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Classes of complete intersection numerical semigroups

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0. Introduction.

The concept of complete intersection is one of the most prominent in algebraic geometry.

Complete intersection numerical semigroups were introduced by Herzog (1970) and have been studied extensively since then by many authors.

Several classes of complete intersection numerical semigroups arise from many different contexts, e.g. algebraic geometry, commutative algebra, coding theory, factorization theory.

The computations were performed by using GAP.

1. Basic definitions and notations.

S submonoind of $\mathbb{N}$ is a numerical semigroup if $|\mathbb{N} \setminus S| < \infty$.

$f = f(S)$:=max$(\mathbb{Z} \setminus S)$ (Frobenius number)

$m = m(S)$:=min$(S \setminus \{0\})$ (multiplicity)

$\nu = \nu(S)$ embedding dimension, $\qquad S = \langle g_1, \ldots, g_\nu \rangle$

The Apéry set of S with respect to m is

$$\mathrm{Ap}(S) = \{s \in S \,|\, s - m \notin S\}$$

Remark. The largest element in Ap(S) is $f(S) + m$.

S is complete intersection if $\Bbbk[[t^S]]$ is complete intersection; equivalently: if the cardinality of any of its minimal presentations equals $\nu(S) - 1$.

A representation of $s \in S$ is $s = \lambda_1 g_1 + \cdots + \lambda_\nu g_\nu$ with $\lambda_i \in \mathbb{N}$.

$\mathrm{ord}(s) = \max\{\sum_{i=1}^{\nu} \lambda_i \mid \sum_{i=1}^{\nu} \lambda_i g_i$ is a representation of $s\}$.

$s = \lambda_1 g_1 + \cdots + \lambda_\nu g_\nu$ is a maximal repres. if $\sum_{i=1}^{\nu} \lambda_i = \mathrm{ord}(s)$.

2. Some new classes.

Let $S = \langle g_1 < \cdots < g_\nu \rangle$. For each $i = 2, \ldots, \nu$, define:

$\alpha_i = \alpha_i(S) = \max\{h \in \mathbb{N} \,|\, hg_i \in \mathrm{Ap}(S)\}$;
$\beta_i = \beta_i(S) = \max\{h \in \mathbb{N} \,|\, hg_i \in \mathrm{Ap}(S)$ and $\mathrm{ord}(hg_i) = h\}$;
$\gamma_i = \gamma_i(S) = \max\{h \in \mathbb{N} \,|\, hg_i \in \mathrm{Ap}(S)$, $\mathrm{ord}(hg_i) = h$ and
hg_i has a unique maximal representation$\}$.

Proposition. $\forall\ i = 2, \ldots, \nu$, we have: $\gamma_i \leq \beta_i \leq \alpha_i$. Moreover,

$$\mathrm{Ap}(S) \subseteq \Gamma := \left\{ \sum_{i=2}^{\nu} \lambda_i g_i \,|\, 0 \leq \lambda_i \leq \gamma_i \right\} \subseteq$$

$$\subseteq B := \left\{ \sum_{i=2}^{\nu} \lambda_i g_i \,|\, 0 \leq \lambda_i \leq \beta_i \right\} \subseteq A := \left\{ \sum_{i=2}^{\nu} \lambda_i g_i \,|\, 0 \leq \lambda_i \leq \alpha_i \right\}$$

In particular, we have

$$m = |\mathrm{Ap}(S)| \leq \prod_{i=2}^{\nu}(\gamma_i + 1) \leq \prod_{i=2}^{\nu}(\beta_i + 1) \leq \prod_{i=2}^{\nu}(\alpha_i + 1)$$

Definition. S has α-rectangular (resp. β-rect. or γ-rect.) Apéry set if $\mathrm{Ap}(S) = A$ (resp. $\mathrm{Ap}(S) = B$ or $\mathrm{Ap}(S) = \Gamma$).

Theorem.
$gr_{\mathfrak{m}}(k[[t^S]])$ c.i. $\Leftrightarrow$ $gr_{\mathfrak{m}}(k[[t^S]])$ CM + γ-rect. Apéry set.

In particular, γ-rectangular Apéry set $\Rightarrow$ S c.i. Hence:

$$\alpha\text{-rect.} \;\Rightarrow\; \beta\text{-rect.} \;\Rightarrow\; \gamma\text{-rect.} \;\Rightarrow\; S \text{ c.i.}$$

3. Other (known) classes.

$S = \langle g_1 < \cdots < g_\nu \rangle$. For each $i = 2, \ldots, \nu$, define:

$\tau_i = \tau_i(S) = \min\{h \in \mathbb{N} \,|\, hg_i \in \langle g_1, \ldots, g_{i-1} \rangle\} - 1$;

Definition. • S is telescopic if $\mathrm{Ap}(S) = \left\{ \sum_{i=2}^{\nu} \lambda_i g_i \,|\, 0 \leq \lambda_i \leq \tau_i \right\}$;
• S is associated to a plane branch if S is telescopic and
$(\tau_i + 1)g_i < g_{i+1}$ (for all $i = 2, \ldots, \nu - 1$).

If $\mathbf{n} = \{n_1, \ldots, n_\nu\}$ is any rearrangement of the minimal generators (i.e. $n_1, \ldots, n_\nu$ are not necessarily in increasing order), define:
$\phi_i = \phi_i(S, \mathbf{n}) = \min\{h \in \mathbb{N} \,|\, hn_i \in \langle n_1, \ldots, n_{i-1} \rangle\} - 1$.

Definition. S is free if there exists a rearrangement $\mathbf{n} = \{n_1, \ldots, n_\nu\}$ of the minimal generators such that

$$\mathrm{Ap}(S, n_1) = \left\{ \sum_{i=2}^{\nu} \lambda_i n_i \,|\, 0 \le \lambda_i \le \phi_i \right\}$$

Notice that the definitions of telescopic and free semigroups are not standard, but are equivalent to the classical definitions.

Well know facts for a numerical semigroup S:

$$S \text{ plane branch} \;\Rightarrow\; S \text{ telescopic} \;\Rightarrow\; S \text{ free} \;\Rightarrow\; S \text{ c.i.}$$

4. Some characterizations.

Proposition. The following conditions are equivalent:

(i) Ap(S) is α-rectangular;

(ii) $f + m = \sum_{i=2}^{\nu} \alpha_i g_i$;

(iii) $m = \prod_{i=2}^{\nu} (\alpha_i + 1)$.

Proposition. The following conditions are equivalent:

(i) Ap(S) is β-rectangular;

(ii) $f + m = \sum_{i=2}^{\nu} \beta_i g_i$;

(iii) $m = \prod_{i=2}^{\nu} (\beta_i + 1)$.

Proposition. The following conditions are equivalent:

(i) Ap(S) is γ-rectangular;

(ii) $f + m = \sum_{i=2}^{\nu} \gamma_i g_i$;

(iii) $m = \prod_{i=2}^{\nu} (\gamma_i + 1)$.

Proposition. (cf. Rosales-Garcia Sanchez, Proposition 9.15) The following conditions are equivalent:

(i) S is telescopic;

(ii) $f + m = \sum_{i=2}^{\nu} \tau_i g_i$;

(iii) $m = \prod_{i=2}^{\nu} (\tau_i + 1)$.

Proposition. (cf. Rosales-Garcia Sanchez, Proposition 9.15) The following conditions are equivalent:

(i) S is free;

(ii) there is a rearrangement $\mathbf{n}$ of the minimal generators such that $f + n_1 = \sum_{i=2}^{\nu} \phi_i n_i$;

(iii) there is a rearrangement $\mathbf{n}$ of the minimal generators such that $n_1 = \prod_{i=2}^{\nu} (\phi_i + 1)$.

5. One more class.

S has a unique Betti element (briefly S u.B.e) if the first syzygies of the semigroup ring $k[[t^S]]$ have all the same degree (in the S-grading).

Proposition. (Garcia-Sanchez, Ojeda, Rosales) $S = \langle g_1, \dots, g_\nu \rangle$ has a unique Betti element $\iff$ $\exists$ pairwise co-prime integers $a_1, \dots, a_\nu$ ($a_i > 1$), such that $g_i = \prod_{j \neq i} a_i$.

Proposition. (Barucci-Froberg) S u.B.e. $\Rightarrow$ $gr_{\mathfrak{m}}(k[[t^S]])$ c.i.

Hence:

S u.B.e. $\Rightarrow$ Ap(S) γ-rectangular

6. The main result.

Theorem. Let S be a numerical semigroup. Then:

$$
\begin{tikzcd}
S \text{ u.B.e.} \arrow[r, Rightarrow] & \alpha\text{-rect.} \arrow[r, Rightarrow] & \beta\text{-rect.} \arrow[r, Rightarrow] & \gamma\text{-rect.} \arrow[r, Rightarrow] & S \text{ free} \arrow[d, Rightarrow] \\
& S \text{ pl. br.} \arrow[u, Rightarrow] \arrow[r, Rightarrow] & S \text{ telesc.} \arrow[u, Rightarrow] & & S \text{ c.i.} \\
& & S \text{ u.B.e.} \arrow[u, Rightarrow] & &
\end{tikzcd}
$$

Moreover, all the implications are strict.

- Unique Betti element $\Rightarrow$ Telescopic.

Let $a_1 > a_2 > \cdots > a_\nu > 1$ be pairwise coprime integers such that $g_i = \prod_{j \neq i} a_j$. Then $\tau_i = a_i - 1$ for each $i \geq 2$ and S is telescopic.

- Unique Betti element $\Rightarrow$ α-rectangular Apéry set.

Let $a_1 > a_2 > \cdots > a_\nu > 1$ be pairwise coprime integers such that $g_i = \prod_{j \neq i} a_j$. Then $\alpha_i + 1 \leq a_i$.

$S = \langle 4, 6, 13 \rangle$: we have $\mathrm{Ap}(S) = \{0, 6, 13, 19\}$, $\tau_2 = \tau_3 = 1$, and $m = (\tau_2 + 1)(\tau_3 + 1)$, $(\tau_2 + 1)g_2 < g_3$. So S is associated to a plane branch and hence telescopic (and with α-rectangular Apéry set), but S does not have a unique Betti element.

• Plane branch $\Rightarrow$ α-rectangular Apéry set.
Use induction on $i \in \{2, \ldots, \nu\}$, to prove that $(\tau_i + 1)g_i \notin \mathrm{Ap}(S)$.
Hence $\alpha_i \leq \tau_i$. Thus $\mathrm{Ap}(S)$ is α-rectangular as

$$m \leq \prod_{i=2}^{\nu} (\alpha_i + 1) \leq \prod_{i=2}^{\nu} (\tau_i + 1) = m$$

(the last equality holds since S is telescopic).

• Telescopic $\Rightarrow$ β-rectangular Apéry set.
$(\tau_i + 1)g_i = \lambda_1 g_1 + \cdots + \lambda_{i-1} g_{i-1} \Rightarrow \mathrm{ord}((\tau_i + 1)g_i) > \tau_i + 1$ and $\beta_i \leq \tau_i$: then $\mathrm{Ap}(S)$ β-rectangular as

$$m \leq \prod_{i=2}^{\nu} (\beta_i + 1) \leq \prod_{i=2}^{\nu} (\tau_i + 1) = m.$$

$S = \langle 4, 5, 6 \rangle$: $\mathrm{Ap}(S) = \{0, 5, 6, 11\}$, $\alpha_2 = \alpha_3 = 1$, $\tau_2 = 3, \tau_3 = 1$;
so $\mathrm{Ap}(S)$ is α-rect. (hence β-rect.), but S is not telesc.

• γ-rectangular Apéry set $\Rightarrow$ Free.

Lemma. Let S have γ-rectangular Apéry set.
For each $i = 2, \ldots, \nu$, there exist relations

$$(\star) \qquad (\gamma_i + 1)g_i = \lambda_{i,1}g_1 + \lambda_{i,2}g_2 + \cdots + \lambda_{i,\nu}g_\nu$$

and $\sigma \in S_\nu$ s.t. $\sigma(1) = 1$ and $\lambda_{\sigma(i),\sigma(j)} = 0$ if $i \leq j$, $j \geq 2$.

Consider the rearrangement $\mathbf{n} = \{n_1, \ldots, n_\nu\}$ with $n_i = g_{\sigma(i)}$.

$$(\gamma_{\sigma(i)} + 1)n_i = (\gamma_{\sigma(i)} + 1)g_{\sigma(i)} = \sum_{j=1}^{\nu} \lambda_{\sigma(i),j}g_j =$$

$$= \sum_{j=1}^{\nu} \lambda_{\sigma(i),\sigma(j)}g_{\sigma(j)} = \sum_{j=1}^{\nu} \lambda_{\sigma(i),\sigma(j)}n_j$$

thus $\phi_i \leq \gamma_{\sigma(i)}$ by the triangularity of the matrix $\mathbf{L}_\sigma = (\lambda_{\sigma(i),\sigma(j)})$ (with $i, j = 2, \ldots, \nu$).

Let $\overline{c}_i = \min \left\{ h \in \mathbb{N} \setminus \{0\} \;\middle|\; \mathrm{gcd}(n_1, \ldots, n_{i-1}) \text{ divides } hn_i \right\}$.

Well known fact: $n_1 = \prod_{i=2}^{\nu} \overline{c}_i$ and $\overline{c}_i \leq \phi_i + 1$.

On the other hand Ap(S) Γ-rect. $\Rightarrow$

$$n_1 = g_1 = m = \prod_{i=2}^{\nu} (\gamma_i + 1) = \prod_{i=2}^{\nu} (\gamma_{\sigma(i)} + 1).$$

Hence $n_1 = \prod_{i=2}^{\nu} \overline{c}_i \leq \prod_{i=2}^{\nu} (\phi_i + 1) \leq \prod_{i=2}^{\nu} (\gamma_{\sigma(i)} + 1) = n_1 \Rightarrow$ S free.

$S = \langle 5, 6, 9 \rangle$. Since 5 is prime Ap($S$) is not γ-rectangular. Consider the rearrangement $\mathbf{n} = \{6, 9, 5\}$: we have $\phi_2 = 1, \phi_3 = 2$ so S is free as $n_1 = (\phi_2 + 1)(\phi_3 + 1)$.

7. More counterexamples.

• Plane branch $\not\Leftarrow$ Telescopic.

$S = \langle 6, 10, 15 \rangle$ is not associated to a plane branch, as $(\tau_2+1)g_2 = 3 \cdot 10 > 15 = g_3$; however S has a unique Betti element, in particular it is telescopic and with α-rectangular Apéry set.

• α-rect. $\not\Leftarrow$ β-rect. $\not\Leftarrow$ γ-rect.

$S = \langle 8, 10, 15 \rangle$ is telescopic, hence Ap(S) is β-rectangular, but it is not α-rectangular: Ap$(S) = \{0, 10, 15, 20, 25, 30, 35, 45\}$, $\alpha_2 = \alpha_3 = 3$ and $\tau_2 = 3, \tau_3 = 1$ so that $m = (\tau_2 + 1)(\tau_3 + 1)$ but $m \neq (\alpha_2 + 1)(\alpha_3 + 1)$.

$S = \langle 8, 10, 11, 12 \rangle$ is γ-rect. but not β-rect.:
Ap$(S) = \{0, 10, 11, 12, 21, 22, 23, 33\}$, $\beta_2 = 1, \beta_3 = 3, \beta_4 = 1$,
$\gamma_2 = \gamma_3 = \gamma_4 = 1$, hence $m = (\gamma_2 + 1)(\gamma_3 + 1)(\gamma_4 + 1)$ and $m \neq (\beta_2 + 1)(\beta_3 + 1)(\beta_4 + 1)$.

8. Gluing.

Let S_1 and S_2 be two numerical semigroups minimally generated by $n_1, \ldots, n_r$ and $m_1, \ldots, m_s$, respectively.
Given $0 \neq d_1 \in S_1 \setminus \{n_1, \ldots, n_r\}$ and $0 \neq d_2 \in S_2 \setminus \{m_1, \ldots, m_s\}$ such that $\gcd(d_1, d_2) = 1$, the semigroup

$$S = d_2 S_1 + d_1 S_2 = \langle d_2 n_1, \ldots, d_2 n_r, d_1 m_1, \ldots, d_1 m_s \rangle$$

is called a gluing of S_1 and S_2. Notice that $\nu(S) = \nu(S_1) + \nu(S_2)$.

Theorem. (Delorme) S c.i. if and only if it is a gluing of two c.i. semigroups.
Theorem. (cf. Rosales-Garcia Sanchez book) S (with emb. dim. ν) is free $\Leftrightarrow$ it is a gluing of $\mathbb{N}$ and a free semigroup of emb. dim. $\nu - 1$.

Using these facts it is possible to produce examples of free, non complete intersection semigroups.

Similar results to the free semigroup case hold for telescopic, plane branch and u.B.e. semigroups (with appropriate conditions on d_1 and d_2).

Theorem. Let T be α-rect. and let $d_1, d_2 \in \mathbb{N}$ such that $d_1 \notin$ Ap(T), $d_1 > d_2 m(T)$; then the gluing $S = d_2 T + d_1 \mathbb{N}$ is α-rect. Conversely, every semigroup $S \neq \mathbb{N}$ α-rect. arises in this way.

Question. Is it possible to characterize semigroups with β-rectangular and γ-rectangular Apéry set in terms of gluing?

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