The Toda system on compact surfaces.

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Sevilla, 16-20 September, 2013.

Outline

The Toda System

In this talk we study existence of solutions for the following version of the Toda system:

$$\begin{cases} -\Delta u_1 = 2\rho_1 \left(h_1 e^{u_1} - 1 \right) - \rho_2 \left(h_2 e^{u_2} - 1 \right) & \text{in } \Sigma, \\ -\Delta u_2 = 2\rho_2 \left(h_2 e^{u_2} - 1 \right) - \rho_1 \left(h_1 e^{u_1} - 1 \right) & \text{in } \Sigma, \end{cases}$$

where Δ is the Laplace-Beltrami operator, $\rho_i > 0$, $h_i(x) > 0$, and Σ is a compact surface with $\int_\Sigma 1 dV_g = 1$.

The Toda system arises in the study of the non-abelian Chern-Simons-Higgs model, when looking for non-topological solutions.

- G. Dunne, Self-dual Chern-Simons Theories, Lecture Notes in Physics, Springer-Verlag, 1995.
- Y. Yang, Solitons in Field Theory and Nonlinear Analysis, Springer-Verlag, 2001.
- G. Tarantello, Self-Dual Gauge Field Vortices: An Analytical Approach, PNLDE 72, Birkhäuser 2007.

By integrating on Σ both equations, we get that $\int_\Sigma h_i e^{u_i}\, dV_g = 1$. Hence, our problem is equivalent to:

$$\begin{cases} -\Delta u_1 = 2\rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right) - \rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right) & \text{in } \Sigma, \\ -\Delta u_2 = 2\rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right) - \rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right) & \text{in } \Sigma. \end{cases} \tag{1}$$

Problem (1) is the Euler-Lagrange equation of the functional:

$$J_\rho(u_1, u_2) = \int_\Sigma Q(u_1, u_2)\, dV_g + \sum_{i=1}^{2} \rho_i \left(\int_\Sigma u_i dV_g - \log \int_\Sigma h_i e^{u_i} dV_g \right).$$

Here $\rho = (\rho_1, \rho_2)$, and:

$$Q(u_1, u_2) = \frac{1}{3} \left(|\nabla u_1|^2 + |\nabla u_2|^2 + \nabla u_1 \cdot \nabla u_2 \right).$$

Moreover, the functions u_i belong to the Sobolev Space

$$H^1(\Sigma) = \{u : \Sigma \to \mathbb{R} : \; u, \; \nabla u \in L^2(\Sigma)\}, \; \|u\|^2 = \|u\|^2_{L^2} + \|\nabla u\|^2_{L^2}.$$

In dimension 2 we have the well-known Moser-Trudinger inequality:

$$\int_\Sigma e^{4\pi u^2} \, dx \leq C = C(\Sigma), \;\; \forall u \in H^1(\Sigma) \text{ with } \; \|\nabla u\|_{L^2} \leq 1, \; \int_\Sigma u = 0.$$

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The direct method of Calculus of Variations proposes to find solutions of our problem as minima of J_ρ. However, as we shall see, sometimes J_ρ is not bounded from below.

Here we will look for critical points of saddle-type by using min-max theory.

Min-max theory

Let us briefly remind Morse Theory. Given $f : \mathbb{R}^k \to \mathbb{R}$ be a C^2 function, and define the sub-level:

$$f^a = \{x \in \mathbb{R}^k : \ f(x) \leq a\}.$$

Then, if x_0 is a non-degenerate critical point of f and $f(x_0) = c$, then the sub-levels $f^{c+\theta}$, $f^{c-\theta}$ have different topology.

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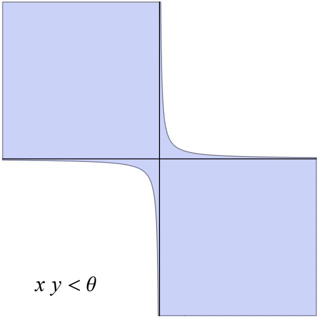

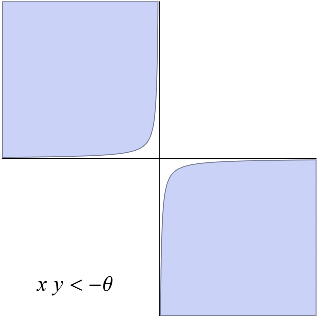

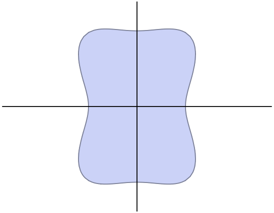

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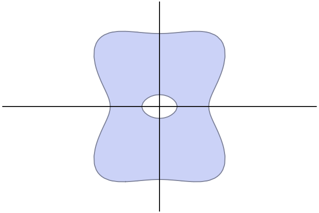

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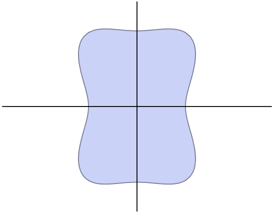

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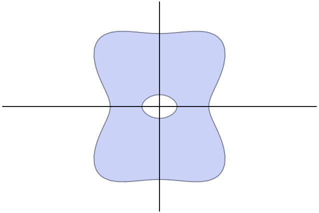

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The idea of min-max theory is to reverse the argument. Roughly speaking, if we find a change of topology of the energy sub-levels, this implies the existence of a critical point.

The min-max principle

Theorem

Let X be a Banach Space, $f : X \to \mathbb{R}$ a C^1 function. Take $a < b$, and $f^a \subset f^b$ the energy sub-levels. Then, either:

1. *f^a is a strong deformaction retract of f^b,*

or

1. *there exists $u_n \in X$, $f'(u_n) \to 0$ and $f(u_n) \to c \in [a, b]$.*

Such sequences are called Palais-Smale sequences. In some cases, one can pass to the limit and prove the existence of a critical point.

A. Bahri, A. Bahri, Critical Points at infinity in some variational problems, Pitman Research Notes in Math. Series 1989.

The scalar case

The scalar counterpart of (1) is a Liouville-type problem in the form:

$$-\Delta u = 2\rho \left(\frac{h(x)e^u}{\int_\Sigma h(x)e^u dV_g} - 1 \right) \quad \text{in } \Sigma, \tag{2}$$

with $\rho \in \mathbb{R}$. This equation has been very much studied. In particular, it appears in the prescribed gaussian curvature problem: in this case, $\rho = 2\pi\chi(\Sigma)$.

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The Euler-Lagrange functional of (2) is:

$$I_\rho(u) = \frac{1}{2}\int_\Sigma |\nabla_g u|^2 dV_g + 2\rho\left(\int_\Sigma u dV_g - \log\int_\Sigma h(x)e^u dV_g\right) \tag{3}$$

for any $u \in H^1(\Sigma)$. By the Moser-Trudinger inequality, I_ρ is bounded from below if and only if $\rho \leq 4\pi$, and is coercive if $\rho < 4\pi$.

If we make $\rho_n \to 4\pi$, the solution u_n could exhibit a blow-up behavior. In such case,

$$u_n \sim U_{\lambda,x}(y) = \log\left(\frac{4\lambda}{\left(1 + \lambda\, d(x,y)^2\right)^2}\right).$$

where $y \in \Sigma$, $d(x,y)$ stands for the geodesic distance and λ is a large parameter. Those functions $U_{\lambda,x}$ are the unique entire solutions of the Liouville equation in $\mathbb{R}^2$:

$$-\Delta U = 2e^U, \qquad \int_{\mathbb{R}^2} e^U\, dx < +\infty.$$

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It has been proved that blowing-up solutions have this behavior around a finite number of points. In particular, if u_n blows up, then $\rho_n \to 4\pi k$, $k \in \mathbb{N}$.

H. Brezis and F. Merle, 1991.

Y. Li and I. Shafrir, 1994.

Theorem

For any $\rho \notin 4\pi\mathbb{N}$ *there exists a solution of* (2).

Z. Djadli and A. Malchiodi, 2008.

The proof is based on the study of the sub-levels I_ρ^{-L} for L large. Assume that $\rho \in (4k\pi, 4(k+1)\pi)$. It can be proved that if $I_\rho(u_n) \to -\infty$, then

$$\frac{e^{u_n}}{\int_\Sigma e^{u_n}\, dV_g} \rightharpoonup \sum_{i=1}^{k} t_i \delta_{x_i}, \quad x_i \in \Sigma,\ t_i \geq 0, \sum_{i=1}^{k} t_i = 1.$$

This allows us to define a continuous map (for $L >> 1$):

$$\Psi : I_\rho^{-L} = \{u \in H^1(\Sigma) : \; I_\rho(u) < -L\} \to \Sigma_k,$$

$$\Sigma_k = \left\{ \sum_{i=1}^{k} t_i \delta_{x_i}, \quad x_i \in \Sigma, \; t_i \geq 0, \sum_{i=1}^{k} t_i = 1 \right\}.$$

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Here the weak topology of measures is considered.
Moreover, one can define a reversed map $\Phi_\lambda : \Sigma_k \to H^1(\Sigma)$,

$$\sigma = \sum_{i=1}^k t_i \delta_{x_i} \mapsto \Phi_\lambda[\sigma](y) = \log \left(\sum_{i=1}^k t_i \frac{4\lambda}{\left(1 + \lambda\, d(x_i, y)^2\right)^2} \right).$$

If λ is chosen large enough, $\Phi_\lambda[\sigma] \in I_\rho^{-L}$.

End of the proof

We have the composition:

$$\Sigma_k \xrightarrow{\Phi_\lambda} I_\rho^{-L} \xrightarrow{\Psi} \Sigma_k.$$

Moreover, as $\lambda \to +\infty$, $\Psi \circ \Phi_\lambda$ tends to the the identity map; so, λ can be used to define an homotopy between $\Psi \circ \Phi_\lambda$ and the identity map. In other words, I_ρ^{-L} covers Σ_k.

It can be shown that Σ_k is not contractible, hence $\Phi_\lambda(\Sigma_k)$ is not contractible in I_ρ^{-L}. But it is clearly contractible in I_ρ^L.

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By the min-max principle, there exists a Palais-Smale sequence. Passing from this sequence to a true critical point is not trivial, but it is a technical issue that will be skipped in this talk.

The Toda System

It is known that if both $\rho_i \leq 4\pi$, then J_ρ is bounded from below. In particular, if $\rho_i < 4\pi$, J_ρ is coercive and achieves its minimum.

J. Jost and G. Wang, 2001.

If we make now $\rho_i \to 4\pi k$, $k \in \mathbb{N}$, the solution (u_1, u_2) could exhibit a blow-up behavior. Solutions may blow-up in different ways, but their energy at the blow-up point is quantized.

J. Jost, C.-S. Lin and G. Wang, 2006.

In particular, the set of solutions is compact for $\rho_i \notin 4\pi\mathbb{N}$.

Assume $\rho_1 \in (4k\pi, 4(k+1)\pi)$, $\rho_2 \in (4l\pi, 4(l+1)\pi)$, $k, l \in \mathbb{N}$.
Reasoning as in the scalar case, if $J_\rho(u_{1,n}, u_{2,n}) \to -\infty$, then either

$$\frac{e^{u_{1,n}}}{\int_\Sigma e^{u_{1,n}}\, dV_g} \rightharpoonup \sum_{i=1}^{k} t_i \delta_{x_i}, \;\; x_i \in \Sigma,$$

or

$$\frac{e^{u_{2,n}}}{\int_\Sigma e^{u_{2,n}}\, dV_g} \rightharpoonup \sum_{j=1}^{l} t_j \delta_{y_j}, \;\; y_j \in \Sigma.$$

Hence, there are two problems:

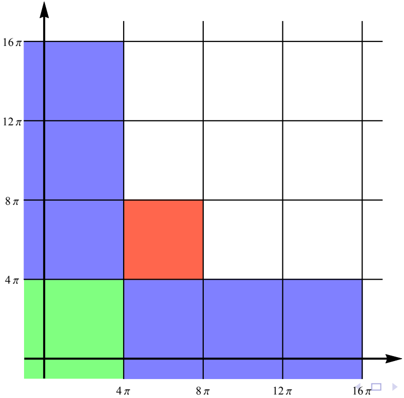
1. The alternative of the concentration phenomena.
2. The interaction between the concentration points.

Previous results

A. Malchiodi and C. B. Ndiaye, 2007.

A. Malchiodi and D. R., 2013.

In the first, $\rho_i < 4\pi$, $\rho_j \in (4k\pi, 4(k+1)\pi)$. Here the sub-levels are described via the second component, as in the scalar case. In the second, $\rho_i \in (4\pi, 8\pi)$; both components may have one peak.



In this talk, we will deal with the alternative difficulty and avoid the interaction problem by using a topological argument.

Theorem

Assume $\rho_i \notin 4\pi\mathbb{N}$, and Σ a compact surface not homeomorphic to $\mathbb{S}^2$ or $\mathbb{RP}^2$. Then J_ρ has a critical point.

L. Battaglia, A. Jevnikar, A. Malchiodi and D. R., preprint.

Dealing with the alternative: the topological join

Given any topological spaces A and B, we define the join $A \star B$ as:

$$A \star B = \{(1-r)a + r\,b; r \in [0,1],\ a \in A,\ b \in B\}.$$

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Given any topological spaces A and B, we define the join $A \star B$ as:

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For $u \in J_\rho^{-L}$, define:

$$\mathbf{d}_1 = \mathbf{d}\left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g}, \Sigma_k\right), \qquad \mathbf{d}_2 = \mathbf{d}\left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g}, \Sigma_l\right).$$

Here $\mathbf{d}$ is the Kantorovich-Rubinstein distance. Moreover, we define $r = r(\mathbf{d}_1, \mathbf{d}_2)$, such that:

$$r = \begin{cases} 0 & \text{if } \mathbf{d}_1 << \mathbf{d}_2, \\ 1 & \text{if } \mathbf{d}_1 >> \mathbf{d}_2. \end{cases} \tag{4}$$

The parameter r measures which component is closer to its respective barycenter space.

In this way, one can define a map:

$$\tilde{\Psi} : J_\rho^{-L} \to \Sigma_k \star \Sigma_l,$$
$$u = (u_1, u_2) \mapsto (1 - r)\psi_k(u_1) + r\psi_l(u_2).$$

Here ψ_k, ψ_l are the continuous maps onto Σ_k, Σ_l. Those are defined only when $\mathbf{d}_1$, $\mathbf{d}_2$ are small, respectively.

Observe that if $\mathbf{d}_1$ is not small, ψ_k is not well-defined but then $r = 1$. Analogously, if $\mathbf{d}_2$ is not small then $r = 0$; so the map is well defined.

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However, $\Sigma_k \star \Sigma_l$ does not seem to be a right space for describing the low sub-levels of the energy. The problem is that we cannot define test functions when the concentration points coincide.

Avoiding the interaction problem

Lemma

There exist two closed curves γ_i in Σ with $\gamma_1 \cap \gamma_2 = \emptyset$ and two continuous retractions $\Pi_i : \Sigma \to \gamma_i$.

Observe that this lemma is not true for $\Sigma = \mathbb{S}^2$ or $\mathbb{RP}^2$.

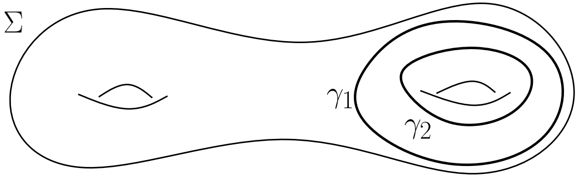

Figura: The curves γ_i

We now define the map:

$$\Psi : J_\rho^{-L} \to (\gamma_1)_k \star (\gamma_2)_l,$$

$$\Psi(u_1, u_2) = (1-r)(\Pi_1)_* \psi_k \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} \right) + r(\Pi_2)_* \psi_l \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} \right),$$

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where $(\Pi_i)_*$ stands for the push-forward of the map Π_i.

Observe that $\gamma_i \sim \mathbb{S}^1$; then, it has been proved that $(\mathbb{S}^1)_k \sim \mathbb{S}^{2k-1}$, and hence $(\gamma_1)_k \star (\gamma_2)_l \sim \mathbb{S}^{2k+2l-1}$.

The reversed map

Let $\zeta = (1-r)\sigma_2 + r\sigma_1 \in (\gamma_1)_k * (\gamma_2)_l$, where:

$$\sigma_1 := \sum_{i=1}^{k} t_i \delta_{x_i} \in (\gamma_1)_k \qquad \text{and} \qquad \sigma_2 := \sum_{j=1}^{l} s_j \delta_{y_j} \in (\gamma_2)_l.$$

Given $\lambda > 0$ large enough, we define:

$$\Phi_\lambda : (\gamma_1)_k * (\gamma_2)_l \to H^1(\Sigma) \times H^1(\Sigma), \; \Phi_\lambda(\zeta) = \varphi_{\lambda,\zeta},$$

with components given by:

$$\varphi_{\lambda,\zeta} = \begin{pmatrix} \log \sum_{i=1}^{k} t_i \left(\frac{1}{1+\lambda_{1,r}^2 d(x,x_i)^2} \right)^2 - \frac{1}{2} \log \sum_{j=1}^{l} s_j \left(\frac{1}{1+\lambda_{2,r}^2 d(x,y_j)^2} \right)^2 \\ -\frac{1}{2} \log \sum_{i=1}^{k} t_i \left(\frac{1}{1+\lambda_{1,r}^2 d(x,x_i)^2} \right)^2 + \log \sum_{j=1}^{l} s_j \left(\frac{1}{1+\lambda_{2,r}^2 d(x,y_j)^2} \right)^2 \end{pmatrix}.$$

Here

$$\lambda_{1,r} = (1-r)\lambda; \qquad \lambda_{2,r} = r\lambda.$$

Proposition

Φ_λ is well defined and for any $L > 0$ there exists $\lambda > 0$ so that

$$\Phi_\lambda((\gamma_1)_k * (\gamma_2)_l) \subset J_\rho^{-L}.$$

Moreover, the composition

$$(\gamma_1)_k * (\gamma_2)_l \quad \xrightarrow{\Phi_\lambda} \quad J_\rho^{-L} \quad \xrightarrow{\Psi} \quad (\gamma_1)_k * (\gamma_2)_l$$

is homotopically equivalent to the identity map on $(\gamma_1)_k * (\gamma_2)_l$ for large λ.

Some applications in geometry

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In the prescribed gaussian curvature problem, $\rho = 2\pi\chi(\Sigma)$. Therefore, the cases $\rho > 4\pi$ do not appear there. However, those arguments apply to some intrinsically geometric problems:

a) The prescribed Q-curvature problem. Here Σ is a 4-dimensional manifold, Δ is replaced by the Paneitz operator P, and K is substituted with the Q-curvature. If Σ is not locally conformally flat, ρ is not quantized, and can take high values.

S. Y. Chang and P. Yang, 1995.

Z. Djadli and A. Malchiodi, 2008.

b) The prescribed gaussian curvature problem with conical points. Here ρ can take values so that the corresponding Euler-Lagrange functional is not bounded from below.

M. Troyanov, 1991.

A. Malchiodi and D. R., 2011.

A. Malchiodi and A. Carlotto, 2012.

A. Carlotto, preprint.

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The problems considered here exhibit some concentration phenomena for blowing-up solutions. Moreover, also low sub-levels of the energy functional concentrate around some points.

Because of that, the topology of the energy sub-levels are affected by the topology of the surface Σ. If the sub-level is not contractible, there may exist a critical point.

Thank you for your attention!