisenberg space* Nil_3, and the case $\kappa < 0$ corresponds to the universal cover of $\mathrm{SL}_2(\mathbb{R})$, the special linear group (or equivalently, the universal cover of the unit tangent bundle of $\mathbb{H}^2$). The above description could be extended for $\kappa = 4\tau^2$, obtaining the Euclidean space $\mathbb{R}^3$ (when $\kappa = \tau = 0$) or a round 3-sphere $\mathbb{S}^3$ (when $\kappa = 4\tau^2 \neq 0$), although these space forms have isometry group of dimension 6 and we will not consider them in this section.

Consider a (two-sided) immersion $X\colon \Sigma \looparrowright \mathbb{E}(\kappa, \tau)$ with CMC $H \geq 0$. We exclude the case $H = \tau = 0$, which was treated in the Section 3. Denote by $ds^2 = \lambda |dz|^2$ its induced metric (as before, $z = x + iy$ is a local conformal coordinate), and by K, N to the Gaussian curvature of ds^2 and the unit normal vector field to X, respectively. The bounded Jacobi function $\nu = \langle N, E_3 \rangle$ is called the *angle function* of X. Another geometrically relevant object in this setting is the complex valued smooth 1-form $A\, dz = \langle X_z, E_3 \rangle dz$. With these ingredients, we can consider the globally defined Abresch-Rosenberg quadratic differential $Q_{AR} = \phi\,(dz)^2$, where

$$\phi = 2(H + i\tau)p - (\kappa - 4\tau^2)A^2 \tag{13}$$

and $\sigma^{2,0} = p\,(dz)^2$ is the Hopf differential of X, defined as $p = \sigma(\partial_z, \partial_z)$. Note that the factor 4 appearing in (1) does not occur now; here we follow the notation in Daniel, Hauswirth and Mira [3]. Abresch and Rosenberg [1, 2] proved that Q_{AR} is holomorphic for every CMC surface $\Sigma \looparrowright \mathbb{E}(\kappa, \tau)$. The structure equations for ψ are given by (see for instance [3]):

$$
\begin{array}{llll}
(\mathbf{C},\mathbf{1}) & p_{\overline{z}} & = & \dfrac{\kappa-4\tau^2}{2}\lambda\nu A, \\
(\mathbf{C},\mathbf{2}) & A_{\overline{z}} & = & \dfrac{H+i\tau}{2}\lambda\nu, \\
(\mathbf{C},\mathbf{3}) & \nu_z & = & -(H-i\tau)A-2\dfrac{p}{\lambda}\overline{A}, \\
(\mathbf{C},\mathbf{4}) & 4\dfrac{|A|^2}{\lambda} & = & 1-\nu^2.
\end{array}
$$

The zero set of Q_{AR} consists of isolated points unless Σ is invariant by a one-parameter group of ambient isometries (Lemma 2.3.6 in [3]). In the sequel we will assume that Q_{AR} is not identically zero, and work outside the discrete set $\mathfrak{U}$ of zeros of Q_{AR}. We will also assume that $4H^2+\kappa>0$. Consider the metric

$$ds_0^2 = b|Q_{AR}| \qquad \text{in } \Sigma-\mathfrak{U},$$

where $b>0$ is a constant to be determined. ds_0^2 is flat and conformal to ds^2. Hence, there exists a smooth, real valued function $w\in C^\infty(\Sigma-\mathfrak{U})$ such that

$$ds^2 = e^{2w}\,ds_0^2 \qquad \text{in } \Sigma-\mathfrak{U}. \tag{14}$$

We define the constants

$$\delta = 4H^2+\kappa>0, \qquad \mu=\kappa-4\tau^2\neq 0,$$

(hence $4(H^2+\tau^2)=\delta-\mu$) and the following strictly increasing diffeomorphism:

$$
\left.
\begin{array}{lll}
\text{If } \mu>0, & f\colon(-\sqrt{\delta/\mu},\sqrt{\delta/\mu})\to\mathbb{R}, & f(x)=\frac{\operatorname{arg\,tanh}\left(\sqrt{\frac{\mu}{\delta}}x\right)}{\sqrt{\delta\mu}} \\
\text{If } \mu<0, & f\colon\mathbb{R}\to\left(-\frac{\pi}{2\sqrt{-\delta\mu}},\frac{\pi}{2\sqrt{-\delta\mu}}\right), & f(x)=\frac{\arctan\left(\sqrt{\frac{-\mu}{\delta}}x\right)}{\sqrt{-\delta\mu}}
\end{array}
\right\} \tag{15}
$$

We remark that the function $\delta-\mu\nu^2$ is smooth and positive on Σ, since otherwise μ is necessarily positive and then,

$$\nu^2 \le 1 \le \frac{4H^2+\kappa}{\kappa-4\tau^2} = \frac{\delta}{\mu}, \tag{16}$$

with equality only if $H=\tau=0$ at those points in Σ where $\nu^2=1$. Since we are assuming $H^2+\tau^2\neq 0$, then $\delta-\mu\nu^2>0$. Note that (16) also implies that ν always lies in the domain of definition of f (regardless of the sign of μ), hence $f\circ\nu$ makes sense.

Lemma 4.1 *In the above situation, assume that $\delta\neq\mu$ (i.e. $H^2+\tau^2\neq 0$). Then, $f\circ\nu$ satisfies the following PDE in $\Sigma-\mathfrak{U}$:*

$$\Delta(f\circ\nu)+\left(\frac{1}{2}+\frac{8e^{-4w}}{b^2}\frac{1}{(\delta-\mu\nu^2)^2}\right)\nu=0.$$

Proof. By the Gauss equation,

$$\tau^2 + \mu\nu^2 = K - \det S = K + \frac{4|p|^2}{\lambda^2} - H^2, \tag{17}$$

where we have used that (see page 4 of [4])

$$\frac{4|p|^2}{\lambda^2} = H^2 - \det S. \tag{18}$$

Hence,

$$\frac{\delta - \mu}{4} + \mu\nu^2 = K + \frac{4|p|^2}{\lambda^2} = -e^{-2w}\Delta_0 w + \frac{4|p|^2}{\lambda^2}, \tag{19}$$

Where Δ_0 refers to ds_0^2. On the other hand,

$$\lambda\,|dz|^2 = ds^2 = e^{2w}\,ds_0^2 = e^{2w} b\,|Q_{AR}| = e^{2w} b|\phi|\,|dz|^2,$$

hence

$$\frac{|\phi|}{\lambda} = \frac{e^{-2w}}{b}. \tag{20}$$

Squaring (20) and using (13) and (**C.4**),

$$\frac{|\phi|^2}{\lambda^2} = \frac{e^{-4w}}{b^2} = (\delta - \mu)\frac{|p|^2}{\lambda^2} + \frac{\mu^2}{16}(1-\nu^2)^2 - 4\mu\Re\left[(H + i\tau)\frac{p\overline{A}^2}{\lambda^2}\right]. \tag{21}$$

Since $\delta \neq \mu$, we can solve for $\frac{4|p|^2}{\lambda^2}$ in (21) obtaining

$$4\frac{|p|^2}{\lambda^2} = \frac{4e^{-4w}}{b^2(\delta-\mu)} - \frac{\mu^2}{4(\delta-\mu)}\left(1-\nu^2\right)^2 + \frac{16\mu}{\delta-\mu}\Re\left[(H+i\tau)\frac{p\overline{A}^2}{\lambda^2}\right]. \tag{22}$$

On the other hand, the last term in the RHS of (21) is related to $|\nabla\nu|^2$. To find the exact expression, first use (13) and (**C.4**) to obtain

$$2(H+i\tau)\frac{p}{\lambda}\overline{A} = \frac{\mu}{4}(1-\nu^2)A + \frac{\phi}{\lambda}\overline{A}. \tag{23}$$

Now (**C.3**) gives

$$\nu_z = -(H - i\tau)\left[1 + \frac{\mu}{\delta-\mu}(1-\nu^2)\right]A - \frac{1}{H+i\tau}\frac{\phi}{\lambda}\overline{A} \tag{24}$$

(compare with equation (2.3.4) in [3]). Now (13), (**C.4**) and (24) give

$$\begin{aligned} |\nabla\nu|^2 = \tfrac{4}{\lambda}|\nu_z|^2 \quad = \quad & \tfrac{\delta-\mu}{4}\left[1 - \tfrac{\mu^2}{(\delta-\mu)^2}(1-\nu^2)^2\right](1-\nu^2) \\ & + \tfrac{4}{\delta-\mu}\tfrac{e^{-4w}}{b^2}(1-\nu^2) + 16\tfrac{\delta-\mu\nu^2}{\delta-\mu}\Re\left[(H+i\tau)\tfrac{p\overline{A}^2}{\lambda^2}\right]. \end{aligned}$$

Solving for $\Re\left[(H+i\tau)\frac{p\overline{A}^2}{\lambda^2}\right]$ in the last expression and substituting the resulting value first in (21) and then in (19), we obtain

$$\frac{\delta-\mu}{4}+\mu\nu^2=-e^{-2w}\Delta_0 w+\frac{4e^{-4w}}{b^2\left(\delta-\mu\nu^2\right)}+\frac{\mu}{\delta-\mu\nu^2}|\nabla\nu|^2-\frac{\mu}{4}\left(1-\nu^2\right),$$

hence

$$\Delta w-\frac{4e^{-4w}}{b^2(\delta-\mu\nu^2)}-\frac{\mu}{\delta-\mu\nu^2}|\nabla\nu|^2+\frac{\delta+3\mu\nu^2}{4}=0. \tag{25}$$

Since ν is a Jacobi function,

$$\Delta\nu=2K\nu-(\delta+\mu\nu^2)\nu=-2\nu\Delta w-(\delta+\mu\nu^2)\nu. \tag{26}$$

Multiplying (25) by 2ν and using (26) we get

$$\frac{\delta+3\mu\nu^2}{2}\nu=\Delta\nu+(\delta+\mu\nu^2)\nu+\frac{8e^{-4w}}{b^2}\frac{\nu}{\delta-\mu\nu^2}+\frac{2\mu\nu}{\delta-\mu\nu^2}|\nabla\nu|^2. \tag{27}$$

On the other hand, by direct computation in (15) (regardless of whether μ positive or negative) we have

$$f'(x)=\frac{1}{\delta-\mu x^2},\qquad f''(x)=\frac{2\mu x}{(\delta-\mu x^2)^2}$$

whenever x lies in the domain of definition of f. Recall that $f\circ\nu$ makes sense. Multiplying by $f'(\nu)\neq 0$ in (27) gives

$$\frac{\delta+3\mu\nu^2}{2}\nu f'(\nu)\overset{(27)}{=}f'(\nu)\Delta\nu+(\delta+\mu\nu^2)\nu f'(\nu)+\frac{8e^{-4w}}{b^2}\frac{\nu f'(\nu)}{\delta-\mu\nu^2}+f''(\nu)|\nabla\nu|^2.$$

The first and fourth terms in the RHS of the last expression add up to $\Delta(f\circ\nu)$. Grouping the LHS with the third term of the RHS, we get

$$\begin{aligned} 0 &= \Delta(f\circ\nu)+\left(\tfrac{\delta-\mu\nu^2}{2}+\tfrac{8e^{-4w}}{b^2}\tfrac{1}{\delta-\mu\nu^2}\right)\nu f'(\nu)\\ &= \Delta(f\circ\nu)+\left(\tfrac{1}{2}+\tfrac{8e^{-4w}}{b^2}\tfrac{1}{(\delta-\mu\nu^2)^2}\right)\nu, \end{aligned}$$

and the lemma is proved. □

Next we manipulate the PDE in Lemma 4.1 to obtain an equation of Sinh-Gordon type if $\mu>0$ (resp. Sin-Gordon type if $\mu<0$).

Assume that $\mu>0$. These $\mathbb{E}(\kappa,\tau)$ spaces are $\mathbb{S}^2(\kappa)\times\mathbb{R}$ and the Berger spheres which lie in the complex projective space $\mathbb{CP}^2$ as geodesic spheres. Then, $\sqrt{\frac{\mu}{\delta}}\nu=\tanh\left[\sqrt{\delta\mu}f(\nu)\right]$ and thus,

$$\cosh^2\left[\sqrt{\delta\mu}f(\nu)\right]=\frac{\delta}{\delta-\mu\nu^2},\qquad \sinh^2\left[\sqrt{\delta\mu}f(\nu)\right]=\frac{\mu\nu^2}{\delta-\mu\nu^2}.$$

Since $\nu^2 < \frac{\delta}{\mu}$ as explained before Lemma 4.1, then $\delta - \mu\nu^2 > 0$ and we can extract square roots in the last two equalities. Therefore, $\sinh\left[\sqrt{\delta\mu} f(\nu)\right] \cosh\left[\sqrt{\delta\mu} f(\nu)\right] = \frac{\sqrt{\delta\mu}}{\delta-\mu\nu^2}\nu$ from which the PDE in Lemma 4.1 reads

$$\begin{aligned} 0 &= \Delta_0(f\circ\nu) + \left(\frac{e^{2w}}{2} + \frac{8e^{-2w}}{b^2}\frac{1}{(\delta-\mu\nu^2)^2}\right)\nu \\ &= \Delta_0(f\circ\nu) + \frac{1}{2\sqrt{\delta\mu}}\sinh\left[2\sqrt{\delta\mu} f(\nu)\right]\left(\frac{e^{2w}}{2}(\delta-\mu\nu^2) + \frac{8e^{-2w}}{b^2}\frac{1}{\delta-\mu\nu^2}\right). \end{aligned}$$

(note that $\nu, f(\nu), \sinh[\sqrt{\delta\mu} f(\nu)]$ all have the same sign). It is natural to write the parenthesis in the RHS in the form $u + \frac{1}{u}$. To do this, we take $b = 2$ and define the positive smooth function

$$u = \frac{e^{2w}}{2}(\delta - \mu\nu^2). \tag{28}$$

Then, $\Delta_0(\sqrt{\delta\mu} f\circ\nu) + \frac{1}{2}\sinh\left[2\sqrt{\delta\mu} f(\nu)\right]\left(u + \frac{1}{u}\right) = 0.$

Assume that $\mu < 0$**.** These $\mathbb{E}(\kappa,\tau)$ spaces are $\mathbb{H}^2(\kappa)\times\mathbb{R}$, Nil_3 and those Berger spheres which lie in the complex hyperbolic space $\mathbb{HP}^2$ as geodesic spheres. We continue assuming $4H^2+\kappa > 0$, which in Nil_3 only excludes the minimal case and in fibrations over the hyperbolic plane $\mathbb{H}^2(\kappa)$ constraints the mean curvature to $|H| > \frac{-\kappa}{2}$. Then, $\sqrt{\frac{-\mu}{\delta}}\nu = \tan\left[\sqrt{-\delta\mu} f(\nu)\right]$ and thus,

$$\cos^2\left[\sqrt{-\delta\mu} f(\nu)\right] = \frac{\delta}{\delta-\mu\nu^2}, \qquad \sin^2\left[\sqrt{-\delta\mu} f(\nu)\right] = \frac{-\mu\nu^2}{\delta-\mu\nu^2}.$$

Extracting square roots (again (note $\nu, f(\nu), \sin[\sqrt{-\delta\mu} f(\nu)]$ all have the same sign), we have $\sin\left[\sqrt{-\delta\mu} f(\nu)\right]\cos\left[\sqrt{-\delta\mu} f(\nu)\right] = \frac{\sqrt{-\delta\mu}}{\delta-\mu\nu^2}\nu$. Taking $b = 2$ and reasoning as before, we transform the PDE in Lemma 4.1 into

$$\Delta_0(\sqrt{-\delta\mu} f\circ\nu) + \frac{1}{2}\sin\left[2\sqrt{-\delta\mu} f(\nu)\right]\left(u + \frac{1}{u}\right) = 0,$$

with u as before (note that $u > 0$ is again positive in this setting). In summary:

Proposition 4.2 *Let $\Sigma \looparrowright \mathbb{E}(\kappa,\tau)$ be a CMC H surface with $H^2+\tau^2 \neq 0$ and $4H^2 + \kappa > 0$. Let ds^2, ν, Q_{AR} be its induced metric, angle function and Abresch-Rosenberg differential, respectively. Consider the flat metric $ds_0^2 = 2|Q_{AR}|$ (defined away from the zero set $\mathfrak{U}$ of Q_{AR}) and the functions*

$$\psi = \begin{cases} \operatorname{arg\,tanh}\left(\sqrt{\frac{\kappa-4\tau^2}{4H^2+\kappa}}\nu\right) & \text{if } \kappa - 4\tau^2 > 0, \\ \arctan\left(\sqrt{\frac{-\kappa+4\tau^2}{4H^2+\kappa}}\nu\right) & \text{if } \kappa - 4\tau^2 < 0, \end{cases} \qquad u = \frac{e^{2w}}{2}\left[4H^2+\kappa-(\kappa-4\tau^2)\nu^2\right],$$

where $w \in C^\infty(\Sigma - \mathfrak{U})$ *is defined by* $ds^2 = e^{2w}\, ds_0^2$. *Then,* $u > 0$ *and*

$$\Delta_0 \psi + \frac{1}{2}\mathbf{S}(2\psi)\left(u + \frac{1}{u}\right) = 0,$$

where $\mathbf{S}(\cdot) = \begin{cases} \sinh(\cdot) & \text{if } \kappa - 4\tau^2 > 0, \\ \sin(\cdot) & \text{if } \kappa - 4\tau^2 < 0. \end{cases}$

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Referencias

[1] U. Abresch and H. Rosenberg, A Hopf differential for constant mean curvature surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, *Acta Math.* **193(2)** (2004), 141–174.

[2] U. Abresch and H. Rosenberg, Generalized Hopf differentials, *Mat. Contemp.* **28** (2005), 1–28.

[3] B. Daniel, L. Hauswirth and P. Mira, *Constant mean curvature surfaces in homogeneous manifolds*, Korea Institute for Advanced Study, Seoul, 2009.

[4] J. M. Espinar and H. Rosenberg, Complete constant mean curvature surfaces in homogeneous spaces, to appear in Comment. Math. Helvet.

[5] L. Hauswirth, R. Sa Earp and E. Toubiana, Associate and conjugate minimal immersions in $M \times \mathbb{R}$, *Tohoku Math. J.* **60(2)** (2008), 267–286.

[6] M. Kilian and U. Schmidt, On the moduli of constant mean curvature cylinders of finite type in the 3-sphere, preprint, 2008, `arXiv:0712.0108`.

[7] U. Pinkall and I. Sterling, On the classification of constant mean curvature tori, *Ann. of Math.* **130** (1989), 407–451.

[8] M. Ritoré and A. Ros, Stable constant mean curvature tori and the isoperimetric problem in three space forms, *Comment. Math. Helvet.* **67** (1992), 293–305.

[9] R. Sa Earp and E. Toubiana, Screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$, *Illinois J. Math.* **49(4)** (2005), 1323–1362.

[10] R. Schoen and S. T. Yau, *Lectures on harmonic maps*, International Press, 1997.