Uniqueness of the Riemann minimal surfaces

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Abstract.-The Riemann minimal examples constitute the only known properly embedded genus zero minimal surfaces in $\mathbb{R}^3$ with infinitely many planar ends. They have been studied over the years, giving rise to many statements directed to conclude their uniqueness. Here we revise some of them, and we point out some new results in that direction.

1 Posing the problem. The solution in the periodic setting.

The surfaces we will handle appeared by first time in a posthumously paper by Riemann [21], where he classified all minimal surfaces in $\mathbb{R}^3$ foliated by circles and straight lines in parallel planes. These surfaces are the plane, the helicoid, the catenoid and a one-parameter family $\{R_t\}_{t>0}$ of surfaces with infinitely many ends asymptotic to parallel planes uniformly distributed, that since then have been known as the *Riemann minimal surfaces.* With more detail, each R_t is a properly embedded minimal surface in $\mathbb{R}^3$, it intersects horizontal planes at integer heights in parallel straight lines $\{l_k\}_{k\in\mathbb{Z}}$ orthogonal to the plane $\{x_2 = 0\}$, the remaining horizontal sections being circles of radius varying from $+\infty$ at the lines $\{l_k\}_k$ to a minimum radius exactly at heights in $1/2 + \mathbb{Z}$. The symmetry group of R_t consists of a reflection symmetry respect to the plane $\{x_2 = 0\}$, 180^o rotations around the lines l_k, a translation by vector $T_t = (t, 0, 2)$ and 180^o rotations around lines $l_k + \frac{1}{4}T_t$ that cut the surface orthogonally at antipodal points of the circles of minimum radius, see Figure 1. Each R_t is conformally a vertical cylinder punctured at integer heights that correspond to the ends. In particular, R_t has genus zero (i.e. R_t is a *planar domain*) and two limit ends. Moreover, for any positive integer n the quotient surface R_t/nT_t has genus one in $\mathbb{R}^3/nT_t$ and $2n$ planar ends.

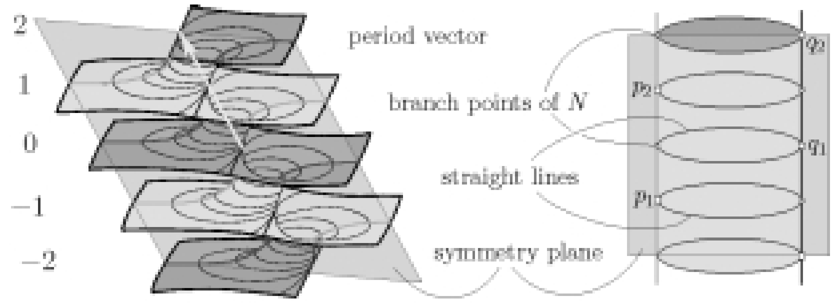

Figure 1. Left: One of the Riemann examples. Right: A conformal representation.

The main open question to be solved that concerns these surfaces is

Question 1 *Are $\{R_t\}_{t>0}$ the unique properly embedded minimal planar domains in $\mathbb{R}^3$ with two limit ends?*

The difficulty of Question 1 arises in that there are no assumptions on the size of the symmetry group of the considered surfaces. This problem has been attacked by many mathematicians assuming different symmetry properties, see for instance [4, 6, 9, 10, 17, 18, 19, 22]. Among this type of results, perhaps the strongest one is in Meeks, Pérez and Ros [12]:

Theorem 1 *Let $M \subset \mathbb{R}^3$ be a properly embedded periodic minimal planar domain with two limit ends. Then, M is one of the Riemann minimal examples.*

Proof. As M has more than one end, it admits a tangent plane at infinity[1], which in the sequel will be supposed horizontal. In this setting, Frohman and Meeks [5] showed that the ends are ordered by heights. By periodicity, M has a top and a bottom limit end, thus all middle ends are simple and annular. As the symmetry group of M is infinite, the structure theorem of Callahan, Hoffman and Meeks [1] guarantees the existence of a nontrivial screw motion or a translation that preserves M, and in both cases the quotient surface of M modulo this symmetry has genus one and a finite number of planar ends. Using properties of the flux, Pérez and Ros [19] excluded the screw motion symmetry in this setting, thus Theorem 1 follows directly from

Theorem 2 (Meeks, Pérez & Ros [12]) *Let $M \subset \mathbb{R}^3/T$ be a properly embedded minimal surface with genus one and a finite number of planar ends, T being a nontrivial translation. Then, M is a quotient of a Riemann minimal surface.*

Sketch of proof. Fix a positive integer n. Let $\mathcal{S}$ be the space of all properly embedded minimal tori in a quotient of $\mathbb{R}^3$ by a translation T (depending on the surface) with $2n$ horizontal planar ends. Also denote by $\mathcal{R} \subset \mathcal{S}$ the family of Riemann minimal examples, so the statement reduces to proving $\mathcal{R} = \mathcal{S}$.

The general description of an element in $M \in \mathcal{S}$ is as follows: The $2n$ planar ends of M are separated by a positive distance (maximum principle at infinity [8] or [14]). Because of embeddedness, consecutive ends of M have reversed (vertical) limit normal vectors,

[1] Callahan, Hoffman and Meeks [1] proved that if $M \subset \mathbb{R}^3$ is a properly embedded minimal surface with more than one end, then there exists a complete embedded minimal end with finite total curvature E in one of the regions of $\mathbb{R}^3 - M$, which defines a plane passing through the origin and orthogonal to the limit normal vector to E. This plane depends only on M and is called its *limit tangent plane at infinity.*

so we can label $(p_1, q_1, \ldots, p_n, q_n)$ the ends of M in increasing order of the height with $N(p_i) = (0, 0, -1)$, $N(q_i) = (0, 0, 1)$, $1 \leq i \leq n$, see Figure 2.

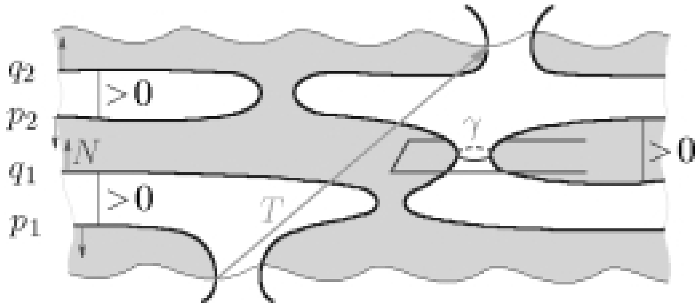

Figure 2: A generic $M \in \mathcal{S}$.

The stereographically projected Gauss map g of M extends as a meromorphic map of degree $2n$ to the torus $\overline{M}$ obtained by attaching to M its planar ends, thus $g(p_i) = 0^{2)}$, $g(q_i) = \infty^{2)}$, M has no points with vertical normal vector and each compact horizontal section γ is a Jordan curve. As the flux around a planar end is zero, the flux of M along γ, denoted by $\text{Flux}(M, \gamma)$, does not depend on the height of γ. By transversality, $\text{Flux}(M, \gamma)$ cannot be horizontal, what suggests to introduce in $\mathcal{S}$ the following normalization: We rescale and orient all surfaces $M \in \mathcal{S}$ so that $\text{Flux}(M, \gamma)$ has third coordinate equal one. Thus, $\text{Flux}(M, \gamma)$ is completely determined by the so called *Flux map*

$$F : \mathcal{S} \to \mathbb{R}^2 - \{(0, 0)\} \ / \ F(M) = (a, b) \text{ such that } \text{Flux}(M, \Gamma) = (a, b, 1).$$

The fact that F omits the value $(0, 0)$ follows from application of the López-Ros deformation, see [19]. This Flux map has three important properties. Two of them are of topological nature, when we endow $\mathcal{S}$ with the uniform topology on compact sets:

- F is a proper map,
- F is an open map,
- There exists $\varepsilon > 0$ such that if $|F(M)| < \varepsilon$, then $M \in \mathcal{R}$.

The proofs on the above three properties are delicate and we will not reproduce them here. They strongly depend on curvature estimates for sequences $\{M_i\}_i \subset \mathcal{S}$ such that $\{F(M_i)\}_i$ is bounded. These curvature estimates together with an analysis of the possible limits of sequences $\{M_i\}_i \subset \mathcal{S}$ give the properness of F. The openness follows after extending F to a holomorphic map P from the complex n-dimensional manifold $\mathcal{W}$ of all allowed Weierstrass data (without imposing to kill the periods) to $\mathbb{C}^n$ (P essentially expresses the conditions for a generic point in $\mathcal{W}$ to produce a minimal immersion, recall that the number of ends is $2n$), and applying to P the local open mapping theorem for holomorphic finite[2] maps of several variables. The fact that P is a finite map around any $M \in \mathcal{S} \subset \mathcal{W}$ uses again that F is proper. Finally, the uniqueness result for almost vertical flux stated in the third property of F above is proved by extending P holomorphically to a suitable point in the closure of the moduli space $\mathcal{W}$ (more precisely, to the limit point of any sequence $\{M_i\}_i \subset \mathcal{S}$ with $\{F(M_i)\}_i \to 0$), and using the inverse mapping theorem to conclude bijectivity of P in a neighborhood of that point.

On the other hand, the topological subspace $\mathcal{R} \subset \mathcal{S}$ of Riemann surfaces has two key properties:

- $\mathcal{R}$ is closed in $\mathcal{S}$, and
- $\mathcal{R}$ is open in $\mathcal{S}$.

The closeness of $\mathcal{R}$ in $\mathcal{S}$ follows because being a Riemann example is characterized by being foliated by circles and lines in parallel planes, a condition preserved by limits in the uniform topology on compact sets. Concerning why $\mathcal{R}$ is open in $\mathcal{S}$, Pérez [18] characterized $\mathcal{R}$ as the only *nondegenerate* minimal surfaces in $\mathcal{S}$, this nondegeneracy being defined in terms of the spaces of Jacobi functions on such surfaces. As the condition to be nondegenerate is open in the generic setting (see [18, 20]), $\mathcal{R}$ must be open in $\mathcal{S}$.

With these ingredients in mind, the proof of Theorem 2 is easy. By contradiction, if $\mathcal{R} \neq \mathcal{S}$ then $\mathcal{S} - \mathcal{R}$ is open and closed in $\mathcal{S}$, thus the restriction $F : \mathcal{S} - \mathcal{R} \to \mathbb{R}^2 - \{(0,0)\}$ must be an open and proper map, in particular it is surjective. This contradicts the third property of F above and finishes the proof of Theorem 2.

2 Some progresses in the nonperiodic setting.

In the sequel we will give some ideas that seem to be useful to face the solution of Question 1 when no periodicity is assumed. All what follows is part of a work in progress [13].

Let $\mathcal{S}$ be the space of all properly embedded minimal surfaces with genus zero and two limit ends. The description of any $M \in \mathcal{S}$ is as follows. As M has more than one end, it admits a tangent plane at infinity, which we will assume horizontal. By the work of Frohman and Meeks [5], the ends of M are ordered by heights. Thanks to a recent paper of Collin, Kusner, Meeks and Rosenberg [3] all middle ends are simple, thus the top and bottom ends are limit ends, and the remaining middle ends can be labeled as E_i, $i \in \mathbb{Z}$ (E_i is above E_j whenever $i > j$). Topologically, M is an infinite vertical cylinder $\mathcal{C}$ punctured in a discrete set that diverges to $+\infty$ and $-\infty$ in height.

The following easy argument shows that M has no catenoidal ends: suppose that M contains a catenoidal end E_0, say with positive logarithmic growth $c_0 > 0$. As M is

[2] A holomorphic map between complex manifolds of the same dimension is said to be *finite* if the inverse image of any point is finite.

embedded, all ends E_i with $i > 0$ must be catenoidal with logarithmic growth $c_i \geq c_0$. Now take a sequence of closed curves $\{\gamma_i \mid i \geq 0\} \subset M$, all homologous to the generator of $H^1(\mathcal{C}, \mathbb{Z})$, and such that each γ_i divides M in two noncompact components Ω_i^-, Ω_i^+, with $\Omega_0^+ \supset \Omega_1^+ \supset \Omega_2^+ \supset ...$ and $E_i \subset \Omega_i^+$ for all $i \geq 0$, see Figure 3. We can also assume that the Gauss map is never vertical along the γ_i-curves.

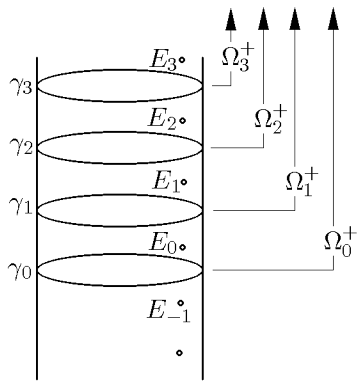

Figure 3: A topological picture of M.

For each $i > 0$, the domain $\Omega_0^+ - \Omega_i^+$ has boundary $\gamma_0 \cup \gamma_i$ and contains i catenoidal ends, all with positive third coordinate of flux. A direct application of divergence theorem shows that in order to vanish the total flux of $\Omega_0^+ - \Omega_i^-$, the sum of the logarithmic growths of ends inside $\Omega_0^+ - \Omega_i^-$ must converge to some finite value, which contradicts that $c_i \geq c_0 > 0$ for all i. Thus, all middle ends of M must be planar.

To discover the conformal structure of M, divide the surface by cutting with a transversal horizontal plane $\{x_3 = c\}$ between consecutive ends. This intersection defines a decomposition $M = M^+ \cup M^-$, with $M^+ \subset \{x_3 \geq c\}$. By a Theorem of Collin, Kusner, Meeks and Rosenberg [3], M^+ is parabolic in the sense that bounded harmonic functions on M^+ are determined by their boundary values. After attaching the simple ends inside M^+, we will obtain a new parabolic surface $\overline{M^+}$ that is topologically a punctured disk, thus $\overline{M^+}$ is also conformally a punctured disk and the same holds for the part of M below $\{x_3 = c\}$. Now we glue these two parts along the compact section at height c, so we get $M \cup \{\text{planar ends}\} = \mathbb{C} - \{0\}$ conformally. In particular, the conformal structure of M is $\mathbb{C} - [\{0\} \cup \{p_i\}_i]$, where $\{p_i\}_i \subset \mathbb{C} - \{0\}$ is a sequence that accumulates at 0 and ∞, each p_i corresponding to a planar end E_i.

Furthermore, the third coordinate function x_3 extends to a proper harmonic function on the punctured disk $\overline{M^+} = \{0 < |z| \leq 1\}$ with constant boundary values, thus $x_3(z) = \lambda \log |z|$ for certain $\lambda < 0$ and consequently, g misses $0, \infty$ in M^+ (analogously for M^-). From here is not difficult to arrive, up to a rescaling of M so that the third coordinate of the flux along any compact horizontal section is one, to the Weierstrass data $(g, \phi_3 =$

$\frac{\partial x_3}{dz}dz = \frac{dz}{z}$), $z \in \mathbb{C} - [\{0\} \cup \{p_i\}_i]$. Finally, the orders of poles and zeros of the Gauss map at the ends are all double, because otherwise the gradient of x_3 would vanish at those points.

All the proven properties for any $M \in \mathcal{S}$ let us consider the *Schiffman function* of M, defined as

$$u = \lambda \frac{\partial \kappa}{\partial y} = \text{ Im}\left[\frac{3}{2}\left(\frac{g'}{g}\right)^2 - \frac{g''}{g} - \frac{1}{|g|^2+1}\left(\frac{g'}{g}\right)^2\right],$$

where $(g(\xi), \phi_3 = d\xi)$, $\xi = x + iy \in \mathbb{C}$ is the Weierstrass data of M in the ξ-plane (the relation between this ξ-coordinate and the above z is $z = e^{\xi}$), $ds^2 = \lambda^2|d\xi|^2$ is the induced metric, κ is the curvature of the planar section $\{x = \text{constant}\}$ and the prime stands for derivation with respect to the ξ-parameter. As λ is a positive function, the zeros of u coincide with the zeros of the derivative of κ respect to the variable y (or respect to any other parameter in the planar section $\{x = \text{constant}\}$). As consequence, the condition on u to vanish identically is equivalent to say that M is foliated by circles or lines in horizontal planes, i.e. $M \in \mathcal{R}$.

The following statement shows that the Riemann minimal examples admit another characterization in terms of the Schiffman function.

Proposition 1 *Let $M \in \mathcal{S}$. If the Schiffman function of M is a linear function of the Gauss map, then $M \in \mathcal{R}$.*

Proof. Let $f = \frac{3}{2}\left(\frac{g'}{g}\right)^2 - \frac{g''}{g} - \frac{1}{|g|^2+1}\left(\frac{g'}{g}\right)^2$ on M, so $u = \text{Im}(f)$ is the Schiffman function of M. We claim that if $u = \langle N, a\rangle$ where N is the Gauss map of M and $a \in \mathbb{R}^3$, then $f = \langle N, z_0\rangle$ for certain $z_0 \in \mathbb{C}$.

To see this, recall that there exists a well-known correspondence between the linear space $\mathcal{J}(M)$ of all Jacobi functions on a given minimal surface M and the linear space $B(N)$ of branched minimal immersions with the same Gauss map N as M: each $v \in \mathcal{J}(M)$ defines a conformal harmonic map $X_v = \nabla v + vN : M \to \mathbb{R}^3$ (thus a branched minimal immersion) whose Gauss map is N and whose support function is $\langle X_v, N\rangle = v$. Furthermore, $v \in \mathcal{J}(M)$ is linear if and only if X_v is constant and if $v^* \in \mathcal{J}(M)$ is Jacobi-conjugate[3] of $v \in \mathcal{J}(M)$ then X_v, X_{v^*} are conjugate branched minimal immersions. For details, see Montiel and Ros [16]. Coming back to our setting, if the Schiffman function u is linear on M, then the corresponding branched minimal immersion X_u is constant, and thus its minimal conjugate $(X_u)^* = X_{u^*}$ is also constant, which implies that $u^* = -\text{Re}(f)$ is also linear. This proves our claim.

Once we know that $f = \langle N, z_0\rangle$, a simple calculation that uses the expression of f and the relationship between N and g let us arrive to an equation of the type

$$(g')^2 = g(\alpha g^2 + \beta g + \delta) \tag{1}$$

[3] Two Jacobi functions $v, v^* \in \mathcal{J}(M)$ are called *Jacobi-conjugate* if there exists a globally defined complex solution h of the Jacobi equation on M such that $v = \text{Re}(h)$ and $v^* = \text{Im}(h)$.

for certain $\alpha, \beta, \delta \in \mathbb{C}$. As consequence, (g, g') is a branched holomorphic covering from the cylinder $\mathcal{C} = M \cup \{\text{planar ends}\}$ onto the compact Riemann surface $\Sigma = \{(z, w) \in \overline{\mathbb{C}}^2 \mid w^2 = z(\alpha z^2 + \beta z + \delta)\}$. On the other hand, the uniqueness of solutions of the holomorphic differential equation of first order (1) with a given initial value shows that if a, b are distinct points in $\mathcal{C}$ with $g(a) = g(b) \neq 0, \infty$, then $g(z + b - a) = g(z)$ for all z. As we can always find distinct points $a, b \in \mathcal{C}$ with this property (take points close enough to different ends with the same normal limit vector), we conclude that $b - a$ is a period of g and also of g'. This implies that (g, g') factorizes through the torus $\mathcal{C}/(b - a)$, giving rise to a holomorphic map $\pi : \mathcal{C}/(b - a) \to \Sigma$.

If Σ is a sphere, then the total ramification number of π is $B(\pi) = 2\text{degree}(\pi)$. But the pullback by π of the meromorphic differential $\frac{dz}{w}$ on Σ is $\frac{dg}{g'} = d\xi$, which has no poles nor zeros on $\mathcal{C}$. In particular, branch points of π occur exactly at $\pi^{-1}(\text{poles of } \frac{dz}{w})$ and thus $B(\pi) = \text{degree}(\pi)$, a contradiction. Therefore, Σ is also a torus and $\pi : \mathcal{C}/(b - a) \to \Sigma$ is unbranched. Finally, the Weierstrass data $(g, d\xi)$ of M can be induced on Σ as $\left(z, c\frac{dz}{w}\right)$, $c \in \mathbb{C} - \{0\}$, producing a properly embedded minimal torus with two planar ends in certain $\mathbb{R}^3/T$, which has to be a Riemann example by Theorem 1. Now the Proposition is proved.

Definition 1 *A minimal planar domain $M \in \mathcal{S}$ is called quasiperiodic if there exist sequences $\{p(j)^+\}_j, \{p(j)^-\}_j \subset M$ with $\{x_3(p(j))\} \to \pm\infty$, and surfaces $M_\infty^+, M_\infty^- \in \mathcal{S}$ such that $\{M - p(j)^\pm\}_j$ converges to $M_\infty^\pm$ on compact subsets of $\mathbb{R}^3$.*

Obviously quasiperiodicity generalizes periodicity, so a remarkable advance in the solution of Question 1 would be characterizing the Riemann minimal examples among all quasiperiodic minimal planar domains. On the other hand, being quasiperiodic is closely related to having global curvature estimates, in the following manner.

Lemma 1 *If the absolute Gaussian curvature of $M \in \mathcal{S}$ is bounded by above, then M is quasiperiodic.*

Sketch of proof. As M cuts transversely every horizontal plane at height different of the end heights and the vertical part of the flux is normalized to be 1, Lemma 2 in [12] insures that for any divergent sequence $\{p(i)\}_i \subset M$ with $N(p(i))$ horizontal (here N is the Gauss map of M), the minimum of the absolute Gaussian curvature $|K|$ in disks $D(i) \subset M$ centered at $p(i)$ with certain positive radius independent of i is bounded away from zero. As by hypothesis $|K|$ is globally bounded by above, a subsequence of $\{M - p(i)\}_i$ must converge to a properly embedded nonflat minimal surface M_∞ in $\mathbb{R}^3$ with genus zero and bounded curvature. Note that we can find points where N is horizontal between any two consecutive planar ends of M, thus to finish the proof if suffices to check that M_∞ has two limit ends.

By contradiction, firstly suppose that M_∞ has only one end, in particular it is simply connected. As M_∞ is properly embedded, simply connected and has bounded Gaussian curvature, Xavier's work [23] insures that it will be a vertical helicoid provided that every horizontal plane cuts the surface transversely in just one component. This last property comes from the fact that the Gauss map of M_∞ omits the vertical direction and from elementary Morse theory (one could also use here a very recent Theorem of Meeks and Rosenberg [15] to conclude that M_∞ is a vertical Helicoid). Now the contradiction comes from the fact that the third component of the flux of $M - p(i)$ equals one while the same component in a vertical helicoid is infinite.

Now assume that the number of ends of M_∞ is finite. By the paper of Collin [2], M_∞ will have finite total curvature, thus it is a catenoid, López-Ros [11]. The above observation on the Gauss map shows that this catenoid is vertical, hence its flux is also vertical. This is also impossible, because the flux of $M - p(i)$ coincides with the flux of M, which is not vertical (this nonverticality follows from application of the López-Ros deformation to pieces of M obtained by intersection with horizontal slabs, bounded by two compact convex planar curves $\gamma(i), \gamma(i+1)$ in horizontal planes, where $\gamma(i)$ converges to the neck of M_∞).

Finally, assume that M_∞ has only one limit end. Denote by $\Pi = \{a\}^\perp$ the tangent plane at infinity of M_∞, orthogonal to a unit vector $a \in \mathbb{S}^2(1)$. If a is vertical, then M_∞ has a highest or lowest end, which must be catenoidal by the Strong Halfspace Theorem [7]. From here it is not difficult to deduce that M_∞ has vertical flux, which is impossible as demonstrates a suitable modification of the argument in the last paragraph. If a is not vertical, then the embeddedness of M_∞ insures that all its ends have limit normal vector $\pm a$. With a little work one can show that the Gauss map N_∞ of M_∞ takes a vertical value, so the open mapping property of N_∞ insures that the same holds for $M - p(i)$ with i large, a contradiction. This completes the proof.

The converse of Lemma 1 was unknown before the time this meeting held. During the weeks here in MSRI, the authors were able to conclude its validity. In fact, they proved a stronger result:

Theorem 3 *Let $\{M_i\}_i \subset \mathcal{S}$ be a sequence of surfaces with $|F(M_i)|$ bounded. Then, the Gaussian curvature of the M_i is uniformly bounded.*

The proof of this result is based in the argument used by the authors when obtained curvature estimates in the periodic setting [12], joint with an appropriate use of recent results by Colding and Minicozzi about curvature estimates for embedded minimal disks and about minimal laminations (see [15] and references therein).

Theorem 3 immediately gives that any surface in $\mathcal{S}$ has bounded Gaussian curvature, thus Lemma 1 shows that quasiperiodicity is shared by any surface in $\mathcal{S}$. The boundedness of the curvature for any surface in $\mathcal{S}$ should in fact hold in much more generality. Let us state it as a

Conjecture 1 *Every properly embedded minimal surface in $\mathbb{R}^3$ with finite genus has bounded Gaussian curvature.*

To finish these notes, we present a situation in which several of the tools explained above can be applied to give an uniqueness result for the Riemann minimal examples.

Theorem 4 *There exists an $\varepsilon > 0$ such that if $|F(M)| < \varepsilon$ for $M \in \mathcal{S}$, then M is a Riemann minimal example.*

Remark 1 *Note that Theorem 4 is nothing but the generalization to the nonperiodic setting of point three in the properties of the periodic Flux map listed in page 3. At this point, the reader could ask if the remaining points remain valid, with the goal of solving Question 1 in the affirmative: the corresponding properness of F follows directly from Theorem 3, while the proof of the closedness of $\mathcal{R}$ in $\mathcal{S}$ remains valid now. Concerning the openness of $\mathcal{R}$ in $\mathcal{S}$, the authors have also proved it with an argument similar in nature to the proof that we present below. All these facts reduce the affirmative answer of Question 1 to proving the openness of the Flux map in the quasiperiodic setting, still work in progress.*

Sketch of proof. The first step consists of proving that given any sequence $\{M_i\}_i \subset \mathcal{S}$ with $F(M_i) \to 0$ and any choice of points $p_i \in M_i$ where the Gauss map of M_i is horizontal, the sequence $M_i - p_i$ has a partial that converges to a vertical catenoid with flux $(0, 0, 1)$ on compact subsets of $\mathbb{R}^3$. This follows from Theorem 3 plus an appropriate analysis of the possible limits of $\{M_i\}_i$. From here one arrives to the following situation: for i large enough, the surface M_i is arbitrarily closed to an infinite stack of vertical catenoids (all with neck lengths equal 2π), joined by almost flat horizontal graphs, and each one of this graphs contains one end and one finite branch point where the Gauss map takes an almost vertical value. Hence our Theorem reduces to the following

Theorem 5 *If $M \in \mathcal{S}$ is a quasiperiodic planar domain and M is close enough to an infinite discrete collection of vertical catenoids joined by almost flat graphs containing the ends, then $M \in \mathcal{R}$.*

Sketch of proof. Given $M \in \mathcal{S}$ with Gauss map g, we will call a *nodal domain* of $\{|g| = 1\}$ to any component Ω of $[M \cup \{\text{planar ends}\}] - \{|g| = 1\}$. Nodal domains will play a key role in the proof, so we need a detailed description of their geometry. Fix such a nodal domain $\Omega \subset M$. As M is extremely close to a stack of vertical catenoids with vertical flux one joined by almost flat graphs, Ω must be bounded by two almost round circles extremely close to the necks of consecutive vertical catenoids, and contains exactly one planar end of M, where g has a double zero or pole. $g|_\Omega$ is a holomorphic branched two-sheeted covering of a halfsphere with two branch points (both of the type z^2), one at the end contained in Ω and the other on a nonvertical value of g extremely close to vertical, see Figure 4.

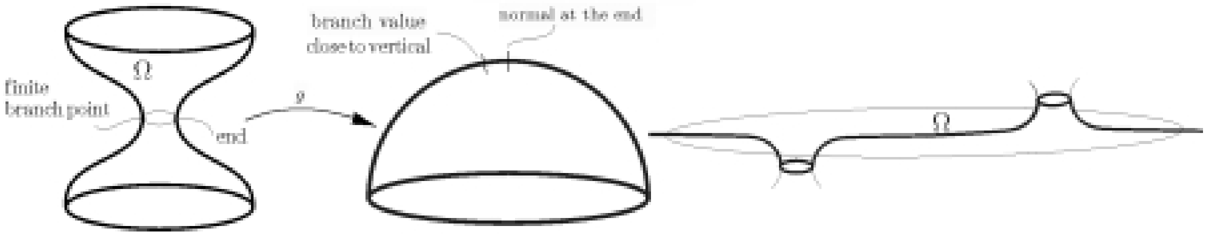

Figure 4: A nodal domain Ω of $\{|g| = 1\}$.

The argument for the Proof of Theorem 5 divides into three steps.

STEP 1. *Let $M \in \mathcal{S}$ be quasiperiodic, with Gauss map g and Schiffman function u. If M is close enough to an infinite discrete collection of vertical catenoids joined by almost flat graphs containing the ends, then for any nodal domain Ω of $\{|g| = 1\}$ it holds*

$$Q_\Omega(u, u) \geq 0, \quad \textit{with equality if and only if } M \in \mathcal{R}.$$

Sketch of proof of Step 1. The statement is equivalent to proving that if $\{M_i\}_i \subset \mathcal{S} - \mathcal{R}$ is a sequence of quasiperiodic minimal planar domains (none being a Riemann minimal example) that converges to an infinite stack of vertical catenoids, then for i large enough it holds $Q_{\Omega_i}(u_i, u_i) > 0$, where u_i is the Schiffman function of M_i, $\Omega_i \subset M_i$ is any nodal domain of $\{|g_i| = 1\}$ (g_i is the Gauss map of M_i) and Q_{Ω_i} is the index form for the Jacobi operator on Ω_i. By contradiction, assume that $\{M_i\}_i \subset \mathcal{S} - \mathcal{R}$ converges to an infinite stack of vertical catenoids and for each i we find a nodal domain Ω_i of $\{|g_i| = 1\}$ such that $Q_{\Omega_i}(u_i, u_i) \leq 0$.

As $i \to \infty$, Ω_i degenerates into the union of two opposite halfspheres Ω^+, Ω^- joined by a common point e (the south pole in Ω^+, the north pole in Ω^-) where both the end and the finite branch point on each Ω_i collapse. We will rescale u_i, which is not identically zero by hypothesis, to have a good limit (up to a subsequence) on $(\Omega^+ \cup \Omega^-) - \{e\}$: exchanging the induced metric ds_i^2 by the spherical metric on Ω_i, the index form $Q_{\Omega_i}(u_i, u_i)$ writes as

$$0 \geq \int_{\Omega_i} (\|\nabla_i u_i\|_i^2 - \|\nabla_i N_i\|_i^2 u_i^2) ds_i^2 = \int_{\Omega_i} (\|\nabla u_i\|^2 - 2u_i^2) dA, \tag{2}$$

where N_i is the Gauss map of M_i, all the objects in the first integral are computed with respect to the induced metric ds_i^2 and the objects in the second integral refer to the spherical metric. Note that by analyticity of the Jacobi functions, u_i cannot be identically zero on Ω_i and it makes sense to define $v_i = \frac{1}{\|u_i\|_{L^2(\Omega_i, dA)}} u_i$ on Ω_i, which is a Jacobi function with L^2-norm one in the spherical metric, for which (2) translates into $\||\nabla v_i|\|_{L^2(\Omega_i)}^2 \leq 2$. Standard compactness and regularity theorems for elliptic equations insure that there exists a function $v \in C^\infty(\Omega^+ \cup \Omega^-)$ such that, after extracting a subsequence, $\{v_i\}_i$ converges to v smoothly on compact subsets of $(\Omega^+ \cup \Omega^-) - \{e\}$, and v satisfies $\Delta v + 2v = 0$ in $\Omega^+ \cup \Omega^-$. Call $v^+ = v|_{\Omega^+}$, $v^- = v|_{\Omega^-}$. From the convergence of v_i to v one can deduce $\|v\|_{L^2(\Omega^+ \cup \Omega^-)} = 1$ and $\||\nabla v|\|_{L^2(\Omega^+ \cup \Omega^-)}^2 \leq 2$, which translate into

$$Q_{\Omega^+}^{\Delta+2}(v^+, v^+) + Q_{\Omega^-}^{\Delta+2}(v^-, v^-) \leq 0, \tag{3}$$

where $Q_{\Omega^\pm}^{\Delta+2}$ stands for the index form associated to the operator $\Delta+2$ in the corresponding round halfsphere. In particular, one of the two terms in the left-hand-side of (3) must be nonpositive, say $Q^{\Delta+2}_{\Omega^+}(v^+,v^+)\leq 0$. Let $\varphi=v^+-cf_3$, where $f_3(x)+\langle x,e_3\rangle$ on Ω^+ (here $e_3=(0,0,1)$) and $c=\left(\int_{\Omega^+}f_3\,dA\right)^{-1}\left(\int_{\Omega^+}v^+\,dA\right)\in\mathbb{R}$. Thus,

$$Q^{\Delta+2}_{\Omega^+}(\varphi,\varphi)=Q^{\Delta+2}_{\Omega^+}(v^+,v^+)-2cQ^{\Delta+2}_{\Omega^+}(v^+,f_3)+Q^{\Delta+2}_{\Omega^+}(f_3,f_3).$$

But

$$Q^{\Delta+2}_{\Omega^+}(f_3,f_3)=\int_{\Omega^+}(\|\nabla f_3\|^2-2f_3^2)dA=\int_{\Omega^+}[\mathrm{div}(f_3\nabla f_3)-f_3(\Delta f_3+2f_3)]\,dA=0,$$

the last equality being true because f_3 vanishes at $\partial\Omega^+$ and because $\Delta f_3+2f_3=0$ in the sphere. A similar argument shows that $Q^{\Delta+2}_{\Omega^+}(v^+,f_3)=0$ from where one has $Q^{\Delta+2}_{\Omega^+}(\varphi,\varphi)=Q^{\Delta+2}_{\Omega^+}(v^+,v^+)\leq 0$. As $\int_{\Omega^+}\varphi\,dA=0$ and the second eigenvalue for the Neumann problem of the Laplacian in the round halfsphere $\mathbb{S}^2(1)\cap\{x_3\leq 0\}$ is two, with eigenfunctions $\langle x,a\rangle$ (here a is orthogonal to e_3), one gets that $Q^{\Delta+2}_{\Omega^+}(\varphi,\varphi)=0$ thus $\varphi(x)=\langle x,a\rangle$ for certain $a\perp e_3$. This shows that v^+ must be linear on Ω^+. Plugging this information into (3), it follows that $Q^{\Delta+2}_{\Omega^-}(v^-,v^-)=0$ thus v^- must also be linear on Ω^-.

The following argument shows that both v^+,v^- are identically zero. By the four vertex theorem, the Schiffman function u_i has at least four zeros on each compact horizontal section of Ω_i, so the same holds for v_i. Passing to the limit both v_i and the compact horizontal sections on Ω_i, it can be deduced that v^+ must have at least one zero on each horizontal circle in Ω^+ (with at most one exception). By contradiction, assume that the linear function v^+ is not identically zero. The above distribution of zeros forces v^+ to be of the form $v^+=\langle x,a\rangle$ with $a\in\mathbb{S}^2(1)$ horizontal. But such v^+ has exactly two zeros on each horizontal circle $\Gamma^+\subset\Omega^+$, which obliges the (at least four) zeros of v_i on the compact horizontal section Γ_i^+ converging to Γ^+ to collapse into the two zeros of v^+ in Γ^+. In particular, the gradient of v^+ must vanish at $\Gamma^+\cap\{v^+=0\}=\Gamma^+\cap\{a\}^\perp$, a contradiction. This proves that $v^+=0$ in Ω^+, and similarly $v^-=0$ in Ω^-.

To finish the proof of Step 1, denote by p_i a maximum of $|u_i|$ in Ω_i (this maximum exists because the Schiffman function extends smoothly through the ends of M). Clearly, p_i is also a maximum for $|v_i|$ in Ω_i. Assume for the moment that $p_i\in\Omega_i$ does not collapse into e when $i\to\infty$. As $\{v_i\}_i$ converges smoothly to $v=0$ on compact subsets of $(\Omega^+\cup\Omega^-)-\{e\}$, it follows

$$1=\|v_i\|_{L^2(\Omega_i)}\leq|v_i(p_i)|\int_{\Omega_i}dA=4\pi|v_i(p_i)|\to 0,$$

a contradiction. Thus, it remains to demonstrate that p_i does not collapse into e as $i\to\infty$. Ω_i is conformally equivalent to a compact cylinder $[a_i^-,a_i^+]\times\mathbb{S}^1$, where $a_i^-<0<a_i^+$ and $a_i^\pm\to\pm\infty$ as $i\to\infty$. Reasoning by contradiction, if $\{p_i\}_i\to e$ as $i\to\infty$, then we can place p_i in the model of $\Omega_i=[a_i^-,a_i^+]\times\mathbb{S}^1$ at a fixed point $p_0\in\{0\}\times\mathbb{S}^1$ for all i. Define $w_i=\frac{1}{|u_i(p_i)|}u_i$ (note that $|u_i(p_i)|\neq 0$ because $M_i\in\mathcal{S}-\mathcal{R}$). Then w_i is a

solution of the Jacobi equation in $\Omega_i = [a_i^-, a_i^+] \times \mathbb{S}^1$, which in the flat metric writes as $\Delta_0 w_i + \|\nabla_0 N_i\|_0^2 w_i = 0$. Using that $\{\|\nabla_0 N_i\|_0^2\}_i$ converges to zero on compact subsets of $\mathbb{R} \times \mathbb{S}^1$ (this follows because any compact subset of $\mathbb{R} \times \mathbb{S}^1$, when viewed in the spherical covering model of Ω_i, must collapse entirely into e thus the spherical area of its image by N_i goes to zero) and that $|w_i|$ is globally bounded by one in $[a_i^-, a_i^+] \times \mathbb{S}^1$, we deduce that, after passing to a subsequence, $\{w_i\}_i$ converges smoothly on compact subsets of $\mathbb{R} \times \mathbb{S}^1$ to a harmonic function $w_\infty : \mathbb{R} \times S^1 \to [-1, 1]$. As $\mathbb{R} \times \mathbb{S}^1$ is parabolic, w_∞ must be constant. As $|w_i(p_0)| = 1$, we get $w_\infty = \pm 1$ on $\mathbb{R} \times \mathbb{S}^1$. Finally, the contradiction follows from a counting argument of the zeros of w_i in each compact horizontal section, similar to the one we did above. Now Step 1 is proved.

STEP 2. *Let $M \in \mathcal{S}$ be any quasiperiodic minimal planar domain for which Step 1 holds. By quasiperiodicity, there exist points $p(j)^+, p(j)^- \in M$ with $\{x_3(p(j)^\pm)\}_j \to \pm\infty$, such that $\{M - p(j)^\pm\}_i$ converges on compact subsets of $\mathbb{R}^3$ to minimal surfaces $M_\infty^+, M_\infty^- \in \mathcal{S}$. Then, M_∞^+ and M_∞^- are the same Riemann minimal example.*

*Proof of Step 2.*For $j \in \mathbb{N}$ fixed, denote by u_j^+ the Schiffman Jacobi function of $M - p(j)^+$, which is nothing but $p \in M - p(j)^+ \mapsto u_j^+(p) = u(p + p(j)^+)$, u being the Schiffman function of M. The smooth convergence of $\{M - p(j)^+\}_j$ to M_∞^+ implies that $\{u_j^+\}_j$ converges smoothly on compact subsets of M_∞^+ to the Schiffman function u_∞^+ of M_∞^+.

Note that as Step 1 holds for M, it also holds for any translation of M and after taking smooth limits, it holds for M_∞^+. Fix a nodal domain $\Omega_\infty \subset M_\infty \cup \{\text{planar ends}\}$ of $\{|g_\infty| = 1\}$, where g_∞ represents the Gauss map of M_∞^+. Let γ_∞ be one of the boundary curves of Ω_∞. As γ_∞ is compact, we find a sequence $\gamma(j)$ of curves inside M such that $\{\gamma(j) - p(j)^+\}_j$ converges smoothly to γ_∞. Now call $\Omega(j)$ to the domain in M bounded by $\gamma(j) \cup \gamma(j+1)$, see Figure 5.

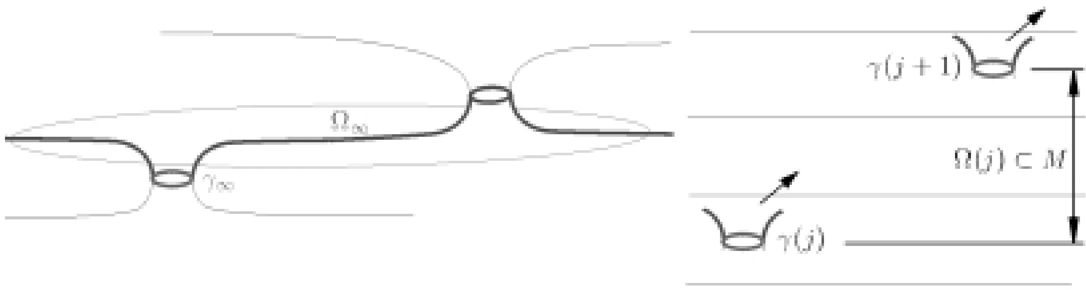

Figure 5. Left: The nodal domain $\Omega_\infty \subset M_\infty^+$. Right: The domain $\Omega(j) \subset M$.

Clearly, we can choose the curves $\gamma(j)$ $(j \in \mathbb{N})$ so that $\Omega(j)$ is a (finite) union of nodal domains of $\{|g| = 1\}$, $\Omega(j) = \Omega_{1,j} \cup \ldots \cup \Omega_{k(j),j}$. Using Step 1 in each nodal domain and integrating by parts,

$$0 \leq \sum_{i=1}^{k(j)} Q_{\Omega_{i,j}}(u,u) = \sum_{i=1}^{k(j)} \int_{\partial \Omega_{i,j}} u \frac{\partial u}{\partial \eta} ds = \int_{\partial \Omega(j)} u \frac{\partial u}{\partial \eta} ds,$$

where η is the unit conormal vector exterior to $\Omega_{i,j}$ along its boundary, and the second equality follows from the observation that if two nodal domains among $\Omega_{1,j}, \ldots, \Omega_{k(j),j}$ share a common part of their boundaries, then the corresponding conormal vectors along such common part are opposite. Furthermore, the last integral equals

$$\int_{\gamma(j+1)} u \frac{\partial u}{\partial \eta} ds - \int_{\gamma(j)} u \frac{\partial u}{\partial \eta} ds = \int_{\gamma(j+1)-p(j+1)^+} u_{j+1}^+ \frac{\partial u_{j+1}^+}{\partial \eta} ds - \int_{\gamma(j)-p(j)^+} u_j^+ \frac{\partial u_j^+}{\partial \eta} ds.$$

Taking limits,

$$0 \leq \lim_{j \to \infty} \sum_{i=1}^{k(j)} Q_{\Omega_{i,j}}(u,u) = \int_{\gamma_\infty} u_\infty^+ \frac{\partial u_\infty^+}{\partial \eta} ds - \int_{\gamma_\infty} u_\infty^+ \frac{\partial u_\infty^+}{\partial \eta} ds = 0.$$

In particular, $Q_{\Omega_{i,j}}(u,u)$ can be made arbitrarily small for j large and for all $i = 1, \ldots, k(j)$. Translating by $p(j)$ and taking limits on the nodal domains inside $M - p(j)^+$ that converge to the nodal domain $\Omega_\infty \subset M_\infty^+$, we obtain $Q_{\Omega_\infty}(u_\infty^+, u_\infty^+) = 0$. Using Step 1 again for M_∞^+, it follows that $M_\infty^+ \in \mathcal{R}$. Similarly we deduce $M_\infty^- \in \mathcal{R}$. Finally, that $M_\infty^+ = M_\infty^-$ comes from the fact that both surfaces are limits of translations of the same surface M, thus they share the same flux that M, and the flux characterizes the element in $\mathcal{R}$.

STEP 3. *If Step 1 holds for $M \in \mathcal{S}$ quasiperiodic, then $M \in \mathcal{R}$.*

Proof of Step 3. By Step 2, there exist points $p(j)^+, p(j)^- \in M$, $\{x_3(p(j)^\pm)\}_j \to \pm\infty$, such that $\{M - p(j)^\pm\}_j$ converges smoothly to the same Riemann example $M_\infty^+ = M_\infty^- = R \in \mathcal{R}$. As before, let $\Omega_\infty \subset R \cup \{\text{planar ends}\}$ be a nodal domain of $\{|g_\infty| = 1\}$ and let $\gamma_\infty \subset \partial\Omega_\infty$ be one of the boundary curves of Ω_∞. As $M_\infty^+ = M_\infty^-$, we can find curves $\gamma(j), \Gamma(j) \subset M$ such that both sequences $\{\gamma(j) - p(j)^+\}_j$ and $\{\Gamma(j) - p(j)^-\}_j$ converge smoothly to γ_∞. Denote by $\Omega'(j) \subset M$ the domain bounded by $\gamma(j) \cup \Gamma(j)$, see Figure 6.

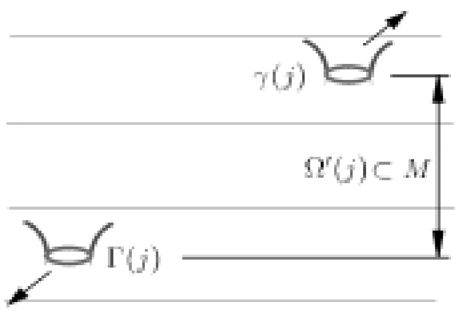

Figure 6. The domain $\Omega'(j) \subset M$.

Again, we can choose the curves $\gamma(j), \Gamma(j)$ so that $\Omega'(j)$ is a finite union of nodal domains of $\{|g| = 1\}$. Reasoning as in the proof of Step 2, we have

$$0 \leq Q_{\Omega'(j)}(u,u) = \int_{\partial\Omega'(j)} u \frac{\partial u}{\partial \eta} ds = \int_{\gamma(j)} u \frac{\partial u}{\partial \eta} ds - \int_{\Gamma(j)} u \frac{\partial u}{\partial \eta} ds.$$

But the first integral in the right-hand-side is

$$\int_{\gamma(j)} u \frac{\partial u}{\partial \eta} ds = \int_{\gamma(j)-p(j)^+} u_j^+ \frac{\partial u_j^+}{\partial \eta} ds \overset{(j\to\infty)}{\longrightarrow} \int_{\gamma_\infty} u_\infty^+ \frac{\partial u_\infty^+}{\partial \eta} ds,$$

which is zero because u_∞^+ is identically zero ($M_\infty^+ \in \mathcal{R}$). Analogously, $\lim_{j\to\infty} \int_{\Gamma(j)} u \frac{\partial u}{\partial \eta} ds = 0$, and hence $\lim_{j\to\infty} Q_{\Omega'(j)}(u,u) = 0$. Finally, consider a nodal domain $\Omega \subset \Omega'(j)$ of $\{|g| = 1\}$. By Step 1, one has $0 \leq Q_\Omega(u,u) \leq Q_{\Omega'(j)}(u,u)$. Taking limits, $Q_\Omega(u,u) = 0$ and again by Step 1, $M \in \mathcal{R}$. Now the Theorem is proved.

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