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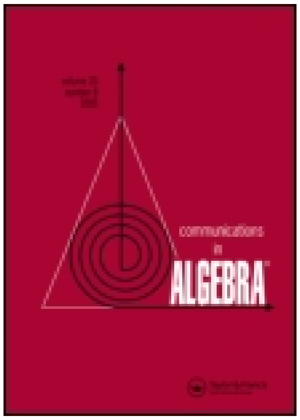

Communications in Algebra
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lagb20

A Characterization of π-Complemented Algebras

J. C. Cabello [a] , M. Cabrera [a] , A. Rodríguez Palacios [a] & R. Roura [a]
[a] Departamento de Análisis Matemático, Facultad de Ciencias , Universidad de Granada , Granada , Spain
Published online: 14 Jun 2013.

To cite this article: J. C. Cabello , M. Cabrera , A. Rodríguez Palacios & R. Roura (2013) A Characterization of π-Complemented Algebras, Communications in Algebra, 41:8, 3067-3079, DOI: 10.1080/00927872.2012.672604

To link to this article: http://dx.doi.org/10.1080/00927872.2012.672604

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Communications in Algebra®, 41: 3067–3079, 2013
Copyright © Taylor & Francis Group, LLC
ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927872.2012.672604

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A CHARACTERIZATION OF π-COMPLEMENTED ALGEBRAS

J. C. Cabello, M. Cabrera, A. Rodríguez Palacios, and R. Roura
Departamento de Análisis Matemático, Facultad de Ciencias,
Universidad de Granada, Granada, Spain

π-complemented algebras are defined as those algebras (not necessarily associative or unital) such that each annihilator ideal is complemented by other annihilator ideal. Let A be a semiprime algebra. We prove that A is π-complemented if, and only if, every idempotent in the extended centroid of A lies in the centroid of A. We also show the existence of a smallest π-complemented subalgebra of the central closure of A containing A. In the case that A is a C*-algebra, this subalgebra turns out to be a norm dense $*$-subalgebra of the bounded central closure of A. It follows that a C*-algebra is boundedly centrally closed if, and only if, it is π-complemented.

Key Words: Complemented algebra; Central closure; Extended centroid; Multiplication algebra; Semiprime algebra.

2010 Mathematics Subject Classification: Primary: 17A01; Secondary: 17A60.

INTRODUCTION

Throughout the article we assume that all algebras considered (associative or nonassociative) are algebras over a field $\mathbb{K}$. This convention will be used without further mention.

Let A be an arbitrary (not necessarily associative or unital) algebra. As usual, for each ideal I of A, the *annihilator* of I in A, denoted here by $\mathrm{Ann}_A(I)$ (or simply $\mathrm{Ann}(I)$ when no confusion can arise), is defined as the largest ideal J of A such that $IJ = JI = 0$. The *π-closure* of I is defined by

$$\overline{I} := \mathrm{Ann}(\mathrm{Ann}(I)).$$

The ideal I is said to be *π-closed* whenever $\overline{I} = I$. The set $\mathcal{I}_A^\pi$ of all π-closed ideals of A is a complete lattice for the meet and join operations given by

$$\bigwedge I_\alpha = \bigcap I_\alpha \quad \text{and} \quad \bigvee I_\alpha = \overline{\sum I_\alpha}.$$

Recall that A is said to be *π-complemented* if for any π-closed ideal I of A there exists a π-closed ideal J of A such that $A = I \oplus J$. It is clear that every π-complemented

Received February 1, 2012. Communicated by M. Bresar.
Address correspondence to Dr. Miguel Cabrera, Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain; E-mail: cabrera@ugr.es

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algebra is *semiprime* (that is, $I^2 \neq 0$ whenever I is a nonzero ideal of A). Moreover, the basic examples of π-complemented algebras are the prime algebras. Recall that A is said to be *prime* if, for ideals I and J of A, the condition $IJ = 0$ implies either $I = 0$ or $J = 0$. A structure theory for π-complemented algebras has been recently developed in [5].

It has long been known that, for a semiprime unital associative and commutative ring R, there exits an isomorphism from the Boolean algebra of all π-closed ideals of R onto the Boolean algebra of all idempotents of the complete ring of quotient of R [10, Section 2.4, Corollary 2]. The main goal of this article is to get an appropriate version of this result in nonassociative and free-unit context. We note that, for a semiprime unital associative ring R, the centre of the complete ring of quotients is nothing other than the extended centroid of R, and that the extended centroid has been considered for (possibly nonassociative and nonunital) algebras.

Different approaches to the concepts of extended centroid and central closure for a semiprime algebra appear in the literature (see [2, 11], and [15]). However, in order to avoid any type of construction, we prefer to take advantage of Razmyslov's characterization given in [12, Proposition 3.1], to introduce these concepts in an axiomatic way (see Definition 1.4 below). The reader is referred to the papers [2, 6, 8, 11, 15], and the books [14], [12, §.3], [3, §.9.2], [16, §.32], and [9, §.13 and 14] for a more detailed account on these concepts.

In the associative setting, the extended centroid C_A and the central closure Q_A of a semiprime associative algebra A can be viewed inside the more familiar rings of quotients for A. We remark that for our purposes it is enough to work with the symmetric Martindale algebra of quotients $Q_s(A)$. It is well-known that C_A is the centre of $Q_s(A)$, and Q_A is the C_A-subalgebra of $Q_s(A)$ generated by A.

Let A be a semiprime algebra. In Section 1, we prove that there exists a natural lattice isomorphism from $\mathcal{B}_A$ (the lattice of all idempotents in C_A) onto $\mathcal{I}_A^\pi$. As a consequence, it is shown that A is π-complemented if, and only if, $\mathcal{B}_A \subseteq \Gamma_A$, where Γ_A stand for the centroid of A. In particular, A is π-complemented whenever A is centrally closed, which implies, as a by-product, that Q_A is π-complemented. In Section 2 we discuss the π-complementation of the subalgebras of Q_A containing A. For such a subalgebra B, we prove that $C_B = C_A$ and $Q_B = Q_A$, and as a consequence there is a smallest π-complemented subalgebra of Q_A containing B.

As an application of the theory of quotient rings for C^*-algebras [1], we prove in Section 3 that, for any C^*-algebra A, the smallest π-complemented subalgebra of Q_A containing A can be seen as a norm dense $*$-subalgebra of the bounded central closure of A. As a consequence, a C^*-algebra is π-complemented if, and only if, it is boundedly centrally closed.

1. π-CLOSED IDEALS IN A SEMIPRIME ALGEBRA

Let A be an algebra, and let $\mathcal{L}(A)$ stand for the algebra of all linear operators on A. For $a \in A$, we will denote by L_a^A and R_a^A (or simply, by L_a and R_a if no confusion can arise) the operators of left and right multiplication by a on A. The *multiplication algebra* $\mathcal{M}(A)$ of A is defined as the subalgebra of $\mathcal{L}(A)$ generated by the identity operator Id_A and the set $\{L_a, R_a : a \in A\}$. It is clear that A is a left $\mathcal{M}(A)$-module for the evaluation action. If Q is an algebra extension of A, then A is said to be a *dense subalgebra* of Q whenever the condition $T(A) = 0$, for $T \in \mathcal{M}(Q)$, implies $T = 0$.

Given a subalgebra A of an algebra Q, it is easy to show that the following assertions are equivalent:

(i) A is a dense subalgebra of Q;
(ii) For each $F \in \mathcal{M}(A)$, there exists a unique $F' \in \mathcal{M}(Q)$ such that $F'(a) = F(a)$ for every $a \in A$,

(see [7, Corollary 3.2] or Proposition 2.1 below for a generalization). In this case, the map $F \mapsto F'$ becomes a canonical algebra embedding $\mathcal{M}(A) \hookrightarrow \mathcal{M}(Q)$. As a consequence, Q has a natural structure of left $\mathcal{M}(A)$-module given by

$$F.q := F'(q) \qquad \text{for all } F \in \mathcal{M}(A) \text{ and } q \in Q.$$

By abuse of notation, we will sometimes write $F(q)$ instead of $F.q$, and $\mathcal{M}(A)(q)$ instead of $\mathcal{M}(A).q$

Proposition 1.1. *Let A be a dense subalgebra of an algebra Q. We have the following statements*:

(1) *If M is a subspace of Q, then M is an $\mathcal{M}(A)$-submodule of Q if, and only if, $AM + MA \subseteq M$.*
(2) *If M is an $\mathcal{M}(A)$-submodule of Q and $f : M \to Q$ is a linear map, then f is an $\mathcal{M}(A)$-homomorphism if, and only if, f is an A-centralizer, that is, $f(aq) = af(q)$ and $f(qa) = f(q)a$ for all $a \in A$ and $q \in M$.*

Proof. (1) Let M be a $\mathcal{M}(A)$-submodule of Q. For all $a \in A$ and $q \in M$, we have

$$aq = L_a^Q(q) = L_a^A(q) \in M \qquad \text{and} \qquad qa = R_a^Q(q) = R_a^A(q) \in M.$$

Therefore, $AM + MA \subseteq M$. In order to prove the converse, suppose that M is a subspace of Q satisfying $AM + MA \subseteq M$, and consider the set $\mathcal{S}$ consisting of all $F \in \mathcal{M}(A)$ such that $F(M) \subseteq M$. It is clear that $\mathcal{S}$ is a subalgebra of $\mathcal{M}(A)$ containing Id_A, L_a^A, R_a^A $(a \in A)$. Therefore, $\mathcal{S} = \mathcal{M}(A)$, as required.

(2) Let M be an $\mathcal{M}(A)$-submodule of Q and $f : M \to Q$ be a linear map. If f is an $\mathcal{M}(A)$-homomorphism, then, for all $a \in A$ and $q \in M$, we have

$$f(aq) = f(L_a^Q(q)) = f(L_a^A(q)) = L_a^A f(q) = af(q),$$

and

$$f(qa) = f(R_a^Q(q)) = f(R_a^A(q)) = R_a^A f(q) = f(q)a,$$

and consequently, f is an A-centralizer. In order to prove the converse, suppose that f is an A-centralizer, and consider the set $\mathcal{T}$ consisting of all $F \in \mathcal{M}(A)$ such that $f(F(q)) = F(f(q))$ for all $q \in M$. It is clear that $\mathcal{T}$ is a subalgebra of $\mathcal{M}(A)$ containing Id_A, L_a^A, R_a^A $(a \in A)$. Therefore, $\mathcal{S} = \mathcal{M}(A)$, and consequently, f is an $\mathcal{M}(A)$-homomorphism. □

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Let A be an algebra. Recall that an ideal D of A is said to be *essential* if $D \cap I \neq 0$ for any nonzero ideal I of A. If A is a dense subalgebra of an algebra Q, then an $\mathcal{M}(A)$-submodule M of Q is said to be *large* in Q if $M \cap N \neq 0$ for any nonzero $\mathcal{M}(A)$-submodule N of Q.

Corollary 1.2. *Let A be a dense subalgebra of an algebra Q. Then we have the following statements*:

(1) *The ideals of A are precisely the $\mathcal{M}(A)$-submodules of Q contained in A*;
(2) *If M is a* (*large*) *$\mathcal{M}(A)$-submodule of Q, then $M \cap A$ is an* (*essential*) *ideal of A.*

Now we point out a context in which dense subalgebras appear, and that will be fundamental in what follows. Given a unital commutative associative algebra C, by a *C-algebra* we mean an algebra Q endowed with a bilinear map $(\lambda, q) \mapsto \lambda q$ from $C \times Q$ to Q satisfying the properties

$$\lambda(pq) = (\lambda p)q = p(\lambda q), \qquad (\lambda\mu)q = \lambda(\mu q), \qquad \text{and} \qquad 1q = q$$

for all $\lambda, \mu \in C$ and $p, q \in Q$.

Proposition 1.3. *Let C be a unital commutative associative algebra, and let Q be a C-algebra. Then $T(\lambda q) = \lambda T(q)$ for all $T \in \mathcal{M}(Q)$, $\lambda \in C$, and $q \in Q$. As a consequence, if A is a subalgebra of Q such that Q is generated by A as a C-algebra, then A is a dense subalgebra of Q.*

Proof. Consider the set $\mathcal{S}$ consisting of all $T \in \mathcal{M}(Q)$ satisfying $T(\lambda q) = \lambda T(q)$ for all $\lambda \in C$ and $q \in Q$. It is clear that $\mathcal{S}$ is a subalgebra de $\mathcal{M}(Q)$ containing Id_Q, L_q^Q, R_q^Q $(q \in Q)$. Therefore, $\mathcal{S} = \mathcal{M}(Q)$, as required. Now, assume that A is a subalgebra of Q such that Q is generated by A as a C-algebra. If $T \in \mathcal{M}(Q)$ satisfies $T(A) = 0$, then we have $T(\sum_{i=1}^n \lambda_i a_i) = \sum_{i=1}^n \lambda_i T(a_i) = 0$ for all $n \in \mathbb{N}$, $\lambda_i \in C$, $a_i \in A$, and hence $T = 0$. Thus A is a dense subalgebra of Q. □

The following definition is in accordance with the characterization given by Razmyslov in [12, Proposition 3.1].

Definition 1.4. Let A be a semiprime algebra. The *extended centroid* C_A and the *central closure* Q_A of A are determined by the following properties:

(P1) C_A is a unital semiprime commutative associative algebra, Q_A is a semiprime algebra extension of A, and Q_A is generated by A as a C_A-algebra (hence A is a dense subalgebra of Q_A because of Proposition 1.3).
(P2) A is a large $\mathcal{M}(A)$-submodule of Q_A.
(P3) For any $\mathcal{M}(A)$-homomorphism f of an $\mathcal{M}(A)$-submodule M of Q_A into Q_A, there exists an element $\lambda \in C_A$ such that $f(q) = \lambda q$ for every $q \in M$. Moreover, if M is a large $\mathcal{M}(A)$-submodule in Q_A, then λ is uniquely determined by f.

We start by collecting some well-known facts which become a part of the classical construction of the extended centroid and the central closure.

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Proposition 1.5. *Let A be a semiprime algebra. Then*

(1) *For every ideal I of A, $C_A I$ is an ideal of Q_A and* $C_A\mathrm{Ann}(I) \cap A = \mathrm{Ann}(C_A I \cap A) = \mathrm{Ann}(I)$.
(2) *For every ideal I of A, we have* $C_A\overline{I} \cap A = \overline{C_A I \cap A} = \overline{I}$.
(3) *The essential ideals of A are precisely the large $\mathcal{M}(A)$-submodules of Q_A contained in A. As a consequence, $C_A D$ is an essential ideal of Q_A, whenever D is an essential ideal of A.*
(4) *If U is an essential ideal of Q_A, then $U \cap A$ is an essential ideal of A and U is a large $\mathcal{M}(A)$-submodule of Q_A.*
(5) *If $\lambda \in C_A$ satisfies $\lambda D = 0$ for some essential ideal D of A, then $\lambda = 0$.*
(6) *For each $\lambda \in C_A$, the set $D_\lambda := \{a \in A : \lambda a \in A\}$ is an essential ideal of A.*
(7) $T(\lambda q) = \lambda T(q)$ *for all $T \in \mathcal{M}(Q_A)$, $\lambda \in C_A$, and $q \in Q_A$.*

Proof. (1) For a given ideal I of A, it is clear that $C_A I$ is an ideal of Q_A and that $C_A I \cap A$ and $C_A\mathrm{Ann}(I) \cap A$ are ideals of A satisfying

$$(C_A I \cap A)(C_A\mathrm{Ann}(I) \cap A) = (C_A\mathrm{Ann}(I) \cap A)(C_A I \cap A) = 0,$$

and consequently $C_A\mathrm{Ann}(I) \cap A \subseteq \mathrm{Ann}(C_A I \cap A)$. Now, the desired equalities follow from the following chain of inclusions

$$\mathrm{Ann}(I) \subseteq C_A\mathrm{Ann}(I) \cap A \subseteq \mathrm{Ann}(C_A I \cap A) \subseteq \mathrm{Ann}(I).$$

(2) This assertion is a consequence of (1). Indeed, by replacing I with $\mathrm{Ann}(I)$ in the equality $C_A\mathrm{Ann}(I) \cap A = \mathrm{Ann}(I)$, we obtain $C_A\overline{I} \cap A = \overline{I}$, and by taking annihilators in the equality $\mathrm{Ann}(C_A I \cap A) = \mathrm{Ann}(I)$, we obtain $\overline{C_A I \cap A} = \overline{I}$.

(3) Since, by property (P2), A is a large $\mathcal{M}(A)$-submodule of Q_A, it follows immediately that the essential ideals of A are precisely the large $\mathcal{M}(A)$-submodules of Q_A contained in A. As a consequence, if D is an essential ideal of A, then $C_A D$ is a large $\mathcal{M}(A)$-submodule of Q_A, and hence an essential ideal of Q_A.

(4) Let U be an essential ideal of Q_A. If I is an ideal of A such that $U \cap A \cap I = 0$, then $U \cap A \subseteq \mathrm{Ann}(I)$, and so $U \cap A \subseteq \mathrm{Ann}(C_A I \cap A)$ because of assertion (1). Therefore $(U \cap A) \cap (C_A I \cap A) = 0$. Keeping in mind property (P2), we deduce that $U \cap C_A I = 0$, hence $C_A I = 0$, and in particular $I = 0$. Thus $U \cap A$ is an essential ideal of A. Finally, it follows from (3) that U is a large $\mathcal{M}(A)$-submodule of Q_A.

(5) Let $\lambda \in C_A$ satisfying $\lambda D = 0$ for some essential ideal D of A. Then $\lambda q = 0$ for every $q \in C_A D$. Since, by assertion (3), $C_A D$ is a large $\mathcal{M}(A)$-submodule of Q_A, we deduce that $\lambda = 0$ because of property (P3).

(6) Let $\lambda \in C_A$. If we define $D_\lambda := \{a \in A : \lambda a \in A\}$, then it is clear that D_λ is an ideal of A and $\lambda\mathrm{Ann}(D_\lambda) \cap A = 0$. Therefore, by property (P2), we have $\lambda\mathrm{Ann}(D_\lambda) = 0$, hence $\mathrm{Ann}(D_\lambda) \subseteq D_\lambda$, and so $\mathrm{Ann}(D_\lambda) = 0$. Thus D_λ is an essential ideal of A.

(7) It follows from Proposition 1.3, taking into account that, by property (P1), A is a dense subalgebra of Q_A. □

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Now, our goal is to prove a nonassociative and nonunital version of some previously known results about rings of quotients of semiprime rings [3, Sect. 2.3]. Given a semiprime algebra A, for each nonempty subset S of Q_A, the *annihilator of S in C_A* is defined by

$$\mathrm{Ann}_{C_A}(S) := \{\lambda \in C_A : \lambda S = 0\}.$$

The next result can be deduced from the the theory of polyform modules (see [16, §.32.3]). For the sake of completeness, we are going to include a proof, which is similar in spirit to that of [3, Theorem 2.3.9].

Proposition 1.6. *Let A be a semiprime algebra and let S be a nonempty subset of Q_A.*

(1) *There exists a unique idempotent $E(S)$ in C_A such that*

$$\mathrm{Ann}_{C_A}(S) = (1 - E(S))C_A;$$

moreover, if U denotes the ideal of Q_A generated by S, then

$$\mathrm{Ann}_{Q_A}(U) = (1 - E(S))Q_A \quad \text{and} \quad E(S)p = p \text{ for every } p \in U.$$

(2) *For any idempotent $e \in C_A$, $E(eS) = eE(S)$.*

Proof. (1) Since Q_A is a semiprime algebra, $D := U \oplus \mathrm{Ann}_{Q_A}(U)$ is an essential ideal of Q_A, and hence D is a large $\mathcal{M}(A)$-submodule of Q_A because of Proposition 1.5.(4). Consider the map $f : D \to Q_A$ defined by

$$f(p + q) := p \qquad \text{for all } p \in U \text{ and } q \in \mathrm{Ann}_{Q_A}(U).$$

Clearly, f is an $\mathcal{M}(A)$-homomorphism. Hence, by property (P3), there exists a unique $e \in C_A$ satisfying $e(p + q) = p$ for all $p \in U$ and $q \in \mathrm{Ann}_{Q_A}(U)$. Moreover, since $e^2 z = ez$ for every $z \in D$, by Proposition 1.5.(5), we see that $e^2 = e$. Also note that, by Proposition 1.5.(7), $\mathrm{Ann}_{C_A}(S) = \mathrm{Ann}_{C_A}(U)$. We claim that $\mathrm{Ann}_{C_A}(U) = (1 - e)C_A$. Indeed, let $\lambda \in C_A$ be such that $\lambda U = 0$. Then $\lambda e(p + q) = \lambda p = 0$ for all $p \in U$ and $q \in \mathrm{Ann}_{Q_A}(U)$. Since D is an essential ideal of Q_A, by Proposition 1.5.(5), we see that $\lambda e = 0$, and so $\lambda = \lambda(1 - e) \in (1 - e)C_A$. On the other hand, the equality $(1 - e)U = 0$ implies that $(1 - e)C_A \subseteq \mathrm{Ann}_{C_A}(U)$. Therefore, $\mathrm{Ann}_{C_A}(U) = (1 - e)C_A$, and consequently, $ep = p$ for every $p \in U$. Being an identity element of the algebra $(1 - e)C_A$, the element $1 - e$ (and so e) is uniquely determined. Clearly, $(1 - e)Q_A \subseteq \mathrm{Ann}_{Q_A}(U)$. Conversely, since $U = eD$, we see that $\mathrm{Ann}_{Q_A}(U)eD = 0$, and we deduce that $\mathrm{Ann}_{Q_A}(U)e = 0$ because D is an essential ideal of Q_A. Therefore, $\mathrm{Ann}_{Q_A}(U) \subseteq (1 - e)Q_A$.

(2) Let $\lambda \in \mathrm{Ann}_{C_A}(eS)$. Then $0 = \lambda eS = \lambda eE(S)S$, and hence

$$\lambda eE(S) \in \mathrm{Ann}_{C_A}(S) = (1 - E(S))C_A.$$

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Therefore, $\lambda e E(S) = \lambda e E(S)(1 - E(S)) = 0$, and consequently,

$$\lambda = (1 - eE(S))\lambda \in (1 - eE(S))C_A.$$

Thus

$$\mathrm{Ann}_{C_A}(eS) \subseteq (1 - eE(S))C_A.$$

The converse inclusion follows from the fact that

$$(1 - eE(S))eS = (e - eE(S))S = e(1 - E(S))S = 0.$$

Therefore, $\mathrm{Ann}_{C_A}(eS) = (1 - eE(S))C_A$ and, by (1), we conclude that $E(eS) = eE(S)$. □

Now we can state a variant of [3, Lemma 2.3.10].

Corollary 1.7. *Let A be a semiprime algebra, and let I, J be ideals of A. Then the following conditions are equivalent*:

(i) $IJ = 0$;
(ii) $E(I)J = 0$;
(iii) $E(I)E(J) = 0$.

Proof. (i) $\Rightarrow$ (ii). Since Q_A is semiprime and $(C_AI)(C_AJ) = 0$, it follows that $C_AJ \subseteq \mathrm{Ann}_{Q_A}(C_AI) = (1 - E(I))Q_A$; Hence $E(I)C_AJ = 0$, and in particular $E(I)J = 0$.

(ii) $\Rightarrow$ (iii). $0 = E(E(I)J) = E(I)E(J)$.

(iii) $\Rightarrow$ (i). $IJ = (E(I)I)(E(J)J) = (E(I)E(J))(IJ) = 0$. □

The *centroid* Γ_A of an algebra A is defined as the subalgebra of $\mathscr{L}(A)$ consisting of all $\mathscr{M}(A)$-endomorphisms of A. For a semiprime algebra A, from properties (P2) and (P3), we can see Γ_A contained in C_A. More precisely, for each $f \in \Gamma_A$, there exists a unique $\lambda \in C_A$ such that $f(a) = \lambda a$ for every $a \in A$. Thus we can regard Γ_A as the subalgebra of C_A consisting of all elements $\lambda \in C_A$ such that $\lambda A \subseteq A$. Also recall that the set $\mathscr{B}_A$ of all idempotents in C_A has a partial orden given by $e \leq f$ if $e = ef$. Moreover, $\mathscr{B}_A$ is a Boolean algebra for the operations

$$e \wedge f = ef, \qquad e \vee f = e + f - ef, \qquad \text{and} \qquad e^* = 1 - e.$$

Now, we are ready to formulate and prove our main result.

Theorem 1.8. *Let A be a semiprime algebra. Then the map $e \mapsto eA \cap A$ is a lattice isomorphism from $\mathscr{B}_A$ onto $\mathscr{I}_A^{\pi}$, and its inverse is the map $I \mapsto E(I)$. As consequences, we have the following statements*:

(1) $\overline{I} = E(I)A \cap A$ *for every ideal I of A*;

(2) $\mathcal{B}_A$ *is a complete Boolean algebra*;
(3) A *is* π*-complemented if, and only if,* $\mathcal{B}_A \subseteq \Gamma_A$. *In this case,*

$$\mathcal{I}_A^\pi = \{eA : e \in \mathcal{B}_A\}.$$

Proof. It is clear that, for $e, f \in \mathcal{B}_A$ such that $e \leq f$, the sets $eA \cap A$ and $fA \cap A$ are ideals of A such that $eA \cap A \subseteq fA \cap A$. Given $e \in \mathcal{B}_A$, consider $D_e = \{a \in A : ea \in A\}$, and note that

$$e\mathrm{Ann}(eA \cap A)D_e = \mathrm{Ann}(eA \cap A)(eD_e) \subseteq \mathrm{Ann}(eA \cap A)(eA \cap A) = 0,$$

hence $(C_A e\mathrm{Ann}(eA \cap A))(C_A D_e) = 0$, and so

$$(C_A e\mathrm{Ann}(eA \cap A)) \cap (C_A D_e) = 0.$$

Since, by Proposition 1.5.(6) and (3), $C_A D_e$ is an essential ideal of Q_A, it follows that $C_A e\mathrm{Ann}(eA \cap A) = 0$, hence $e\mathrm{Ann}(eA \cap A) = 0$, and so $\mathrm{Ann}(eA \cap A) \subseteq (1-e)A \cap A$. Since the converse inclusion is clear, we conclude that $\mathrm{Ann}(eA \cap A) = (1-e)A \cap A$. It follows, by interchanging the roles of e and $1-e$, that $\mathrm{Ann}((1-e)A \cap A) = eA \cap A$. Thus, $eA \cap A \in \mathcal{I}_A^\pi$. Note also that $(1-e)C_A \subseteq \mathrm{Ann}_{C_A}(eA \cap A)$ and $e\mathrm{Ann}_{C_A}(eA \cap A)D_e = 0$. Keeping in mind Proposition 1.5.(5), from this last equality we derive that $e\mathrm{Ann}_{C_A}(eA \cap A) = 0$, and hence $\mathrm{Ann}_{C_A}(eA \cap A) = (1-e)C_A$. So $e = E(eA \cap A)$. Moreover, for each ideal I of A, by Corollary 1.7, we see that $E(I)\mathrm{Ann}(I) = 0$, and hence $\mathrm{Ann}(I) \subseteq (1 - E(I))A \cap A$. Since the converse inclusion is an obvious consequence of Proposition 1.6.(1), we conclude that $\mathrm{Ann}(I) = (1 - E(I))A \cap A$, and consequently, $I = E(I)A \cap A$ whenever I is π-closed. Thus the map $e \mapsto eA \cap A$ is a lattice isomorphism from $\mathcal{B}_A$ onto $\mathcal{I}_A^\pi$ with inverse map $I \mapsto E(I)$.

Now, let us show the consequences in the statement.

(1) Let I an ideal of A. If $e \in \mathcal{B}_A$ satisfies $I \subseteq eA \cap A$, then $I = eI$. Therefore, by Proposition 1.6.(2), $E(I) = eE(I)$, that is, $E(I) \leq e$, and hence $E(I)A \cap A \subseteq eA \cap A$. Hence $\overline{I} = E(I)A \cap A$.

(2) This assertion follows from the fact that $\mathcal{I}_A^\pi$ is a complete Boolean algebra (with the operation $I \mapsto \mathrm{Ann}(I)$ as complementation) [5, Corollary 1.4].

(3) If A is π-complemented, then for each $e \in \mathcal{B}_A$ we have

$$A = (eA \cap A) \oplus \mathrm{Ann}(eA \cap A) = (eA \cap A) \oplus ((1-e)A \cap A),$$

hence $eA = eA \cap A$, and in particular $eA \subseteq A$. Thus $\mathcal{B}_A \subseteq \Gamma_A$. Conversely, if $\mathcal{B}_A \subseteq \Gamma_A$, then from the first conclusion in the statement, we see that $\mathcal{I}_A^\pi = \{eA : e \in \mathcal{B}_A\}$ and $A = eA \oplus (1-e)A$ for every $e \in \mathcal{B}_A$. Thus A is π-complemented. □

Recall that a semiprime algebra A is said to be *centrally closed* whenever $Q_A = A$, that is to say whenever $C_A = \Gamma_A$.

Corollary 1.9. *Every centrally closed semiprime algebra* A *is* π*-complemented and* $\mathcal{I}_A^\pi = \{eA : e \in \mathcal{B}_A\}$.

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2. π-COMPLEMENTATION IN SUBALGEBRAS OF Q_A CONTAINING A

For a given semiprime algebra A, we will study the π-complementation for subalgebras of Q_A containing A. We begin by showing that the inclusions for such subalgebras transfer to the corresponding multiplication algebras.

Proposition 2.1. *Let Q be an algebra, and let A be a dense subalgebra of Q. Assume that B and B' are subalgebras of Q containing A such that $B \subseteq B'$. Then, for each $F \in \mathcal{M}(B)$, there exists a unique $F' \in \mathcal{M}(B')$ such that $F'(a) = F(a)$ for every $a \in A$, and the map $F \mapsto F'$ becomes an algebra monomorphism from $\mathcal{M}(B)$ into $\mathcal{M}(B')$.*

Proof. Consider the set $\mathcal{S}$ consisting of all $F \in \mathcal{M}(B)$ for which there exists $T \in \mathcal{M}(B')$ such that $T(a) = F(a)$ for every $a \in A$. It is immediate to verify that $\mathcal{S}$ is a subspace of $\mathcal{M}(B)$ and that $\mathcal{U} := \{G \in \mathcal{M}(B) : G\mathcal{S} \subseteq \mathcal{S}\}$ is a subalgebra of $\mathcal{M}(B)$. Moreover, if $F \in \mathcal{S}$ and $T \in \mathcal{M}(B')$ satisfy $T(a) = F(a)$ for every $a \in A$, then

$$\mathrm{Id}_B F(a) = \mathrm{Id}_{B'} T(a), \qquad L_b^B F(a) = L_b^{B'} T(a), \qquad \text{and} \qquad R_b^B F(a) = R_b^{B'} T(a)$$

for all $a \in A$ and $b \in B$. Therefore, $\mathrm{Id}_B, L_b^B, R_b^B \in \mathcal{U}$ for every $b \in B$, hence $\mathcal{U} = \mathcal{M}(B)$, and so $\mathcal{S}$ is a left ideal of $\mathcal{M}(B)$. Since clearly $\mathrm{Id}_B \in \mathcal{S}$, it follows that $\mathcal{S} = \mathcal{M}(B)$. Thus, for each $F \in \mathcal{M}(B)$, there exists $T \in \mathcal{M}(B')$ such that $T(a) = F(a)$ for every $a \in A$.

Now, assume that for $F \in \mathcal{M}(B)$ there exist $T_1, T_2 \in \mathcal{M}(B')$ such that $T_1(a) = T_2(a) = F(a)$ for every $a \in A$. By the above part in the proof, with the chain of subalgebras $A \subseteq B \subseteq B' \subseteq Q$ replaced by $B' \subseteq B' \subseteq Q \subseteq Q$, we can assert that there exists $T \in \mathcal{M}(Q)$ such that $T(b') = (T_1 - T_2)(b')$ for every $b' \in B'$. Since $T(A) = (T_1 - T_2)(A) = 0$ and A is a dense subalgebra of Q, it follows that $T = 0$, and consequently, $T_1 = T_2$. Thus, for each $F \in \mathcal{M}(B)$, there exists a unique $F' \in \mathcal{M}(B')$ such that $F'(a) = F(a)$ for every $a \in A$. It is immediate to check that $F \mapsto F'$ is a linear map from $\mathcal{M}(B)$ into $\mathcal{M}(B')$. Next we will prove the following claim:

$$F'(b) = F(b) \qquad \text{for all } F \in \mathcal{M}(B) \text{ and } b \in B.$$

Indeed, given $F \in \mathcal{M}(B)$, by the above part in the proof, with the chain of algebras $A \subseteq B \subseteq B' \subseteq Q$ replaced by $B \subseteq B \subseteq Q \subseteq Q$, we can assert the existence of $T_1 \in \mathcal{M}(Q)$ such that $F(b) = T_1(b)$ for every $b \in B$. Analogously, by considering the chain $B \subseteq B' \subseteq Q \subseteq Q$, we can confirm the existence of $T_2 \in \mathcal{M}(Q)$ such that $F'(b) = T_2(b)$ for every $b \in B$. Since, for each $a \in A$, we have $T_1(a) = F(a) = F'(a) = T_2(a)$, the denseness of A in Q yields to $T_1 = T_2$, and hence $F(b) = T_1(b) = T_2(b) = F'(b)$ for every $b \in B$, and the claim is proved.

Now, it is clear that the equality $F_1' = F_2'$, for $F_1, F_2 \in \mathcal{M}(B)$, implies $F_1 = F_2$. Moreover, for all $F_1, F_2 \in \mathcal{M}(B)$, we can assert that $F_1'F_2' \in \mathcal{M}(B')$ satisfies $F_1'F_2'(b) = F_1F_2(b)$ for every $b \in B$, and in particular $F_1'F_2'(a) = F_1F_2(a)$ for every $a \in A$. Therefore $F_1'F_2' = (F_1F_2)'$. As a result, the map $F \mapsto F'$ is an algebra monomorphism from $\mathcal{M}(B)$ into $\mathcal{M}(B')$. □

Corollary 2.2. *If A is a semiprime algebra and if B, B' are subalgebras of Q_A such that $A \subseteq B \subseteq B' \subseteq Q_A$, then the evaluation at elements of A determines the corresponding inclusions for the multiplication algebras*:

$$\mathcal{M}(A) \subseteq \mathcal{M}(B) \subseteq \mathcal{M}(B') \subseteq \mathcal{M}(Q_A).$$

We can now formulate the main result in this section.

Theorem 2.3. *Let A be a semiprime algebra, and let B be a subalgebra of Q_A containing A. Then B is semiprime, $C_B = C_A$, and $Q_B = Q_A$.*

Proof. If I is an ideal of B such that $I^2 = 0$, then $I \cap A$ is an ideal of A such that $(I \cap A)^2 = 0$, and hence $I \cap A = 0$. Since I is an $\mathcal{M}(A)$-submodule of Q_A and A is a large $\mathcal{M}(A)$-submodule of Q_A, we conclude that $I = 0$. Thus B is semiprime. According to Definition 1.4, we need to prove that the algebras C_A and Q_A satisfy properties (P1)–(P3) with respect to the algebra B.

(P1) We know that C_A is a semiprime commutative associative algebra with a unit, Q_A is a semiprime algebra extension of B, and Q_A is a C_A-algebra generated by A, and hence by B.

(P2) By Corollary 2.2, every $\mathcal{M}(B)$-submodule of Q_A is an $\mathcal{M}(A)$-submodule. Since A is a large $\mathcal{M}(A)$-submodule of Q_A, it follows that B is also a large $\mathcal{M}(B)$-submodule of Q_A.

(P3) Let M be an $\mathcal{M}(B)$-submodule of Q_A, and let f be an $\mathcal{M}(B)$-homomorphism from M to Q_A. Taking into account Corollary 2.2, we can assert that M is an $\mathcal{M}(A)$-submodule of Q_A and f is an $\mathcal{M}(A)$-homomorphism. Therefore, there exists an element $\lambda \in C_A$ such that $f(q) = \lambda q$ for every $q \in M$. Now, assume that in addition M is a large $\mathcal{M}(B)$-submodule in Q_A. Let N be an $\mathcal{M}(A)$-submodule of Q_A such that $M \cap N = 0$. Fix $q \in M \cap C_A N$, write $q = \sum_{i=1}^n \lambda_i q_i$ for some $n \in \mathbb{N}$, $\lambda_i \in C_A$, $q_i \in N$, and consider the essential ideal of A given by $D := \cap_{i=1}^n D_{\lambda_i}$ (Proposition 1.5.(6)). Keeping in mind Proposition 1.5.(7), for each $x \in D$ and $F \in \mathcal{M}(A)$ we see that

$$xF(q) = xF\left(\sum_{i=1}^n \lambda_i q_i\right) = \sum_{i=1}^n \lambda_i xF(q_i) = \sum_{i=1}^n L^A_{\lambda_i x} F(q_i) \in M \cap N.$$

Therefore, $D\mathcal{M}(A)(q) = 0$, and hence $q = 0$ because of Proposition 1.5.(8). Thus $M \cap C_A N = 0$. Note that $C_A N$ is an ideal of Q_A, and so an $\mathcal{M}(B)$-submodule of Q_A. Therefore, $C_A N = 0$, and in particular $N = 0$. Thus M is a large $\mathcal{M}(A)$-submodule in Q_A, and consequently, λ is uniquely determined by f. □

On account of Corollary 1.9, as a direct consequence of Theorem 2.3, we have the following result.

Corollary 2.4. *If A is a semiprime algebra, then Q_A is a π-complemented algebra and $\mathcal{I}^{\pi}_{Q_A} = \{eQ_A : e \in \mathcal{B}_A\}$.*

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Another consequence of Theorems 1.8 and 2.3 is the following result.

Corollary 2.5. *Let A be a semiprime algebra.*

(1) *If $\{B_\alpha\}$ is a family of π-complemented subalgebras of Q_A containing A, then $\cap B_\alpha$ is a π-complemented algebra.*
(2) *For each subalgebra B of Q_A containing A, there is a smallest π-complemented subalgebra of Q_A containing B.*
(3) *$A^\pi := \sum_{e\in\mathscr{B}_A} eA$ is the smallest π-complemented subalgebra of Q_A containing A.*

Proof. (1) Let $\{B_\alpha\}$ be a family of π-complemented subalgebras of Q_A containing A. By Theorem 2.3, $C_{B_\alpha} = C_A$ for every α, and $C_{\cap B_\alpha} = C_A$. Given $e \in \mathscr{B}_A$, by Theorem 1.8.(3), we have $eB_\alpha \subseteq B_\alpha$ for every α, and consequently $e(\cap B_\alpha) \subseteq \cap B_\alpha$. Therefore, again by Theorem 1.8.(3), $\cap B_\alpha$ is π-complemented.

(2) Let B be a subalgebra of Q_A containing A. By Corollary 2.4, the family of all π-complemented subalgebras of Q_A containing B is nonempty. Now, by part (1) above, the intersection of this family is a π-complemented algebra. Clearly, this algebra is the smallest π-complemented subalgebra of Q_A containing B.

(3) It is clear that $A^\pi := \sum_{e\in\mathscr{B}_A} eA$ is a subalgebra of Q_A containing A. By Theorem 2.3, $C_{A^\pi} = C_A$. Moreover, for each $e \in \mathscr{B}_A$, we see that $eA^\pi \subseteq A^\pi$, and hence A^π is a π-complemented algebra because of Theorem 1.8.(3). Finally, given a π-complemented subalgebra B of Q_A containing A, again keeping in mind Theorems 2.3 and 1.8.(3), we see that $eA \subseteq eB \subseteq B$ for every $e \in \mathscr{B}_A$, and consequently, $A^\pi \subseteq B$. Thus A^π is the smallest π-complemented subalgebra of Q_A containing A. □

The algebra A^π associated to each semiprime algebra A in the above corollary appears in the literature with the name of *idempotent closure* of A [16, §.32.5].

Corollary 2.6. *A semiprime algebra A is π-complemented if, and only if, $A = A^\pi$.*

3. THE IDEMPOTENT CLOSURE OF A C^*-ALGEBRA

Let A be an algebra. Given elements a, b, c in A, we put $[a, b] := ab - ba$ for the commutator and $[a, b, c] := (ab)c - a(bc)$ for the associator. Recall that the *centre* of A is defined as the set

$$Z_A := \{z \in A : [z, a] = [z, a, b] = [a, z, b] = [a, b, z] = 0 \text{ for all } a, b \in A\}.$$

Z_A is a commutative associative subalgebra of A. If A has zero annihilator, then the map $z \mapsto L_z$ allows us to regard Z_A as a subalgebra of Γ_A. Also recall that the *symmetric Martindale algebra of quotients* of a semiprime associative algebra A, denoted here by $Q_s(A)$, can be introduced as the associative algebra which is the maximal extension Q of A satisfying the following conditions:

(Q1) For each $q \in Q$ there exists an essential ideal D of A such that

$$qD + Dq \subseteq A;$$

(Q2) If $q \in Q$ satisfies $qD = 0$ for some essential ideal D of A, then $q = 0$.

It is well-known that, if A is a semiprime associative algebra, then C_A is the centre of $Q_s(A)$, and Q_A is the C_A-subalgebra of $Q_s(A)$ generated by A.

For any C^*-algebra A, $Q_s(A)$ becomes a unital algebra with positive-definite involution $*$, so that it is possible to consider the $*$-subalgebra $Q_b(A)$ of $Q_s(A)$ consisting of all order-bounded elements. $Q_b(A)$ is called the *bounded symmetric algebra of quotients* of A, and its centre $C_b(A)$ is called the *bounded extended centroid* of A. Moreover, $Q_b(A)$ is a pre-C^*-algebra, whose completion is the *algebra of local multipliers* of A, which will be denoted here by $\mathrm{Mult}_{\mathrm{loc}}(A)$. The C^*-subalgebra ${}^cA$ of $\mathrm{Mult}_{\mathrm{loc}}(A)$ generated by $C_b(A)A$ is called the *bounded central closure* of A. For a comprehensive treatment and for references to the extensive literature on the subject, we refer to the book [1] by P. Ara and M. Mathieu. Now, some results in [1, Chapter 3] allow us to realize that the idempotent closure of a C^*-algebra is within the bounded central closure.

Proposition 3.1. *Let A be a C^*-algebra. Then A^π is a norm dense $*$-subalgebra of ${}^cA$.*

Proof. For convenience in the following, we shall abbreviate $Z_{\mathrm{Mult}_{\mathrm{loc}}(A)}$ by Z. By [1, Remark 2.2.9.1 and Lemma 3.1.2], $\mathscr{B}_A$ is the set of all projections in Z. Since, by [1, Proposition 3.1.5], Z is an AW^*-algebra, Proposition 8.1 of [4] applies, so that Z is the norm closed linear span of $\mathscr{B}_A$. On the other hand, by the local Dauns–Hofmann theorem [1, Theorem 3.1.1], ${}^cA$ equals the norm closure of ZA. Keeping in mind that $A^\pi = \mathscr{B}_A A$ (by Corollary 2.5.(3)), it follows that A^π becomes a norm dense $*$-subalgebra of ${}^cA$. □

A C^*-algebra A is said to be *boundedly centrally closed* if ${}^cA = A$. Since ${}^cA$ is a C^*-algebra containing A as a C^*-subalgebra, we derive the following corollary.

Corollary 3.2. *A C^*-algebra is π-complemented if, and only if, it is boundedly centrally closed.*

Relevant examples of boundedly centrally closed C^*-algebras are the AW^*-algebras, and in particular the W^*-algebras [1, Example 3.3.1.2]. As shown in [13, Corollary 2.9] (see also [1, Corollary 6.3.5]), boundedly centrally closed C^*-algebras become the better C^*-algebras concerning the behaviour of surjective Jordan-homomorphisms.

ACKNOWLEDGMENTS

This work is partially supported by Junta de Andalucía grants FQM 0199, FQM 0290, and FQM 3737, and Projects I+D MCYT-2006-15546-C02-02, MTM-2007-65959, and MTM-2009-12067.

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