

MATHEMATICAL CONCEPTS, THEIR MEANINGS, AND UNDERSTANDING¹

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"Try not to think of understanding as a 'mental process' at all.- For *that* is the expression which confuses you. (...) In the sense in which there are processes (including mental processes) which are characteristic of understanding, understanding is not a mental process". (Wittgenstein, 1953, *Philosophical Investigations*, p. 61)

Abstract

Some key elements for developing a theory for understanding mathematical concepts are outlined. These elements are derived from the theory of mathematical objects and their meanings developed by Godino and Batanero (1994; 1996). We shall argue for the need to complement the psychological facets of understanding - 'as a mental experience', and 'connection of internal representations in information networks'- with the sociocultural approach, that is, understanding as 'correspondence between personal and institutional meanings'. The role of situation-problems and semiotic instruments is also emphasized in both personal and institutional dimensions of understanding processes.

Understanding in Mathematics Education

As Hiebert and Carpenter (1992) asserted, one of the most widely accepted ideas in mathematics education is that students should understand mathematics. Sierpiska (1994) starts her book on understanding in mathematics with similar words: "*how to teach so that students understand?, What exactly don't they understand? What do they understand and how?*" (p. XI). Pirie and Kieren (1994) mention the interest towards teaching and learning mathematics with understanding, which is shown in recent curricular reforms in many countries. This interest is also reflected in conference proceedings and research articles in psychology and artificial intelligence.

The importance of the idea of understanding for mathematics education is emphasized in recent works by Sierpiska, Pirie and Kieren, Koyama (1993), amongst others. Nevertheless, characterizing understanding "*in a way which highlights its growth, and identifying pedagogical acts which sponsor it, however, represent continuing problems*" (Pirie and Kieren, 1994, p. 165).

The use of the term 'understanding' (or 'comprehension') is varied, depending on institutional contexts, although the dominant psychological approach emphasizes the mental facet of understanding, which is strongly challenged by Wittgenstein. The cognitive revolution supported by Vygotsky -who claims the analytical and genetic priority of sociocultural factors when attempt to understand individual psychological

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processes-, Bruner (1990) -with his proposal of a cultural psychology, or Chevallard (1992) -who speaks of cognitive and didactical anthropology- requires a reconceptualization of mathematical knowledge and its understanding.

The book by Sierpiska (1994) represents an important step forward, when discerning between understanding acts and processes and when relating "*good understanding*" of a mathematical situation (concept, theory, problem) to the sequence of acts of overcoming obstacles specific to this situation. Through the historico-empirical approach it is possible to identify meaningful acts for understanding a concept. Nevertheless, we think that taking the notion of object as primitive and deriving meaning from understanding cause some difficulties in analyzing the processes of assessing students' understanding.

From our point of view, a theory of conceptual understanding useful for mathematics education should not be limited to saying, for example, that understanding the concept of function is a person's mental experience assigning some object to the term 'function'. A cultural entity, a very complex, not ostensive object is designated with the term 'function'. Therefore, in order to define what is understanding the concept of function, we need to clarify a previous question: What object must the student assign to the term 'function' so that the teacher may say that he/she understands the object function?

The problem of understanding is, consequently, closely linked to how the nature of mathematical knowledge is conceived. Mathematical terms and expressions denote abstract entities whose nature and origin should be researched for elaborating a useful and effective theory for what it is to understand such objects. This research requires answering questions such as: What is the structure of the object to be understood? What forms or ways of understanding exist for each concept? What are the possible and desirable aspects or components of mathematical concepts for students to learn at a given time and under certain circumstances? How are these components developed?

If, for example, we consider mathematical knowledge as internally represented information, understanding occurs when the representations achieved are connected by progressively more structured and cohesive networks (Hiebert and Carpenter, 1992). However, we consider it excessively reductionist to view mathematical activity as information processing. From our point of view, the theories of understanding derived from this conception do not adequately describe the teaching and learning processes of mathematics, especially the social and cultural aspects involved in these processes.

We think that a theory of understanding mathematical abstractions must be supported by a previous theory concerning the nature of such objects. In this research report, we first present a summary of our previous theoretical articles on the nature of the mathematical objects and its meanings (Godino and Batanero 1994; 1996). After this synthesis, we shall provide some elements deduced from this theory, to be taken into account for the development of a theory of understanding in the teaching and learning of mathematics.

Pragmatic and relativist ontosemantics for mathematics

Our theory is based on the following epistemological and cognitive assumptions about mathematics, which take into account some recent tendencies in the philosophy of mathematics (Tymoczko, 1986, Ernest, 1991):

- a) Mathematics is a human activity involving the solution of problematic situations. In finding the responses or solutions to these external and internal problems, mathematical objects progressively emerge and evolve. According to Piagetian constructivist theories, people's acts must be considered the genetic source of mathematical conceptualization.
- b) Mathematical problems and their solutions are shared in specific institutions or collectives involved in studying such problems. Thus, mathematical objects are socially shared cultural entities.
- c) Mathematics is a symbolic language in which problem-situations and the solutions found are expressed. The systems of mathematical symbols have a communicative function and an instrumental role.
- d) Mathematics is a logically organized conceptual system. Once a mathematical object has been accepted as a part of this system, it can also be considered as a textual reality and a component of the global structure. It may be handled as a whole to create new mathematical objects, widening the range of mathematical tools and, at the same time, introducing new restrictions in mathematical work and language.

To build our model, we take the notion of *problem-situation* as a primitive idea. Problems do not appear in isolation; the systematic variation of the variables intervening in problem-situations yields different *fields of problems*, sharing similar representations, solutions, etc.

The subject performs different types of *practices*, or actions intended to solve a mathematical problem, to communicate the solution to other people or to validate or generalize that solution to other settings and problems. The genesis of a subject's knowledge arises as a consequence of that subject's interaction with the field of problems, which is mediated by institutional contexts.

Two primary units of analysis to study cognitive and didactic processes are the *meaningful practices*, and the *meaning of an object*, for which we postulate two interdependent dimensions: personal and institutional. A practice is meaningful for a person (resp. institution) if it fulfills a function for solving the problem, or for communicating, validating, or extending the solution. This notion is used to conceptualize mathematical objects, both in their psychological and epistemological facets (personal and institutional objects). Mathematical objects -abstractions or empirico and operative generalizations (Dörfler, 1991)- are considered as emergents from the systems of personal (institutional) practices made by a person (or within an institution) when involved with some problem-situations.

The system of meaningful prototype practices, i.e., the system of efficient practices to reach the goal aimed at is defined as the *personal (institutional) meaning of the object*. It is considered to be the genetic (epistemological) origin of personal objects (institutional objects). It is linked to the field of problems from which this object emerges at a given time and it is a compound entity. Its nature is opposed to the intensional character of the object, and it allows us to focus, from another point of view, on the issues for designing teaching situations and assessing subjects' knowledge.

To sum up, we postulate a relativity of the emergent objects, intrinsic to the different institutions involved in the field of problems, and also depending on the available expressive forms. This assumption could be useful to explain the adaptations (or transpositions) and mutual influences that mathematical objects undergo when transmitted between people and institutions.

The systemic complexity we postulate for the meaning of mathematical concepts -understood from a pragmatic perspective - is well illustrated by the problem list that Sierpiska (1994, p. 21) outlines on the meaning of the term 'function', expressed by its use:

How can a function be? Or, what adjectives can we use with the noun 'function' (Defined/ non-defined, defined in a point/ in an interval/ everywhere; increasing, decreasing, invertible, continuous in a point/ in an interval; smooth; differentiable in a point/ in a interval; integrable ..., etc). *What* can a function *have*? (Zeros, values, a derivative, a limit in a point/ in infinity; etc). *What* can be done, with functions? (Plot, calculate the values in points ..., calculate a derivative, an integral, combine functions, take sequences, series of functions, etc). How do we *verify* that a function is ... (continuous, differentiable ... in a point; increasing in an interval ...)? What can functions be *used for*? (Representing relations between variable magnitudes, modelling, predicting, interpolating, approximating, ...).

The socio-epistemic relativism for the meaning of mathematical abstractions that we postulate in our theoretical model is justified, for example, when we study the diversity of *conceptions* of the function concept identified by Ruíz-Higueras (1994), not only from a historical perspective, but also in didactic transpositions at the teaching institutions (curricula, text books, mathematics classroom).

Elements for a model of understanding in Mathematics Education

From our theoretical positions summarized in the previous Section, we identify the following consequences that should be considered in order to elaborate a theory on understanding mathematics.

Institutional and personal dimension

According to our pragmatic and relativist conception of mathematics, a theory of mathematical understanding, which is to be useful and effective to explain teaching and learning processes, should recognize the dialectical duality between the personal and institutional facets of knowledge and its understanding.

The definition of understanding by Sierpiska as the 'mental experience of a subject by which he/she relates an object (sign) to another object (meaning)' emphasizes one of the senses in which the term 'understanding' is used, well adapted for studying the psychological processes involved. Nevertheless, in mathematics teaching the term 'understanding' is also used in the processes for assessing students' learning. School institutions expect subjects to appropriate some culturally fixed objects, and assign the teacher with the task of helping the students to establish the agreed relationships between terms, mathematics expressions, abstractions, and techniques. In this case, understanding is not merely a mental activity, but it is converted into a social process. As an example, we may consider that a pupil sufficiently "understands" the function concept in secondary teaching and that he/she does not understand it, if the judgment is made by a university institution.

Furthermore, from a subjective sense, understanding cannot merely be reduced to a mental experience but it involves the person's whole world. As Johnson (1987) states, our understanding "*is the way we are meaningfully situated in our world through*

our bodily interactions, our cultural institutions, our linguistic tradition, and our historical context" (p. 102).

Systemic nature

Since, in our theoretical model, we start out from the notions of object and meaning, personal understanding of a concept is "grasping or acquiring the meaning of the object". Therefore, the construct 'meaning of an object' is not conceived as an absolute and unitary entity, but rather as compound and relative to institutional settings. Therefore, the subject's understanding of a concept, at a given moment and under certain circumstances, will imply the appropriation of the different elements composing the corresponding institutional meanings:

- extensional elements (recognition of prototype situations of use of the object);
- intensional elements (different characteristic properties and relationships with other entities);
- expressions and symbolic notations used to represent situations, properties, and relationships.

Furthermore, recognizing the systemic complexity of the object's meaning implies a dynamical, progressive, though nonlinear nature of the process of appropriation by the subject (Pirie & Kieren, 1994), due to the different domains of experience and institutional contexts in which he/she participates.

The conception of mathematics underlying our theoretical model is characterized primarily by considering mathematics as a human activity. Concepts and mathematical procedures emerge from a persons' acts for solving some problem fields. This activity is mediated by the semiotic instruments provided by the culture and by our capacity for deductive logical reasoning. Secondly, mathematics is a socially shared and logically structured conceptual system. Consequently, the process axis for personal understanding must contain the following categories: intuitive (operative), declaratory (communicative), argumentative (validating), and structural (institutionalized). The achievement of these levels of understanding for a concept or conceptual field will require the organization of specific didactic moments or situations, as Brousseau (1986) proposes in his *didactic situations theory*.

Human action and intentionality

Our theoretical model also includes, as the primary unit of analysis, the notion of *meaningful prototype practice*, defined as the action that the person carries out in his/her attempts for solving a class of problem-situations and for which he/she recognizes or attributes a purpose (an *intentionality*). Therefore, this is a situated expressive form involving a problem-situation, an institutional context, a person and the semiotic instruments mediating the action. This notion is used to define mathematical objects as emerging from the systems of meaningful prototype practices. Consequently, understanding the object, in its integral or systemic sense, requires the subject, not only the semiotic and relational components, but to identify a *role* -an *intention* (Maier, 1988)- in the problem solving process for the object.

Assessment of understanding

We conceive the processes for assessing understanding as the study of the correspondence between personal and institutional meanings. The evaluation of a subject's understanding is relative to the institutional contexts in which the subject participates. An institution (educational or not) will say that a subject "understands" the meaning of an object - or that he/she 'has grasped the meaning' of a concept, if the subject is able to carry out the different prototype practices that make up the meaning of the institutional object.

It is also necessary to recognize the *unobservable construct character* of personal understanding. Consequently, an individual's personal understanding about a mathematical object may be deduced from the analysis of the practices carried out by the person in solving problematic tasks, which are characteristics of that object. Since, for each mathematical object, the population of such tasks is potentially unlimited, the analysis of the task variables and the selection of the items to design evaluation instruments become of primary interest. The construct 'meaning of an object' we propose, in its two dimensions, personal and institutional, might be a useful conceptual tool to study the evaluation processes, the achievement of the 'good understanding', and the institutional and evolutionary factors conditioning them.

Final remarks

In this report, we explore some elements for a theory of understanding in mathematics, derived from our previous research work into the meaning of mathematical objects.

Our proposal is that mental functioning and sociocultural settings be understood as dialectically interacting moments, or aspects of a more inclusive unit of analysis: *human action*. Nevertheless, it is necessary to recognize the analytical priority of studying institutional meanings. Since each person develops in different institutions and cultural settings, the psychological processes involved in understanding the linguistic or conceptual mathematical objects are mediated by institutional meanings, namely, by situations-problems, semiotic instruments, habits and shared conventions.

Finally, the systemic nature of meaning and understanding highlights the sampling character of teaching and assessment situations, and the inferential problems associated with their study. Therefore, we propose the characterization of the personal and institutional meaning of mathematical objects, and of its mutual interdependence and development as a priority research agenda for Mathematics Education (Godino and Batanero, 1996).

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