Parabolicity and minimal surfaces

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Abstract.- A crucial problem in the understanding of the theory of minimal surfaces is to determine which conformal structures are allowed under appropriate geometric conditions. Parabolicity is a powerful tool in this direction, when minimal surfaces with boundary are considered. We will give an introductory look to this classical concept in the abstract setting, revise some of the known applications to minimal surface theory (due to Collin, Kusner, Meeks and Rosenberg) and explain briefly some new statements in this field.

1 Preliminaries on parabolicity.

In this Section, M will denote a noncompact Riemannian surface with boundary $\partial M \neq \emptyset$, although all what follows is also valid in greater dimensions. Given a point $p \in M$, we can associate to p a measure μ_p on ∂M, called the *harmonic or hitting measure of M respect to p*, in two equivalent ways. The first one comes from a probability point of view:

Definition 1 *Given an interval $I \subset \partial M$, $\mu_p(I) \in [0,1]$ is the probability of a random walk that begins at p of exiting M by first time crossing at a point in I.*

For instance, when $M = \{|z| \leq 1\} - A$ with A a nonvoid closed subset of $\{|z| = 1\}$ and $p = 0$, then the harmonic measure μ_p assigns to each interval $I \subset \partial M$ the value

$$\mu_p(I) = \frac{\text{length}(I)}{\text{length}(\{|z| = 1\})} = \frac{1}{2\pi}\text{length}(I).$$

In this particular case, μ_p coincides with the Lebesgue measure divided by 2π, and $\mu_p(\partial M) = 1 - \frac{\text{length}(A)}{2\pi}$, see Figure 1.

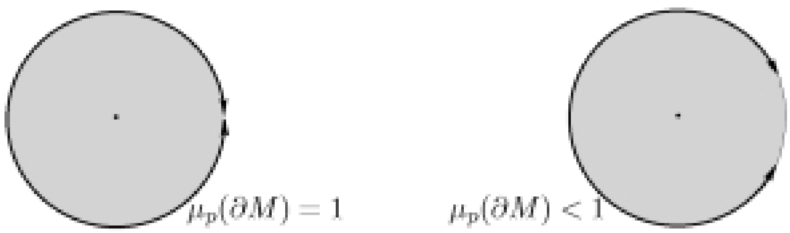

Figure 1. Left: $M = \{|z| \leq 1\} - \{1\}$. Right: $M = \{|z| \leq 1\} -$ Interval.

Note that we have even avoided to define what is a random walk. The reason for this omission is that in what follows, we will only use a second definition for harmonic measure, this one being purely analytic.

Analytic approach to the harmonic measure.

Fix an interval $I \subset \partial M$. Given a relatively compact open set $\Omega \subset M$, the noncompactness of M gives that the first eigenvalue of the Dirichlet problem for the Laplacian in Ω is positive, which implies that the following Dirichlet problem can be uniquely solved:

$$(\star)_{\Omega,I}\begin{cases} \Delta u = 0 & \text{in } \ \Omega, \\ u = 1 & \text{in } \ \partial\Omega \cap I, \\ u = 0 & \text{in } \ \partial\Omega - I. \end{cases}$$

Moreover, the maximum principle implies that $0 \leq u \leq 1$ in Ω.

Now consider an increasing sequence $\{\Omega_k \mid k \in \mathbb{N}\}$ of relatively compact open sets with $\cup_k \Omega_k = M$. Denote by u_k the unique solution of the Dirichlet problem $(\star)_{\Omega_k,I}$ for each k. As $\{u_k\}_k$ is an increasing sequence (by the maximum principle) and $0 \leq u_k \leq 1$ in Ω_k, it follows that $\{u_k\}_k$ converges on compact subsets of M to a harmonic function $h : M \to \mathbb{R}$ that satisfies $h = 1$ in I, $h = 0$ in $\partial M - I$, $0 \leq h \leq 1$ in M, see Figure 2.

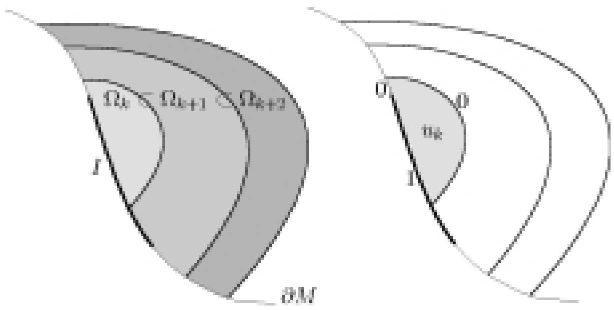

Figure 2: The sequences $\Omega_k \nearrow M$ and $u_k \nearrow h$.

We will express the fact that h is a limit of functions that vanish e (i.e. that vanish outside certain compact set, depending on k) by saying that h *vanishes at the ideal boundary of* M, although we will not discuss rigorously the notion of ideal boundary.

Next we check that the function h does not depend on the increasing sequence $\{\Omega_k\}_k$, but only on the chosen interval $I \subset \partial M$: Let $\widetilde{h}$ be another harmonic function on M with the same boundary values than h, constructed as limit of solutions $\widetilde{u}_k$ of the Dirichlet problem $(\star)_{\widetilde{\Omega}_k,I}$ for another increasing sequence $\{\widetilde{\Omega}_k\}_k \nearrow M$ of relatively compact open sets. As $\widetilde{h} \geq u_k$ in $\partial\Omega_k$, the maximum principle gives the same inequality in Ω_k. This is true for all k, hence taking limits we get $\widetilde{h} \geq h$ in M. The inverse inequality can be deduced analogously, thus equality holds.

Thus, for each interval $I \subset \partial M$ there exists a *unique* harmonic function h_I on M with boundary values $h_I = 1$ in I, $h_I = 0$ in $\partial M - I$, vanishing at the ideal boundary of M.

Definition 2 *In the above setting, the harmonic measure μ_p respect to the point $p \in M$ assigns to I the value $\mu_p(I) = h_I(p) \in [0, 1]$. Now μ_p extends to any (Lebesgue) measurable set $A \subset \partial M$ by the standard procedure.*

Remark 1

- The above uniqueness for h_I does not extend to (bounded) solutions of the following Dirichlet problem
$$(\star)_I \begin{cases} \Delta u = 0 & \text{in } M, \\ u = 1 & \text{in } I, \\ u = 0 & \text{in } \partial M - I, \end{cases}$$
as demonstrates the counterexample of a closed disk minus a nontrivial closed interval: we can fill the ideal boundary with arbitrary data and solve uniquely the corresponding Dirichlet problem in the whole closed disk. With this in mind, h_I is the unique solution when we impose the data at the ideal boundary to be zero.

- When $p \in \partial M$, the definition clearly implies $\mu_p(I) = 1$ if $p \in I$ and $\mu_p(I) = 0$ if $p \in \partial M - I$, that is, $\mu_p(I) = \chi_I(p)$, where χ_I is the characteristic function of I.

- If $I_1 \subseteq I_2 \subseteq \partial M$, then $\mu_p(I_1) \leq \mu_p(I_2)$: this follows from the maximum principle for harmonic functions, applied to the solutions of $(\star)_{\Omega_k, I_1}$, $(\star)_{\Omega_k, I_2}$ for an increasing sequence $\Omega_k \nearrow M$ as before.

- Suppose that there exists an interior point $p \in \text{Int}(M)$ and a measurable subset $I \subset \partial M$ such that $\mu_p(I) = 1$. Then, $\mu_q(I) = 1$ and $\mu_q(\partial M - I) = 0$ for all $q \in M$. To see this, note that $1 = \mu_p(I) = h_I(p)$, hence the harmonic function h_I attains an interior maximum at p, thus it is constant one. Thus, $\mu_q(I) = h_I(q) = 1$ for any $q \in M$. Now the point follows immediately.

- As harmonicity is independent of the Riemannian metric inside a given conformal class, we deduce that the harmonic measure μ_p does not depend on the metric on M but only on its conformal class. In other words, μ_p carries information of the conformal structure of M, not of its metric character. This observation also applies to all what follows.

Definition 3 *The surface M is said to be parabolic if there exists $p \in Int(M)$ such that the harmonic measure μ_p is full, i.e. $\mu_p(\partial M) = 1$.*

By the fourth point in Remark 1, if μ_p is full for some $p \in \text{Int}(M)$ then μ_q is also full for any $q \in M$, so the above definition is independent of p. Equivalently, M is parabolic if and only if the unique bounded solution $h_{\partial M}$ of the Dirichlet problem $(\star)_{\partial M}$ that vanishes at the ideal boundary of M is constant one.

Remark 2

- If the constant one is the unique bounded solution of $(\star)_{\partial M}$ (without any further conditions at the ideal boundary), then M is clearly parabolic. The converse is also true: suppose that M is parabolic and h is a bounded harmonic function on M with $h = 1$ in ∂M. We claim that h is constant one. To see this, take $a > 0$ such that $|h| \leq a$ on M and let $\Omega_k \nearrow M$ be an increasing sequence of relatively compact open sets. The existence and uniqueness of the solution u_k of the Dirichlet problem $(\star)_{\Omega_k, \partial M}$ gives that the problem

$$\begin{cases} \Delta w_k = 0 & \text{in } \ \Omega_k, \\ w_k = 1 & \text{in } \ \partial\Omega_k \cap \partial M, \\ w_k = a & \text{in } \ \partial\Omega_k - \partial M. \end{cases}$$

  has $w_k = f \circ u_k$ as unique solution, where $f : \mathbb{R} \to \mathbb{R}$ is the affine function $f(t) = (1 - a)t + a$. As M is parabolic, $\{u_k\}_k$ must converge to the constant one, hence $\{w_k\}_k$ converges to the constant $f(1) = 1$. But the maximum principle implies $h \leq w_k$ in Ω_k, so after taking limits we have $h \leq 1$. A similar argument shows that $h \geq 1$ (use $-a$ instead of a), thus h must be constant one. Therefore,

  > *M is parabolic if and only if the unique bounded solution of the Dirichlet problem $(\star)_{\partial M}$ is constant one.*

  This fact can be easily generalized to the uniqueness of bounded solutions of any Dirichlet problem on M, no matter what (bounded) boundary values we impose. This is commonly expressed as

  > *M is parabolic if and only if bounded harmonic functions on M are determined by their boundary values.*

- On a parabolic surface M, any bounded harmonic function f satisfies the *mean value property*

$$f(p) = \int_{x \in \partial M} f(x)\, \mu_p, \qquad \text{for any } p \in M.$$

  Sketch of proof: Both sides of the formula are bounded harmonic functions in the variable $p \in M$ (for the right-hand-side, one has to compute the laplacian respect to p under the integral sign, and use the expression of μ_p in terms of the value at p of certain harmonic functions), and they coincide along ∂M (use that when $p \in \partial M$, $\mu_p(I)$ coincides with the value at p of the characteristic function of I for all $I \subset \partial M$). As the surface M is parabolic, bounded harmonic functions are determined by their boundary values, so both sides must coincide. The converse is obvious (apply the mean value property to the constant one), thus

> *M is parabolic if and only if the mean value property is true for any bounded harmonic function.*

- Let $h : M \to [0,1]$ be the (unique) solution of the Dirichlet problem $(\star)_{\partial M}$ that vanishes at the ideal boundary of M.

  > *If M is not parabolic, then $\inf_M h = 0$.*

  Proof: By contradiction, suppose that $\varepsilon \leq h$ in M for some $\varepsilon > 0$. Consider the affine function $f(t) = \frac{t-\varepsilon}{1-\varepsilon}$, whose unique fixed point is $f(1) = 1$. Then, the function $h_1 = f \circ h$ is bounded and harmonic in M, has boundary values $h_1 = 1$ in ∂M. Furthermore, h_1 is limit of the sequence $\{w_k = f \circ u_k\}_k$, where each u_k is the solution of the Dirichlet problem $(\star)_{\Omega_k, \partial M}$ for an increasing sequence of relatively compact open subsets $\Omega_k \nearrow M$. We claim that $h_1 = h$ in M: the boundary conditions of w_k are $w_k = 1$ in $\partial\Omega_k \cap \partial M$, $w_k = -\frac{\varepsilon}{1-\varepsilon}$ in $\partial\Omega_k - \partial M$. Thus, $w_k \leq u_k$ in $\partial\Omega_k$, so $w_k \leq u_k$ in Ω_k by the maximum principle, and taking limits, $h_1 \leq h$ in M. Conversely, $h \geq \varepsilon$ in M implies $h_1 \geq 0$ in M, from where $h_1 \geq u_k$ in $\partial\Omega_k$. Again the maximum principle gives $h_1 \geq u_k$ in Ω_k, thus taking limits $h_1 \geq h$ in M, thereby proving the claim. Finally, as $h = h_1 = f(h)$ in M and 1 is the unique fixed point of f we deduce that $h = 1$ in M, which is impossible because M is not parabolic.

- *Suppose that M is parabolic, and $\Omega \subset M$ is a noncompact proper subdomain. Then, Ω is parabolic.*

  Proof: It suffices to see that the unique harmonic function $h : \Omega \to [0,1]$ with boundary values $h = 1$ in $\partial\Omega$ and that vanishes at the ideal boundary of Ω is the constant one. Take an increasing sequence of relatively compact open sets $\Omega_k \nearrow M$ and let u_k be the corresponding solutions of the problem $(\star)_{\Omega_k, \partial M}$. Comparing u_k with h it holds $u_k \leq h$ in $\partial(\Omega \cap \Omega_k)$, so again the maximum principle gives $u_k \leq h$ in $\Omega \cap \Omega_k$. Taking limits, $\lim_k u_k \leq h$ in Ω. But $\lim_k u_k$ is constant one because M is parabolic, thus $h \geq 1$ which forces h to be one.

- *Parabolicity is not affected by removing or adding compact subsets.*

  Proof: If M is parabolic and $K \subset M$ is compact, then $M - K$ is parabolic by the last point. Conversely, suppose that $M - K$ is parabolic for some $K \subset M$ compact, and let us see that M is also parabolic: On the contrary, assume that M is not parabolic. Thus, the solution h of the Dirichlet problem $(\star)_{\partial M}$ that vanishes at the ideal boundary of M is not constant one. As $0 < h \leq 1$ in M and K is compact, it holds $0 < \varepsilon \leq h|_K \leq 1$ for certain $\varepsilon > 0$. Consider an increasing sequence of relatively compact subsets $\Omega_k \nearrow M$. We can suppose that $K \subset \Omega_k$ for all k. As $M - K$ is parabolic, the sequence $\{u_k\}_k$ of solutions of the Dirichlet problem

$$\begin{cases} \Delta u_k = 0 & \text{in } \Omega_k - K, \\ u_k = 1 & \text{in } \partial(\Omega_k - K) \cap \partial(M - K), \\ u_k = 0 & \text{in } \partial(\Omega_k - K) - \partial(M - K) \end{cases}$$

converges to the constant one in $M - K$. But $\frac{h}{\varepsilon} \geq u_k$ in $\partial(\Omega_k - K)$, thus $\frac{h}{\varepsilon} \geq u_k$ in $\Omega_k - K$ by the maximum principle. Taking limits, $\frac{h}{\varepsilon} \geq 1$ in $M - K$, which contradicts that $\inf_M h = 0$, see the third point in this Remark.

- *If $M = M_1 \cup M_2$ with M_1, M_2 both noncompact and parabolic and $M_1 \cap M_2$ is compact, then M is parabolic.*

  Proof: By contradiction, suppose that M is not parabolic. Repeating the argument in the last point with $M_1 \cap M_2$ instead of K, we deduce that the solution h of the Dirichlet problem $(\star)_{\partial M}$ that vanishes at the ideal boundary of M satisfies: h is not constant one and $0 < \varepsilon \leq h|_{M_1 \cap M_2} \leq 1$ for certain positive ε. On the other hand, $M_1 - (M_1 \cap M_2)$ is a noncompact proper subdomain of the parabolic surface M_1, thus $M_1 - (M_1 \cap M_2)$ is parabolic. Again from the arguments in the proof of the last point (exchange $M - K$ by $M_1 - (M_1 \cap M_2)$), we arrive to $\frac{h}{\varepsilon} \leq 1$ in $M_1 - (M_1 \cap M_2)$. Analogously, $\frac{h}{\varepsilon} \leq 1$ in $M_2 - (M_1 \cap M_2)$ thus $h \geq \varepsilon$ in M, a contradiction.

- *If M carries a proper C^2 function $h : M \to \mathbb{R}$ for which there exists a compact subset $K \subset M$ such that $h > 0$ and $\Delta h \leq 0$ in $M - K$, then M is parabolic.*

  Proof: It suffices to check that if u is a bounded harmonic function on M with $u = 0$ in ∂M, then u is constant zero. If $u|_{M-K}$ were identically zero, then u would also vanish identically in K by the usual maximum principle. Then, suppose that there exists $p \in M - K$ such that $u(p) \neq 0$. Choose $a \in \mathbb{R}$ such that $au(p) > h(p)$, and cnsider the superharmonic function $H = h - au : M - K \to \mathbb{R}$. As h is proper and positive in $M - K$ and u is bounded, H must be also proper in $M - K$ and positive outside a compact subset. But $H(p) < 0$, thus H reaches an interior minimum, a contradiction. As consequence, u must be identically zero.

2 Some known results of parabolicity for minimal surfaces with boundary.

Let $X : M \to \mathbb{R}^3$ be a complete minimal immersion with compact boundary. If X has finite total curvature, then it has finite topology and each end is conformally a punctured disk. Such surface is always parabolic, so we can see the parabolicity as a generalization of having finite total curvature.

The following theorem gives a geometric situation where parabolicity holds. Recently, it has been successfully applied in different settings to control the conformal structure of proper minimal surfaces without boundary [6, 7]. Moreover, the nature of its proof has been a source of inspiration for the statements we will explain later on.

Theorem 1 (Collin, Kusner, Meeks, Rosenberg [2]) *Let $X : M \to \mathbb{R}^3$ be a proper minimal immersion of a surface M with nonempty boundary. If $X(M)$ is contained in a closed halfspace, then M is parabolic.*

Sketch of Proof. After trivial considerations, $X(M)$ can be supposed nonflat and contained in $\{x_3 \geq 0\}$. The function x_3 is harmonic on M, so it has a locally well-defined harmonic conjugate x_3^*. Outside the discrete set of points with vertical normal vector, $z = x_3 + ix_3^*$ is a local conformal coordinate for M, thus the pullback of the flat metric $|dz|^2$ through z is conformal to the induced metric ds^2 by the immersion. We label ∇, Δ as the gradient and laplacian operators respect to $|dz|^2$ and $r = \sqrt{x_1^2 + x_2^2}$ as the horizontal distance to the x_3-axis. Using that the Weierstrass data of X respect to z is $(g(z), \phi = dz)$, one gets $\Delta \log r = r^{-2}\text{Real}\left(\frac{g}{\bar{g}}e^{-2\theta i}\right)$ in $M - \{r = 0\}$, where $\tan\theta = \frac{x_2}{x_1}$. In particular, $|\Delta \log r| \leq \frac{1}{r^2}$ where $r \neq 0$.

Now consider the domain $M_k = x_3^{-1}([0, k])$. As $M_k \cap r^{-1}([0, 1])$ is compact, to prove that M_k is parabolic it suffices to check that $M_k \cap r^{-1}([1, +\infty))$ is parabolic. This is proved by finding a proper eventually (i.e. outside a compact set) positive superharmonic function on $M_k \cap r^{-1}([1, +\infty))$, namely $h = \log r - x_3^2$. That h is proper and eventually positive follows because $\log r$ is proper, positive and x_3 is bounded in $M_k \cap r^{-1}([1, +\infty))$, while that h is eventually superharmonic comes from the estimate

$$\Delta h = \Delta \log r - \Delta(x_3^2) \leq |\Delta \log r| - 2\|\nabla x_3\|^2 \leq \frac{1}{r^2} - 2.$$

Once parabolicity holds for M_k, the argument finishes as follows: As x_3 is bounded and harmonic on M_k, the mean value property says that for a fixed point $p \in M$ at height one,

$$1 = x_3(p) = \int_{\partial M_k} x_3 \mu_p^k = \int_{\partial M_k \cap \{0 \leq x_3 < k\}} x_3\, \mu_p^k + \int_{\partial M_k \cap \{x_3 = k\}} k\, \mu_p^k \geq k \int_{\partial M_k \cap \{x_3 = k\}} \mu_p^k,$$

where μ_p^k stands for the harmonic measure of M_k respect to p. Thus, $\int_{\partial M_k \cap \{x_3=k\}} \mu_p^k \leq \frac{1}{k}$. Again by parabolicity of M_k,

$$1 = \int_{\partial M_k} \mu_p^k = \int_{\partial M_k \cap \{0 \leq x_3 < k\}} \mu_p^k + \int_{\partial M_k \cap \{x_3=k\}} \mu_p^k \leq \int_{\partial M_k \cap \{0 \leq x_3 < k\}} \mu_p^k + \frac{1}{k}.$$

Taking limits, one gets $1 \leq \int_{\partial M} \mu_p$ thus the harmonic measure μ_p in M is full. □

Further refinements of the above Theorem plus arguments firstly used in Xavier [8], give raise to statements as

Theorem 2 (Meeks, Rosenberg [6]) *Let $X : M \to \mathbb{R}^3$ be a proper minimal immersion of a surface M with finite topology and compact boundary. If there exists a plane $\Pi \subset \mathbb{R}^3$ such that $X^{-1}(\Pi)$ has a finite number of components and a finite number of crosses, then any end of M is conformally a punctured disk, and M is conformally a compact Riemann surface minus a finite number of points and a finite number of disks.*

Other conditions that imply the existence of a finite plane, that is, a plane $\Pi \subset \mathbb{R}^3$ in the hypotheses of Theorem 2, are contained in the following

Theorem 3 (López [4]) *Let $X : M \to \mathbb{R}^3$ be a proper minimal immersion of a surface M with finite topology and compact boundary. If $X(M)$ misses a subset $F \subset \mathbb{R}^3$ of one of the three types described below, then M is conformally a compact Riemann surface minus a finite number of points and a finite number of disks.*

1. *F is the disjoint union of three closed vertical halfplanes whose convex hull is $\mathbb{R}^3$.*

2. *F is the disjoint union of three closed vertical halfplanes, two of them being coplanar.*

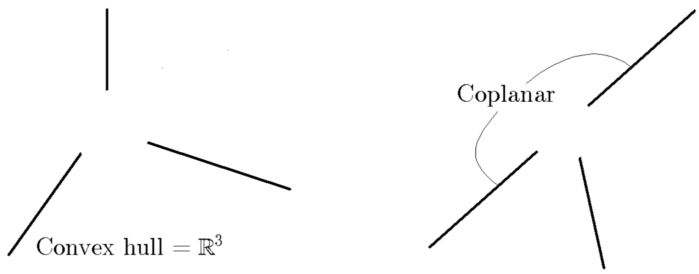

Figure 3: Cases 1 (left) and 2 (right) of Theorem 3.

3. *Take three parallel planes in $\mathbb{R}^3$, draw a city map on one of the three planes and translate it orthogonally into the remaining planes. Then, we define F as the union of the complements of the city maps in the three planes[1]. Moreover, it must be imposed that the streets are narrow enough in terms of the distance between the planes, see Figure 4.*

[1] In other words, we impose the surface to meet these three parallel planes along the city maps.

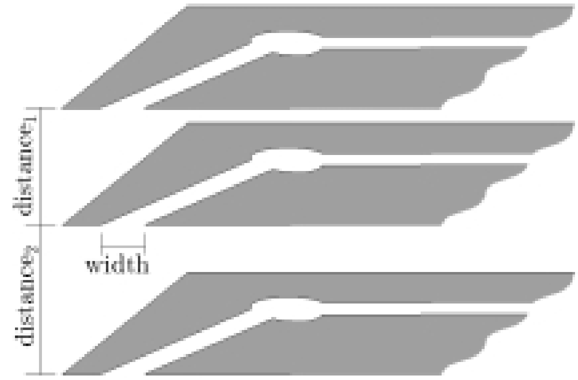

Figure 4: Case 3 of Theorem 3. The ratio $\frac{\text{width}}{\text{distance}_i}$ must be small.

3 Minimal surfaces over a negative sublinear graph.

In this Section we will explain a new result of similar nature as Theorem 1, valid in a strictly larger region than a halfspace, although an additional assumption on the Gauss map is needed. For details, see the paper by the author and López [5]. As before, call $r = \sqrt{x_1^2 + x_2^2}$ to the distance from any point in space to the x_3-axis. Our aim is to conclude parabolicity for certain minimal surfaces properly immersed in the region $W_\alpha = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > -r^\alpha\}$, where $\alpha \in (0, 1)$ (see Figure 5).

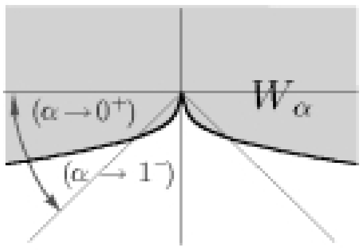

Figure 5: The region W_α, $0 < \alpha < 1$.

With this notation, our main statement says

Theorem 4 *Let $X : M \to \mathbb{R}^3$ be a proper nonflat minimal immersion such that $X(M) \subset W_\alpha$ for some $\alpha \in (0, 1)$. If, up to removing a compact subset of M, the Gauss map of the surface has image set contained in a hyperbolic open subset of the sphere, then M is parabolic.*

Although the term *hyperbolic* mentioned above is classic, we give here a brief explanation for the sake of completeness. Given a Riemann surface Σ without boundary and $q \in \Sigma$, a function G_q is called a *Green's function in Σ with singularity at q* provided that:

i) G_q is positive and harmonic in $\Sigma - \{q\}$, *ii)* If z is a local conformal coordinate centered at q, then $G_q(z) + \log |z|$ is harmonic in a neighborhood of q (this condition is independent on the conformal coordinate), and

iii) G_q is the smallest function satisfying *i),ii)*.

If G_q exists for one such point q, then the Green's function G_x with singularity at x also exists for any $x \in \Sigma$, see [3]. Two consequences of the minimality in *iii)* are that if such a function G_q exists then it is unique, and that if Σ is an open subset of a larger Riemann surface Σ' and G_q exists for Σ, then $G_q|_{\partial\Sigma} = 0$ provided that $\partial\Sigma$ is smooth enough.

The existence of Green's functions is classically related with the classification of Riemann surfaces without boundary: the Green's function exists (with singularity at any point of Σ) if and only if the surface is *hyperbolic*, in the sense that Σ carries a positive nonconstant superharmonic function. In the case $\partial\Sigma \neq \emptyset$, the classification of Riemann surfaces is more involved, see for instance [1]. The reader should be aware with the dichotomy between hyperbolicity and parabolicity in our setting, as parabolicity has been only defined for surfaces with *nonempty boundary*. In this nonempty boundary case, we will say that a Riemann surface is hyperbolic when it is not parabolic.

When $\Sigma = \{|z| < 1\} \subset \mathbb{C}$, the Green's function with singularity at $q = 0$ is $G_0(z) = -\log |z|$. After obvious transformations, this gives the Green's function in any round open disk contained in the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. On the other hand, if G_q is the Green's function in a surface Σ for a point $q \in \Sigma$ and $\Omega \subset \Sigma$ is a relatively compact open set with $q \in \Omega$, then the Green's function G_q^Ω in Ω for the point q is $G_q^\Omega = G_q - u$, where u is the (unique) solution of the Dirichlet problem in Ω with boundary values $u = G_q$ in $\partial\Omega$. These elementary facts show that any open nondense subset Ω of $\overline{\mathbb{C}}$ is hyperbolic. As $\overline{\mathbb{C}}$ minus a closed disk is biholomorphic to $\overline{\mathbb{C}}$ minus a nonconstant curve, this last surface is also hyperbolic. Nevertheless, $\overline{\mathbb{C}}$ minus any finite number of points is not hyperbolic.

Before sketching the proof of Theorem 4, we state here a direct consequence of it.

Corollary 1 *Let $M \subset \mathbb{R}^3$ be a proper minimal graph defined on a domain $D \subset \mathbb{R}^2$. If $M \subset W_\alpha$ for some $\alpha \in (0,1)$, then M is parabolic. In particular, any proper minimal graph lying above a halfcatenoid $\{x_3 = -a \cosh \sqrt{x_1^2 + x_2^2}\}$, $a > 0$, is parabolic.*

Outline of proof of Theorem 4. Let $R = \sqrt{x_1^2 + x_2^2 + x_3^2}$ be the norm of X, a positive proper C^∞-function in $M - X^{-1}(\{0\})$. As before, $z = x_3 + i x_3^*$ is a local conformal coordinate for M around points where the gradient of x_3 does not vanish, where x_3^* stands for a (locally well-defined) harmonic conjugate function x_3^*. Let $|dz|^2, \nabla, \Delta$ be respectively the pullback metric through z of the flat metric, the gradient and the laplacian respect to $|dz|^2$. The Weierstrass representation of X in terms of z gives the following estimates for $\|\nabla R\|, |\Delta R|$ (this length being also computed respect to $|dz|^2$):

$$\|\nabla R\|^2 \leq \frac{1}{2} \left(|g|^2 + |g|^{-2} + 2 \right), \qquad |\Delta R| \leq \frac{1}{R} \left(|g|^2 + |g|^{-2} + 2 \right), \tag{1}$$

where g is the Gauss map of X. We divide the proof of Theorem 4 in four steps.

STEP 1. The Theorem holds when g eventually omits neighborhoods of $0, \infty \in \overline{\mathbb{C}}$.

To see this, define $h = R^a + f(x_3) + cx_3 \in C^2(M - \{R^{-1}(0)\})$, with $a, c > 0$ and f a real C^2 function to be precised later on. This Step 1 will we done by choosing suitable a, c, f so that h is eventually positive, proper and superharmonic on M. We fix $a \in (\alpha, 1)$ and $b \in (1, 2 - a)$. Define $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(u) = -\frac{1}{(u^2+1)^{b/2}}$. This is a smooth strictly negative even function on the real line with a global minimum at $\varphi(0) = -1$ and

$$\lim_{|u|\to\infty} |u|^b \varphi(u) = -1. \tag{2}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be the only smooth even function satisfying $f''(u) = \varphi(u)$, $f(0) = f'(0) = 0$. f has a global maximum at $f(0) = 0$, is strictly decreasing on the positive numbers and

$$\lim_{|u|\to\infty} \frac{f(u)}{|u|} = -C(b),$$

$C(b)$ being a positive constant depending on b (the exact value of b does not matter for our purposes; we can see the functions φ and f in Figure 6 below).

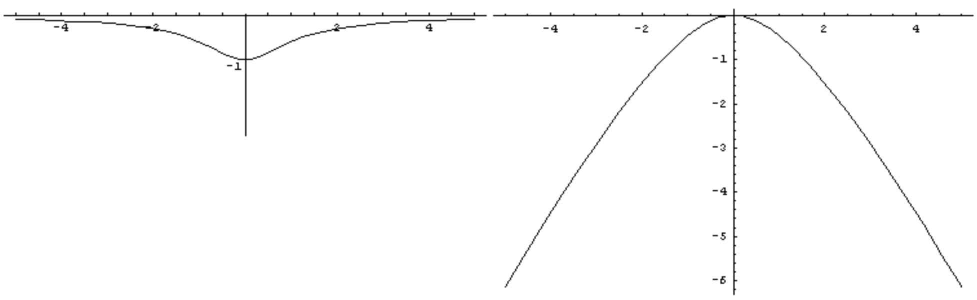

Figure 6: The functions φ (left) and f (right) for $b = 1.5$.

Note that neither f nor $C(b)$ depend of the omitted neighborhoods of the vertical directions by g, but only on the region W_α (b depends on a, and a depends only on α).

As $h = R^a + f(x_3) + cx_3$ and $f(x_3)$ diverges linearly to $-\infty$ as $|x_3| \to \infty$, we can choose $c > 0$ depending only on b such that $f(x_3) + cx_3 > 0$ for all $x_3 > 0$ large. In particular, h diverges to $+\infty$ when $x_3 \to +\infty$.

When $x_3 < 0$ is large in absolute value, we have $h > R^a - C_1|x_3|$ for a suitable constant $C_1 > 0$ that only depends on b. The hypothesis $X(M) \subset W_\alpha$ implies $|x_3|^{2/\alpha} < x_1^2 + x_2^2$, thus

$$h > (x_1^2 + x_2^2 + x_3^2)^{a/2} - C_1|x_3| > \left(|x_3|^{2/\alpha} + x_3^2\right)^{a/2} - C_1|x_3|.$$

As $\frac{2}{\alpha} > 2$ and $a > \alpha$, the function appearing in the last right-hand-side is positive for $|x_3|$ large enough (depending only on α, a, b) and proper as function of $|x_3|$.

Finally, in a region of the type $M \cap x_3^{-1}([k,k])$, h is eventually positive because $f(x_3) + cx_3$ is bounded and R^a is proper and positive. From the above arguments, we conclude that h is eventually positive and proper.

Concerning the $|dz|^2$-laplacian of h, firstly note that both $\|\nabla R\|$ and $R|\Delta R|$ are bounded by a positive constant (depending on the omitted neighborhoods of the vertical directions). Therefore, (1) allows us to write

$$\Delta h = \Delta(R^a) + f'' \leq |\Delta(R^a)| + \varphi \leq \frac{C}{R^{2-a}} + \varphi \leq \frac{C}{|x_3|^{2-a}} + \varphi \approx \frac{C}{|x_3|^{2-a}} - \frac{1}{|x_3|^b} \qquad (3)$$

where we have also used (2) in the last approximation, this last one being valid for $|x_3|$ large. Note that C only depends of the missing neighborhoods by the Gauss map. As $b < 2 - a$, the right-hand-side of (3) is negative for $|x_3|$ large enough, which gives $\Delta h \leq 0$ in $M - x_3^{-1}([-k,k])$ for a certain $k > 0$ (depending on the omitted neighborhoods by g). In the set $M \cap x_3^{-1}([k,k])$, the inequality $\Delta h \leq \frac{C}{R^{2-a}} + \varphi$ together with the facts that R^{2-a} is proper and φ is strictly negative, imply that Δh is eventually negative. This shows that h is proper, eventually positive and superharmonic, which finishes the proof of Step 1.

STEP 2. The Theorem holds when g eventually omits a neighborhood U of 0 in $\overline{\mathbb{C}}$.

As being parabolic is an hereditary property, we can reduce to prove this second step for a sufficiently small open neighborhood of zero. Thus $U \subset \overline{\mathbb{C}}$ can be assumed to be open, containing to zero, with $\infty \notin \overline{U}$ and $g(M) \cap \overline{U} = \emptyset$. As $\overline{\mathbb{C}} - \overline{U}$ is open and nondense, it must be hyperbolic and so, the Green's function G in $\overline{\mathbb{C}} - \overline{U}$ with singularity at ∞ exists. Given $k \in \mathbb{N}$, consider the surface $M_k = (G \circ g)^{-1}([0,k]) \subset M$, whose boundary is the disjoint union $\partial M_k = I_k \cup \{G \circ g = k\}$, where $I_k = \partial M \cap \{0 \leq G \circ g < k\}]$, see Figure 7.

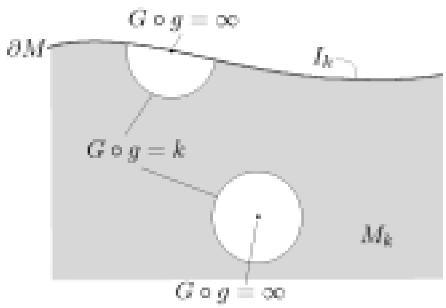

Figure 7: The shaded surface is M_k.

$X|_{M_k}$ is a proper nonflat minimal immersion with boundary, whose Gauss map omits neighborhoods of zero and infinity, and whose image in $\mathbb{R}^3$ is contained in W_α. By Step 1,

M_k is parabolic and this holds for each k. Fix a point $p \in M$ such that $g(p) \neq \infty$. For all k sufficiently large, p is an interior point of M_k and as $G \circ g$ is a bounded harmonic function on M_k, it must satisfy the mean value property

$$(G \circ g)(p) = \int_{I_k} (G \circ g)\mu_p^k + \int_{\{G \circ g = k\}} (G \circ g)\mu_p^k = \int_{\partial M \cap \{0 \leq G \circ g < k\}} (G \circ g)\mu_p^k + k \int_{\{G \circ g = k\}} \mu_p^k,$$

where μ_p^k stands for the harmonic measure of M_k respect to p. The first integral in the last right-hand-side is nonnegative, thus

$$0 \leq \int_{\{G \circ g = k\}} \mu_p^k \leq \frac{1}{k}(G \circ g)(p) \overset{(k \to \infty)}{\longrightarrow} 0. \tag{4}$$

Since μ_p^k is full on ∂M_k,

$$1 = \int_{\partial M_k} \mu_p^k = \int_{I_k} \mu_p^k + \int_{\{G \circ g = k\}} \mu_p^k,$$

which together with (4) imply that $\int_{I_k} \mu_p^k \to 1$ as k tends to ∞. But this last integral can be computed as $\int_{I_k} \mu_p^k = \mu_p^k(I_k) = h_{I_k}(p)$, where h_{I_k} is the (unique) bounded solution of the Dirichlet problem

$$(\star)_{I_k} \begin{cases} \Delta h_{I_k} = 0 & \text{in } M_k, \\ h_{I_k} = 1 & \text{in } I_k, \\ h_{I_k} = 0 & \text{in } \{G \circ g = k\}. \end{cases}$$

vanishing at the ideal boundary. To finish this second step, we only need to check that the (unique) bounded harmonic function h in M with boundary values $h = 1$ in ∂M and that vanishes at the ideal boundary of M is constant one. To see this, recall that such a function satisfies $0 \leq h \leq 1$ in M (because h is constructed as limit of solutions u_k of Dirichlet problems in relatively compact open subsets $\Omega_k \nearrow M$ and $0 \leq u_k \leq 1$ in Ω_k for each k, see Section 1). This inequality for h implies that for $k \in \mathbb{N}$ fixed, it holds $h_{I_k} \leq h$ in ∂M_k. As M_k is parabolic, this implies $h_{I_k} \leq h$ in M_k. Evaluating at p and taking limits as $k \to \infty$, we have $1 = \lim_k h_{I_k}(p) \leq h(p) \leq 1$. Thus h attains an interior maximum at p with value one, so it must be constant one. This finishes Step 2.

STEP 3. The Theorem is valid when $g(M)$ omits an open subset $\emptyset \neq U \subset \overline{\mathbb{C}}$.

If zero is a point of U, then the statement follows directly from Step 2. Thus, we can assume that $0 \notin U$. Again using that parabolicity inherits to subsets, we can assume $0 \notin \overline{U}$ and $g(M) \cap \overline{U} = \emptyset$. As $\overline{\mathbb{C}} - \overline{U}$ is open and nondense, it must be hyperbolic. As $0 \in \overline{\mathbb{C}} - \overline{U}$, the Green's function G in $\overline{\mathbb{C}} - \overline{U}$ with singularity at 0 exists. Consider the surface $M_k = (G \circ g)^{-1}([0, k]) \subset M$. As $X(M_k) \subset W_\alpha$ and the Gauss map restricted to

M_k omits a neighborhood of zero in $\overline{\mathbb{C}}$, Step 2 guarantees that M_k is parabolic. From this point, the proof of Step 2 can be repeated without changes.

Now we can prove the Theorem. Up to removing a compact subset of M, we can assume that $g(M)$ is contained in an open hyperbolic subset $\Omega \subset \overline{\mathbb{C}}$. Fix a point $q \in \Omega$ and let G be the Green's function in Ω with singularity at q. Given $k \in \mathbb{N}$, define $U(k) = G^{-1}((k, +\infty])$, which is an open neighborhood of q. As in previous steps, we consider the surface $M_k = (G \circ g)^{-1}([0, k]) = M - g^{-1}(U(k))$. As $X(M_k) \subset W_\alpha$ and $g|_{M_k}$ omits $U(k)$, Step 3 gives that M_k is parabolic. Following again the arguments in the proof of Step 2 and recalling that the parabolicity is not affected by adding or removing compact subsets, we conclude the proof.

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