II Congreso de Jóvenes Investigadores

RSME - Universidad de Sevilla

Multilinear Harmonic Analysis: some recent results

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Based in two joint works:

- Wei Chen, and W.D., *Weighted estimates for the multisublinear maximal function*, to appear in Rend. Circ. Mat. Palermo, 2013.
- W.D, Andrei Lerner, and Carlos Pérez, *Sharp weighted bounds for multilinear maximal functions and Calderón-Zygmund operators*, preprint, 2012.

Outline

Origin of modern theory of weights

The Hardy–Littlewood maximal function

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CLASSICAL ESTIMATES

$$M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n),\ 1 < p < \infty,$$

and

$$M : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n).$$

Origin of modern theory of weights

In 1972, Muckenhoupt characterized the class of weights (u,v) for which the following weak inequality holds

$$\sup_{\lambda>0} \lambda^p \int_{\{Mf>\lambda\}} u(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x)dx, \quad f \in L^p(v).$$

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A_p CONDITION

$$[u,v]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q u(x)dx\right)\left(\frac{1}{|Q|}\int_Q v(x)^{-\frac{1}{p-1}}\right)^{p-1} < \infty, p > 1$$

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When $u = v$, Muckenhoupt also proved that the strong estimate

$$\int_{\mathbb{R}^n} (Mf(x))^p v(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x)dx, \quad f \in L^p(v),$$

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holds if and only if v satisfies the A_p condition

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$$[v]_{A_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q v(x)dx \right) \left(\frac{1}{|Q|} \int_Q v(x)^{-\frac{1}{p-1}} \right)^{p-1} < \infty, p > 1.$$

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S_p SAWYER'S CONDITION

$$[u,v]_{S_p} = \sup_Q \left(\frac{\int_Q M(\chi_Q \sigma)^p u dx}{\sigma(Q)} \right)^{1/p} < \infty,$$

where $\sigma = v^{1-p'}$ and $1 < p < \infty$.

Finding sharp bounds

GOAL: Determining the sharp dependence of the $L^p(w)$ norm of M in term of the relevant constant involving the weights.

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BUCKLEY, 1993

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$$||M||_{L^p(w)\to L^p(w)} \leq C_n p' [w]_{A_p}^{\frac{1}{p-1}}, \quad 1 < p < \infty.$$

MOEN, 2009

$$||M||_{L^p(v)\longrightarrow L^p(u)} \approx [u,v]_{S_p}.$$

Finding sharp bounds

In 2013, Hytönen and Pérez proved:

B_p THEOREM

Let $p > 1$ and let w and σ be any weights. Then

$$||M(f\sigma)||_{L^p(w)} \leq Cp'(B_p[w,\sigma])^{1/p}||f||_{L^p(\sigma)},$$

where

$$B_p[w,\sigma] := \sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q \sigma\right)^p \exp\left(\frac{1}{|Q|}\int_Q \log \sigma^{-1}\right).$$

Finding sharp bounds

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$$[w]_{A_p} \leq B_p[w,\sigma]$$

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$$[w]_{A_p} \leq B_p[w,\sigma] \leq [w]_{A_p}[\sigma]_{A_\infty}^H$$

HRUSČĚV A_∞ CONSTANT

$$[\sigma]_{A_\infty}^H = \sup_Q \left(\frac{1}{|Q|}\int_Q \sigma\right)\exp\left(\frac{1}{|Q|}\int_Q \log \sigma^{-1}\right)$$

Finding sharp bounds

Hytönen and Pérez also improved Buckley's result.

A_p-A_∞ MIXED BOUND

Let $p > 1$ and let w and σ be any weights. Then

$$||M(f\sigma)||_{L^p(w)} \leq Cp'([w]_{A_p}[\sigma]_{A_\infty})^{1/p}||f||_{L^p(\sigma)},$$

where

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q).$$

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IMPROVEMENT OF BUCKLEY'S THEOREM

Since

$$[\sigma]_{A_\infty} \leq [\sigma]_{A_{p'}} \leq [w]_{A_p}^{\frac{1}{p-1}}.$$

then

$$([w]_{A_p}[\sigma]_{A_\infty})^{1/p} \leq ([w]_{A_p}[w]_{A_p}^{\frac{1}{p-1}})^{1/p} \leq [w]_{A_p}^{\frac{1}{p-1}}.$$

What about Calderón–Zygmund operators

Hilbert transform

$$Hf(x) = \frac{1}{\pi}\,\mathrm{pv}\int_{\mathbb{R}} \frac{1}{x-y} f(y)\,dy, \quad x \in \mathbb{R}.$$

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Definition

A linear operator T is a Calderón–Zygmund operator if:

1. $T : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$.
2. There exists a function K defined off the diagonal such that

$$T(f)(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \quad x \notin supp\,(f), f \in C_c^\infty.$$

3. K satisfies size and regularity conditions.

What about Calderón–Zygmund operators

Control by Hardy–Littlewood maximal function

Let $0 < \delta < 1$ and let T be a Calderón–Zygmund operator. Then for every $f \in L^p(\mathbb{R}^n)$

$$M^{\sharp}_{\delta}(T(f))(x) \leq CM(f)(x), \quad x \in \mathbb{R}^n.$$

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Control by Hardy–Littlewood maximal function

Let $0 < \delta < 1$ and let T be a Calderón–Zygmund operator. Then for every $f \in L^p(\mathbb{R}^n)$

$$M_\delta^\sharp(T(f))(x) \leq CM(f)(x), \quad x \in \mathbb{R}^n.$$

In 2012, Hytönen found the sharp bound for a Calderón–Zygmund operator.

Sharp bound for CZO

$$||T||_{L^p(w)} \leq C_{T,n,p}[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}, \quad 1 < p < \infty.$$

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SHARP BOUND FOR CZO

$$||T||_{L^p(w)} \leq C_{T,n,p}[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}, \quad 1 < p < \infty.$$

Later on, Lerner gave another different proof of the above result showing the relationship between the singular integral T and some special dyadic sparse type operators.

The multilinear maximal function

In [LOPTR] is defined the following maximal operator

MULTILINEAR MAXIMAL FUNCTION

Given $\vec{f} = (f_1, \ldots, f_m)$, we define the maximal operator $\mathscr{M}$ by

$$\mathscr{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{i=1}^{m} \frac{1}{|Q|} \int_Q |f_i(y_i)| dy_i,$$

- $\mathscr{M}$ is sublinear in each entry.

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- $\mathscr{M}$ is sublinear in each entry.
- $\mathscr{M}$ verifies weak and strong classical estimates.

$$\mathscr{M} : L^1(\mathbb{R}^n) \times \ldots \times L^1(\mathbb{R}^n) \longrightarrow L^{1/m,\infty}(\mathbb{R}^n)$$

$$\mathscr{M} : L^{p_1}(\mathbb{R}^n) \times \ldots \times L^{p_m}(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$$

where

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- $\mathscr{M}$ is sublinear in each entry.
- $\mathscr{M}$ verifies weak and strong classical estimates.
- If T is a multilinear CZO, for $0 < \delta < 1/m$

$$M_\delta^\sharp(T(\vec{f}))(x) \leq C\mathscr{M}(\vec{f})(x).$$

Two-weight-weak estimate for $\mathcal{M}$

Theorem (LOPTR, 2009)

Let $1 < p_j < \infty, j = 1, \ldots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Let ν and w_j be weights. Then the inequality

$$\|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(\nu)} \simeq [\nu, \vec{w}]_{A_{\vec{P}}}^{1/p} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}$$

holds for any $\vec{f}$ if and only if

$$[\nu, \vec{w}]_{A_{\vec{P}}} := \sup_Q \left(\frac{1}{|Q|}\int_Q \nu\right) \prod_{j=1}^{m} \left(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\right)^{p/p_j'} < \infty.$$

$A_{\vec{P}}$ condition

Let $\vec{P} = (p_1, \cdots, p_m)$ and let p be a number such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$.

DEFINITION

Let $1 \leq p_1, \ldots, p_m < \infty$. Given $\vec{w} = (w_1, \ldots, w_m)$, set $\nu_{\vec{w}} = \prod_{j=1}^{m} w_j^{p/p_j}$.

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$A_{\vec{P}}$ CONDITION

We say that $\vec{w}$ satisfies the $A_{\vec{P}}$ condition if

$$[\vec{w}]_{A_{\vec{P}}} := \sup_Q \left(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}\right) \prod_{j=1}^m \left(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\right)^{p/p_j'} < \infty.$$

When $p_j = 1$, $\left(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\right)^{p/p_j'}$ is understood as $(\inf_Q w_j)^{-p}$.

One-weight strong estimate for $\mathscr{M}$

THEOREM (LOPTR, 2009)

Let $1 < p_j < \infty, j = 1, \dots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Then the inequality

$$\|\mathscr{M}(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}$$

holds for every $\vec{f}$ if and only if $\vec{w}$ satisfies the $A_{\vec{P}}$ condition.

Sharp estimates for $\mathscr{M}$

GOALS: Find full analogues of the mentioned results in the multilinear setting and find the sharp bounds for $\mathscr{M}$ and *CZOs*.

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THEOREM (D., LERNER & PÉREZ, 2012)

Let $1 < p_i < \infty, i = 1, \ldots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$. Then the inequality

$$\|\mathscr{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C_{n,m,\vec{P}} [\vec{w}]_{A_{\vec{P}}}^{\frac{1}{p}} \prod_{i=1}^{m} ([\sigma_i]_{A_\infty})^{\frac{1}{p_i}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}$$

holds if $\vec{w} \in A_{\vec{P}}$, where $\sigma_i = w_i^{1-p_i'}$, $i = 1, \ldots, m$. Furthermore the exponents are sharp in the sense that they cannot be replaced by smaller ones.

Sharp estimates for $\mathscr{M}$

In the case of generalizing Buckley's result we could get partial results:

THEOREM (DLP, 2012)

Let $1 < p_i < \infty, i = 1, \ldots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$. Denote by $\alpha = \alpha(p_1, \ldots, p_m)$ the best possible power in

$$\|\mathscr{M}(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C_{n,m,p}\, [\vec{w}]_{A_{\vec{P}}}^{\alpha} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}.$$

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In the case of generalizing Buckley's result we could get partial results:

Theorem (DLP, 2012)

Let $1 < p_i < \infty, i = 1, \ldots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$. Denote by $\alpha = \alpha(p_1, \ldots, p_m)$ the best possible power in

$$\|\mathcal{M}(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C_{n,m,p} \, [\vec{w}]^{\alpha}_{A_{\vec{P}}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}.$$

Then we have the following results:

1. For all $1 < p_1, \ldots, p_m < \infty$, $\frac{m}{mp-1} \leq \alpha \leq \frac{1}{p}\left(1 + \sum_{i=1}^{m} \frac{1}{p_i - 1}\right)$;
2. If $p_1 = p_2 = \cdots = p_m = r > 1$, then $\alpha = \frac{m}{r-1}$.

Sharp estimates for $\mathscr{M}$

Later on, Li, Moen and Sun proved the full analogue of Buckley's result.

Theorem (Li, Moen & Sun, 2012)

Suppose $1 < p_1, \ldots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$, and $\vec{w} \in A_{\vec{P}}$. Then

$$\|\mathscr{M}(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C_{n,m,\vec{P}} \, [\vec{w}]_{A_{\vec{P}}}^{\max\{\frac{p_1'}{p}, \ldots, \frac{p_m'}{p}\}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}.$$

Moreover, the exponents are sharp in the sense that they cannot be replaced by smaller ones.

Finding multilinear analogues

Multilinear Sawyer's condition

We say that $(v, \overrightarrow{w})$ satisfies the $S_{\overrightarrow{P}}$ condition if

$$[v, \overrightarrow{w}]_{S_{\overrightarrow{P}}} = \sup_Q \left(\int_Q \mathscr{M}(\overrightarrow{\sigma\chi_Q})^p v dx \right)^{\frac{1}{p}} \left(\prod_{i=1}^m \sigma_i(Q)^{\frac{1}{p_i}} \right)^{-1} < \infty,$$

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Reverse Hölder condition

We say that $\vec{w}$ satisfies the $RH_{\vec{P}}$ condition if there exists a positive constant C such that

$$\prod_{i=1}^m \left(\int_Q \sigma_i dx \right)^{\frac{p}{p_i}} \leq C \int_Q \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx,$$

where $\sigma_i = w_i^{1-p_i'}$ for $i = 1, \ldots, m$. We denote by $[\vec{w}]_{RH_{\vec{P}}}$ the smallest constant C.

Finding multilinear analogues

Sawyer's theorem (Chen & D., 2013)

Let $1 < p_i < \infty$, $i = 1, \ldots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$. Let v and w_i be weights. If we suppose that $\vec{w} \in RH_{\vec{P}}$ then there exists a positive constant C such that

$$||\mathcal{M}(\overrightarrow{f\sigma})||_{L^p(v)} \leq C \prod_{i=1}^{m} ||f_i||_{L^{p_i}(\sigma_i)},\ f_i \in L^{p_i}(\sigma_i), \tag{1}$$

where $\sigma_i = w_i^{1-p_i'}$, if and only if $(v, \vec{w}) \in S_{\vec{P}}$. Moreover, if we denote the smallest constant C in (1) by $||\mathcal{M}||$, we obtain

$$[v, \vec{w}]_{S_{\vec{P}}} \lesssim ||\mathcal{M}|| \lesssim [v, \vec{w}]_{S_{\vec{P}}} [\vec{w}]_{RH_{\vec{P}}}^{1/p}.$$

Finding multilinear analogues

REMARKS ON SAWYER'S THEOREM

When $v = v_{\vec{w}}$, the following are equivalent:

1. $\vec{w} \in A_{\vec{P}}$.
2. $\sigma_i = w_i^{1-p_i'} \in A_{mp_i'}$, for $i = 1, \ldots, m$ and $v_{\vec{w}} \in A_{mp}$.
3. $(v_{\vec{w}}, \vec{w}) \in S_{\vec{P}}$.
4. There exists a positive constant C such that

$$||\mathscr{M}(\vec{f})||_{L^p(v_{\vec{w}})} \leq C \prod_{i=1}^{m} ||f_i||_{L^{p_i}(w_i)}, \, f_i \in L^{p_i}(w_i).$$

Finding multilinear analogues

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When $v = v_{\vec{w}}$, the following are equivalent:

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2. $\sigma_i = w_i^{1-p_i'} \in A_{mp_i'}$, for $i = 1, \ldots, m$ and $v_{\overrightarrow{w}} \in A_{mp}$.
3. $(v_{\overrightarrow{w}}, \overrightarrow{w}) \in S_{\overrightarrow{P}}$.
4. There exists a positive constant C such that

$$||\mathscr{M}(\overrightarrow{f})||_{L^p(v_{\overrightarrow{w}})} \leq C \prod_{i=1}^{m} ||f_i||_{L^{p_i}(w_i)}, \; f_i \in L^{p_i}(w_i).$$

Additionally, $RH_{\vec{P}}$ condition is **not necessary** and

$$[\overrightarrow{w}]_{A_{\overrightarrow{P}}}^{1/p} \lesssim [v_{\overrightarrow{w}}, \overrightarrow{w}]_{S_{\overrightarrow{P}}} \lesssim ||\mathscr{M}|| \lesssim [\overrightarrow{w}]_{A_{\overrightarrow{P}}}^{1/p} \prod_{i=1}^{m} [\sigma_i]_{\infty}^{\frac{1}{p_i}}.$$

Finding multilinear analogues

$B_{\vec{P}}$ constant

$$[v, \vec{w}]_{B_{\vec{P}}} = \sup_Q \frac{v(Q)}{|Q|} \Big(\prod_{i=1}^m \frac{w_i(Q)}{|Q|} \Big)^p \exp \Big(\frac{1}{|Q|} \int_Q \log \prod_{i=1}^m w_i^{-\frac{p}{p_i}} dx \Big).$$

Finding multilinear analogues

$B_{\vec{P}}$ CONSTANT

$$[v, \vec{w}]_{B_{\vec{P}}} = \sup_Q \frac{v(Q)}{|Q|} \Big(\prod_{i=1}^m \frac{w_i(Q)}{|Q|} \Big)^p \exp \Big(\frac{1}{|Q|} \int_Q \log \prod_{i=1}^m w_i^{-\frac{p}{p_i}} dx \Big).$$

$B_{\vec{P}}$ THEOREM (CHEN & D., 2013)

Let $1 < p_i < \infty$, $i = 1, \ldots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$. Let v and w_i be weights. Then

$$||\mathcal{M}(\overrightarrow{f\sigma})||_{L^p(v)} \lesssim [v, \vec{\sigma}]_{B_{\vec{P}}}^{1/p} \prod_{i=1}^m ||f_i||_{L^{p_i}(\sigma_i)}, \, f_i \in L^{p_i}(\sigma_i),$$

where $\sigma_i = w_i^{1-p_i'}$, $\vec{\sigma} = (\sigma_1, \ldots, \sigma_m)$ and $\overrightarrow{f\sigma} = (f_1\sigma_1, \ldots, f_m\sigma_m)$.

Finding multilinear analogues

$W_{\vec{P}}^{\infty}$ constant

$$[\vec{w}]_{W_{\vec{P}}^{\infty}} = \sup_{Q} \left(\int_Q \prod_{i=1}^{m} M(w_i \chi_Q)^{\frac{p}{p_i}} \, dx \right) \left(\int_Q \prod_{i=1}^{m} w_i^{\frac{p}{p_i}} \, dx \right)^{-1}.$$

Finding multilinear analogues

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$A_{\vec{P}} - W_{\vec{P}}^{\infty}$ theorem (Chen & D., 2013)

Let $1 < p_i < \infty$, $i = 1, \ldots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$. Let v and w_i be weights. Then

$$||\mathcal{M}(\overrightarrow{f\sigma})||_{L^p(v)} \lesssim ([v, \vec{w}]_{A_{\vec{P}}} [\vec{\sigma}]_{W_{\vec{P}}^{\infty}})^{1/p} \prod_{i=1}^{m} ||f_i||_{L^{p_i}(\sigma_i)}, \; f_i \in L^{p_i}(\sigma_i),$$

where $\sigma_i = w_i^{1-p_i'}$, $\vec{\sigma} = (\sigma_1, \ldots, \sigma_m)$ and $\overrightarrow{f\sigma} = (f_1\sigma_1, \ldots, f_m\sigma_m)$.

Multilinear Calderón–Zygmund operators

DEFINITION

We say that T is an m-linear **Calderón-Zygmund operator** if, for some $1 \leq q_j < \infty$, it extends to a bounded multilinear operator from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q, where

$$\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m},$$

and if there exists a function K, defined off the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \ldots f_m(y_m)\, dy_1 \ldots dy_m$$

for all $x \notin \cap_{j=1}^m supp f_j$.

Multilinear Calderón–Zygmund operators

Definition

The kernel K must also satisfy these two **conditions**:

$$|K(y_0, y_1, \ldots, y_m)| \leq \frac{A}{\left(\sum_{k,l=0}^{m} |y_k - y_l|\right)^{mn}}$$

and

$$|K(y_0, \ldots, y_j, \ldots, y_m) - K(y_0, \ldots, y_j', \ldots, y_m)| \leq \frac{A|y_j - y_j'|^{\varepsilon}}{\left(\sum_{k,l=0}^{m} |y_k - y_l|\right)^{mn+\varepsilon}},$$

for some $\varepsilon > 0$ and all $0 \leq j \leq m$, whenever $|y_j - y_j'| \leq \frac{1}{2} \max_{0 \leq k \leq m} |y_j - y_k|$.

Sparse families and dyadic operators

Definition

We say that $\mathscr{S} = \{Q_j^k\}$ is a **sparse family** of cubes if:

1. Q_j^k are disjoint in j, with k fixed.
2. If $\Omega_k = \cup_j Q_j^k$, then $\Omega_{k+1} \subset \Omega_k$.
3. $|\Omega_{k+1} \cap Q_j^k| \leq \frac{1}{2}|Q_j^k|$.

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DEFINITION

Given a sparse family $\mathscr{S}$ of a dyadic grid $\mathscr{D}$, the multilinear dyadic sparse operator $\mathscr{A}_{\mathscr{D},\mathscr{S}}$ is defined as follows

$$\mathscr{A}_{\mathscr{D},\mathscr{S}}(\vec{f})(x) = \sum_{j,k} \left(\prod_{i=1}^{m} (f_i)_{Q_j^k} \right) \chi_{Q_j^k}(x).$$

where $\vec{f} = (f_1, \ldots, f_m)$ and $|\vec{f}| = (|f_1|, \ldots, |f_m|)$.

CZO and dyadic operators

Theorem (D., Lerner & Pérez, 2012)

Let T be a $m-CZO$ and let X be a Banach space over $\mathbb{R}^n$ equipped with Lebesgue measure. Then, for any $\vec{f}$,

$$\|T(\vec{f})\|_X \leq c_{T,m,n} \sup_{\mathscr{D},\mathscr{S}} \|\mathscr{A}_{\mathscr{D},\mathscr{S}}(|\vec{f}|)\|_X.$$

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Remarks

- $1/m < p < \infty$ and $L^p(v_{\vec{w}})$ is not a Banach space when $1/m < p < 1$.

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Remarks

- $1/m < p < \infty$ and $L^p(\nu_{\vec{w}})$ is not a Banach space when $1/m < p < 1$.
- Can be X a quasi-Banach space instead of a Banach space in the above theorem?

A_2 multilinear theorem

THEOREM (D., LERNER & PÉREZ, 2012)

Let T be a $m-CZO$. Assume that $p_1 = p_2 = \cdots = p_m = m+1$. Then

$$\|T(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C_{T,m,n}[\vec{w}]_{A_{\vec{P}}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}.$$

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QUESTION

Is it possible to obtain a sharp version of the extrapolation theorem of Rubio de Francia in the multilinear setting?

Sharp bound and open problems

Theorem (Li, Moen & Sun, 2012)

Let T be a $m-CZO$ and let $1 < p_1, \ldots, p_m < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$.

Then

$$\|T(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C_{\vec{P},T,m,n} [\vec{w}]_{A_{\vec{P}}}^{\max\{1, \frac{p_1'}{p}, \ldots, \frac{p_m'}{p}\}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}.$$

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REMARK

The problem is still open when $1/m < p < 1$.

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THANK
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