INdAM workshop: International meeting on numerical semigroups - Roma 2022

Abstracts

Abdallah Assi, Université d'Angers, *Semigroup associated with a free polynomial* arXiv

Given a hypersurface V in $\mathbb{C}^n$, Abyhankar-Sathaye conjecture says the following; if V is isomorphic to a coordinate of $\mathbb{C}^n$, then there exists an automorphism σ of $\mathbb{C}^n$ such that $\sigma(f)$ is a coordinate. The conjecture is known to be true if $n=2$ by Abhyankar-Moh theory, which is based on the arithmetic of the numerical semigroup associated with $f \in \mathbb{C}[X,Y]$, where $V=V(f)$ and f has one place at infinity. The conjecture remains open for $n \geq 3$. The aim of this talk is to prove that, given $F \in A=\mathbb{C}[\![x_1,\cdots,x_n]\!]$, we can associate with F, up to a preparation theorem and under an additional condition on the irreducibility in an extention of A, a semigroup in $\mathbb{N}^n$. It turns out that this semigroup is free. Moreover, it satisfies the fundamental results of Abhyankar-Moh theory.

This is a joint work with *A. Abbas*, Journal of Algebra.

Matheus Bernardini, Universidade de Brasília, *Gapsets and m-extensions* arXiv

In this talk, we discuss Bras-Amorós conjecture and we bring two approaches to deal with the sequence (n_g), where n_g denotes the number of gapsets (or numerical semigroups) with genus g. In the first part, we introduce the concept of pure κ-sparse gapsets and we present some results about the cardinality of the set of κ-sparse gapsets with fixed genus. In the second part, we use the Kunz coordinates of a gapset to identify a gapset with genus g with a tiling of a g-board and, more generally, we obtain a bijective map between the set of m-extensions with genus g with the set of tillings of a g-board. In particular, this method provides lower and upper bounds for the number of gapsets with genus g (and for the number of gapsets with genus g and depth at most 3) and a proved version of Bras-Amorós conjecture for m-extensions.

This is a joint work with *Gilberto B. Almeida Filho*.

Alessio Borzì, University of Warwick, *Cyclotomic Numerical Semigroups and Graded Algebras* arXiv arXiv

The semigroup polynomial of a numerical semigroup S is $P_S(x)=1+(x-1)\sum_{g\notin S} x^g$. A numerical semigroup is cyclotomic if its semigroup polynomial is a product of cyclotomic polynomials, that is, all its roots are roots of unity. It is conjectured that every cyclotomic numerical semigroup is complete intersection.

In this talk we show that the conjecture is true for semigroups with semigroup polynomial with at most two irreducible factors (this part is joint work with *Andrés Herrera-Poyatos* and *Pieter Moree*). Moreover we present a generalization of this problem to graded algebras. While in this more general context it is not hard to find examples of cyclotomic algebras that are not complete intersection, we show that the analogue conjecture holds true for Koszul algebras (this part is joint work with *Alessio D'Alì*).

Maria Bras-Amorós, Universitat Rovira i Virgili, *On the seeds and the great-grandchildren of a numerical semigroup*

We present a revisit of the seeds algorithm to explore the semigroup tree. First, an equivalent definition of seed is presented, which seems easier to manage. Second, we determine the seeds of semigroups with at most three left elements. And third, we find the great-grandchildren of any numerical semigroup in terms of its seeds.

The algorithm has been used to prove that there are no Eliahou semigroups of genus 66, hence proving the Wilf conjecture for genus up to 66. We also found three Eliahou semigroups of genus 67. One of these semigroups is neither of Eliahou-Fromentin type, nor of Delgado's type. However, it is a member of a new family suggested by Shalom Eliahou.

Scott Chapman, Sam Houston State University, *Length Density on Numerical Monoids* arXiv

Length density is a recently introduced factorization invariant, assigned to each element n of a cancellative commutative atomic semigroup S, that measures how far the set of factorization lengths of n is from being a full interval. We examine length density of elements of numerical monoids.

Laura Cossu, University of Graz, *Factorization spaces and their elasticity*

Factorization spaces (F-spaces for short) are a unifying framework for a variety of problems that involve the decomposition of *certain* elements of a monoid into a product of *certain* other elements (below referred to as "building blocks") that are in some sense "minimal". Examples of "factorization problems" of this sort include those (classically studied in factorization theory) in which the building blocks are atoms, i.e., non-unit elements of a monoid that are not products of two non-units, but also cyclic decompositions of permutations in the symmetric group of degree n; factorizations into idempotents of the "singular elements" of a monoid; additive decompositions into multiplicative units in rings; and so on.

Formally, we define an F-space Φ as an ordered triple consisting of a morphism η between two monoids H and K each endowed with a preorder (i.e., a reflexive and transitive binary relation on their underlying set), a set $A \subseteq H$, and a preorder $\sqsubseteq$ on the free monoid $\mathscr{F}(A)$ on A, all satisfying some "natural conditions". Consequentely, a Φ*-factorization* of an element $x \in H$ is a word $\mathfrak{a} \in \mathscr{F}(A)$ that is $\sqsubseteq$-minimal in the inverse image of x under the canonical homomorphism $\mathscr{F}(A) \to H$. The elements of A are then the "building blocks" of a Φ-factorization, while the morphism η (called the *seminorm* of Φ) is used to "measure" its length. For an F-space Φ we address the following questions: Is there a non-trivial characterization of the elements of H that have at least one Φ-factorization? How to qualify and quantify the non-uniqueness properties of a Φ-factorization?

Sensible answers can be provided under specific features of the "components" of Φ but also in some generality. In particular, we define the *elasticity* $\varrho(\Phi)$ of an F-space Φ and, among other things, we show that $\varrho(\Phi)$ is finite if H is commutative, A is finite, and η and $\sqsubseteq$ satisfy some "reasonable properties". Our result generalizes (from commutative unit-cancellative to arbitrary commutative monoids) [3, Proposition 3.4(1)].

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Manuel Delgado, Universidade do Porto, *On the contribution of Herbert Wilf to the development of the theory of Numerical semigroups*

Among many results published in various areas, most of them related to combinatorics, Wilf published in the American Mathematical Monthly an algorithm to compute the Frobenius number of a numerical semigroup.

At the end of the paper, which appeared in 1978, Wilf left two questions. One related to counting numerical semigroups and another currently known in the field as Wilf's conjecture.

Much of the recent work on numerical semigroups is motivated by these problems.

By surveying some results, I will give a personal view on the relevance of the questions stated by Wilf to the development of the area of numerical semigroups, namely in the discovery of connections with other areas (graphs, statistics, algorithms, etc.).

Felix Delgado de la Mata, Universidad de Valladolid, *On the coefficients of the Poincaré series of good semigroups*

For a curve singularity with $r > 1$ branches, its semigroup of values is a submonoid S of $(\mathbb{N}^r, +)$. It is the natural generalization of the well known semigroup (a numerical one in this case) for the case of an irreducible curve. The Poincaré series (in fact a polynomial) in r indeterminates $P_S(t_1, \ldots, t_r)$ of the former also generalizes the characteristic series of the latter and in the case of plane curves is a complete invariant of equisingularity, in particular it contains the same information as the semigroup of values.

Good semigroups are the natural arithmetic extension of the multi-branch semigroups of values and are therefore also they are a natural extension of the numerical semigroups following valuative ideas. The Poincaré series can also be defined for a good semigroup, however in this case it is far from containing the same information as the semigroup. It is also relevant that not all the good semigroups are semigroups of values of curves, in contrast with the case of numerical semigroups and irreducible curve germs. In this context it is natural to deepen the study of the information contained in the Poincaré series and to what extent, or under what conditions, it allows to recover the semigroup.

In this lecture we will see some of the relations between the Poincaré series P_S and the semigroup S, in particular how are the coefficients of the Poincaré series, establishing the analogy with the case of curves and its semigroups of values.

Shalom Eliahou, Université du Littoral Côte d'Opale, *Some remarks on Wilf's conjecture*

Let S be a numerical semigroup with multiplicity m, conductor c and embedding dimension e. In this talk, we shall propose to consider the special case where m divides c as a key target case for future progress on Wilf's conjecture. As motivation, we shall show how this special hypothesis allows for quite simpler proofs in recently settled cases of the conjecture, for instance when $c \leq 3m$ or $e \geq m/3$.

Naoki Endo, Meiji University, *Ulrich ideals and numerical semigroup rings* arXiv

Ulrich ideals with smallest number of generators possess a beautiful structure of minimal free resolutions. In this talk, we investigate the behavior of the Ulrich ideals, especially in numerical semigroup rings. If the semigroups are three-generated but not symmetric, the semigroup rings are Golod, since the Betti numbers of the residue class fields of the semigroup rings form an arithmetic progression; therefore, these semigroup rings are G-regular, possessing no Ulrich ideals. Nevertheless, even in the three-generated case, the situation is different, when the semigroups are symmetric. We shall explore this phenomenon, describing an explicit system of generators, that is the normal form of generators, for the Ulrich ideals in the numerical semigroup rings of multiplicity at most 3. As the multiplicity is greater than 3, in general the task of determining all the Ulrich ideals seems formidable, which we shall experience, analyzing some of the simplest examples.

This talk is based on recent research jointly with **S. Goto**.

Marcelo Escudeiro Hernandes, Universidade Estadual de Maringá, *On the Analytic Invariants and Semiroots of Plane Branches* arXiv

The value semigroup of a k-semiroot C_k of a plane branch C allows us to recover part of the value semigroup $\Gamma = \langle v_0, \ldots, v_g \rangle$ of C, that is, it is related to topological invariants of C. In this talk we consider the set of values of differentials Λ_k of C_k, that is an analytical invariant, and we show how it determine part of the set of values of differentials Λ of C. As a consequence, in a fixed topological class, we relate the Tjurina number τ of C with the Tjurina number of C_k. In particular, we show that $\tau \leq \mu - \frac{3n_g-2}{4}\mu_{g-1}$ where $n_g = gcd(v_0, \ldots, v_{g-1})$, μ and μ_{g-1} denote the Milnor number of C and C_{g-1} respectively. If $n_g = 2$, we have that $\tau = \mu - \mu_{g-1}$ for any curve in the topological class determined by Γ that is a generalization of a result obtained by Luengo and Pfister.

This is a joint work with *Marcelo Osnar Rodrigues de Abreu*.

Leonid Fel, Technion - Israel Institute of Technology, *Genera of numerical semigroups and polynomial identities for degrees of syzygies* arXiv

We derive a polynomial identities of arbitrary degree n for syzygies degrees of numerical semigroups $S_m = \langle d_1, \ldots, d_m \rangle$ and show that for $n \geq m$ they contain higher genera $G_r = \sum_{s \in \mathbb{Z}_> \setminus S_m} s^r$ of semigroup. We find a minimal number $\mathsf{g}_m = B_m - m + 1$ of algebraically independent genera G_r and equations, related any of $\mathsf{g}_m + 1$ genera, where $B_m = \sum_{k=1}^{m-1} \beta_k$ and β_k denote the total and partial Betti numbers of S_m. The number g_m decreases with a growth of semigroup's symmetry and reaches a value $m-2$, when S_m becomes a complete intersection.

Ignacio García Marco, Universidad de La Laguna, *Robustness and related properties on numerical semigroups*

Every numerical semigroup $\mathcal{S} = \langle a_1, \ldots, a_n \rangle \subset \mathbb{N}$ has associated a (toric) ideal $I_{\mathcal{S}}$.

In this work we characterize when $I_{\mathcal{S}}$ satisfies the following properties:

(a) $I_{\mathcal{S}}$ has a complete intersection (CI) initial ideal.

Since $I_{\mathcal{S}}$ is a height $n-1$ ideal. Then, it has a CI initial ideal if and only if there is a monomial order $\prec$ such that the corresponding reduced Gröbner basis $\mathcal{G}_{\prec}$ of $I_{\mathcal{S}}$ has $n-1$ elements.

(b) $I_{\mathcal{S}}$ is robust or generalized robust.

A toric ideal is robust if its universal Gr"obner basis is a minimal set of generators, and is generalized robust if its universal Gröbner basis equals its universal Markov basis (the union of all its minimal sets of binomial generators). Robust and generalized robust toric ideals are both interesting from both a Commutative Algebra and an Algebraic Statistics perspective.

The results of this talk are summarized in the following diagram:

$$\begin{array}{ccc} \mathcal{S} = \langle a_1, a_2 \rangle & \Longleftrightarrow & \mathcal{S} \text{ is robust} \\ \Downarrow & & \\ \mathcal{S} \text{ has a unique Betti element} & \Longleftrightarrow & \mathcal{S} \text{ is generalized robust} \\ \Downarrow & & \\ \mathcal{S} \text{ is free} & \Longleftrightarrow & I_{\mathcal{S}} \text{ has a CI initial ideal} \end{array}$$

We finish with some open questions.

This is a joint work with *Christos Tatakis*.

Alfred Geroldinger, University of Graz, *On monoids of ideals* arXiv

The talk will discuss the algebraic and arithmetic structure of monoids of invertible ideals of Krull domains and of weakly Krull Mori domains. We also investigate monoids of all nonzero ideals of polynomial rings with at least two indeterminates over noetherian domains. Among others, we show that they are not transfer Krull but they share several arithmetic phenomena with Krull monoids having infinite class group and prime divisors in all classes.

This is joint work with *Azeem Khadam*.

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Philippe Giménez, Universidad de Valladolid, *Consequences and applications of gluing semigroup rings in dimension n* arXiv OA

A semigroup $\langle C \rangle$ in $\mathbb{N}^n$ is a gluing of $\langle A \rangle$ and $\langle B \rangle$ if its finite set of generators C splits into two parts, $C = k_1 A \sqcup k_2 B$ with $k_1, k_2 \geq 1$, and the defining ideals of the corresponding semigroup rings satisfy that I_C is generated by $I_A + I_B$ and one extra element. Two semigroups $\langle A \rangle$ and $\langle B \rangle$ can be glued if there exist positive integers k_1, k_2 such that, for $C = k_1 A \sqcup k_2 B$, $\langle C \rangle$ is a gluing of $\langle A \rangle$ and $\langle B \rangle$.

Although any two numerical semigroups can always be glued, it is no longer the case in higher dimensions. In this talk, we will give necessary and sufficient conditions on A and B for the existence of a gluing of $\langle A \rangle$ and $\langle B \rangle$, and give examples to illustrate why they are necessary. These generalize and explain the previous known results on existence of gluing. We also show that the glued semigroup $\langle C \rangle$ inherits the properties like Gorenstein or Cohen-Macaulay from the two parts $\langle A \rangle$ and $\langle B \rangle$.

This talk is based on a series of joint papers and ongoing work with *Hema Srinivasan* (University of Missouri, Columbia, EE.UU.).

Felix Gotti, MIT, *On the arithmetic of positive cyclic semirings* arXiv

Numerical semigroups are additive submonoids of the nonnegative real ray and are closed under multiplication. For a real positive r, we can consider the smallest additive submonoid $S(r)$ consisting of nonnegative real numbers that is closed under multiplication and contains r. We call $S(r)$ the positive cyclic semiring generated by r. Many arithmetic and factorization properties of $S(r)$ have been recently studied by various authors. In this talk, we will discuss several aspects related to the arithmetic and divisibility of positive cyclic semirings.

Most of this talk is based on a joint work with *J. Correa-Morris* and has been recently published in Journal of Pure and Applied Algebra.

Lorenzo Guerrieri, Jagiellonian University, Krakow, *Good semigroups, good ideals and Apéry sets in* $\mathbb{N}^d$ arXiv

Good semigroups are defined as a family of submonoids of N^d, properly containing the family of value semigroups of rings of curves singularities. In the case d=1, they are exactly the numerical semigroups and their properties, in relation with those of the associated monomial curves have been well studied. Very useful objects for these studies are the subsets called Apéry sets.

In this talk we define a finite canonical partition for the Apéry sets (and more in general for any complement set of a good semigroup ideal) and show how various results generalize from the numerical case to the good case. This work also extends results done in a previous work only in the case $d = 2$.

(Joint work with *N. Maugeri* and *V. Micale*.)

Main references

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Raheleh Jafari, Kharazmi University, *On the Gorenstein locus of simplicial affine semigroup rings*

Let S be a simplicial affine semigroup in $\mathbb{N}^d$, $R = \mathbb{K}[S]$ be the affine semigroup ring over a field $\mathbb{K}$, and $\mathrm{cone}(S)$ be the rational polyhedral cone spanned by S. Let $E = \{\mathbf{a}_1, \ldots, \mathbf{a}_d\}$ be the set of componentwise smallest integer vectors in S, situated on each extremal ray of $\mathrm{cone}(S)$, respectively. Then, $\mathrm{Ap}(S,E) = \cap_{i=1}^d \mathrm{Ap}(S,\mathbf{a}_i)$ is a finite set.

The aim of this talk, is to characterize the Gorenstein locus of R, generalizing the Gorenstein characterization given in [1]. The results come from an analysis of Cohen-Macaulay type of homogeneous localizations at monomial prime ideals. Characterizing when the homogeneous localization at a monomial prime ideal is Gorenstein, we can read the dimension of the non-Gorenstein locus in terms of $\mathrm{Ap}(S,E)$. In particular, we
show that R is Gorenstein on the punctured spectrum precisely when, for any $1 \le i \le d$, there exist $\mathbf{m}_i \in \mathrm{Ap}(S,E)$ and $\lambda_i \in \mathbb{N}$ such that $\mathbf{w} \preceq_S \mathbf{m}_i + \lambda_i \mathbf{a}_i$ for all $\mathbf{w} \in \mathrm{Ap}(S,E)$. The later property, when S is a normal affine semigroup, is equivalent to existence of integral points on every extremal ray of the sub-cone of the relative interior of $\mathrm{cone}(S)$, generated by all of its integral points. This result, gives a different argument for [3, Corollary 4.11].

This talk is mainly based on joint work with *A. Taherizadeh* and *M. Yaghmaei*, [3].

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Harold Jiménez Polo, University of Florida,*On the arithmetic of polynomial semidomains* arXiv

A subset S of an integral domain R is called a semidomain provided that the pairs $(S,+)$ and $(S,\cdot)$ are semigroups with identities. The study of factorizations in integral domains was initiated by D. D. Anderson, D. F. Anderson, and M. Zafrullah in 1990, and this area has been systematically investigated since then. In this paper, we study the divisibility and arithmetic of factorizations in the more general context of semidomains. We are specially concerned with the ascent of the most standard divisibility and factorization properties from a semidomain to its semidomain of (Laurent) polynomials. As in the case of integral domains, here we prove that the properties of satisfying ACCP, having bounded factorizations, and having finite factorizations ascend in the class of semidomains. We also consider the ascent of the property of being atomic, and we show that the property of having unique factorization ascends only when a semidomain is an integral domain. Throughout the paper we provide several examples aiming to shed some light upon the arithmetic of factorizations of semidomains.

This is a joint work with *F. Gotti.*

Nathan Kaplan, University of California - Irvine, *Numerical Semigroups Ordered by Genus*

Let n_g denote the number of numerical semigroups of genus g. In a 2008 paper, Bras-Amorós computed the first 50 values of n_g and made several conjectures about the behavior of this sequence. We will survey what is known about counting numerical semigroups by genus and highlight some recent work in this area. We will also discuss some questions about properties of a “typical” numerical semigroup of genus g.

Takao Komatsu, Zhejiang Sci-Tech University, *p-Numerical semigroup and p-Frobenius numbers*

Let $a_1, \ldots, a_k$ be positive integers with $\gcd(a_1, \ldots, a_k) = 1$. For a nonnegative integer p, the p-Frobenius number, which is a generalized Frobenius number, is largest integer n such that the linear equation $a_1x_1 + \cdots + a_kx_k = n$ has at most p nonnegative integer solutions $(x_1, \ldots, x_k)$. This implies that if n is replaced by a larger integer, then the linear equation has more than p nonnegative integer solutions. When $p = 0$, the original and famous Frobenius number is 0-Frobenius number.

In this talk, by introducing p-numerical semigroup, including p-gaps, p-Apery set, p-Hilbert series and so on, we give some p-Frobenius numbers in explicit forms for some triples from Fibonacci numbers, triangular numbers, arithmetic sequences and so on.

David Llena, Universidad de Almería, *Stratification of atoms as a way to solve Elliott's Problem* arXiv

This talk is based on a joint paper with *P.A. García-Sánchez* and *U. Krause*, [3].

In a paper from 1903, E. B. Elliott proposed the following problem: given the linear Diophantine equation $ax + by = cz$, with a, b and c positive integers, is it possible to represent all nonnegative integer solutions in a unique manner by means of a "simple set of solutions"? He was able to describe parametrically the solutions for small c with the help of generating functions: every solution to $ax + by = cz$ had a unique expression as a linear combination of some basic solutions with some coefficients free and others bounded or fulfilling some linear constraints. In [2] we gave a possible approach to this problem based on Apéry sets, but still the solution proposed was not, in general, in the spirit of Elliott's paper. In this talk, based on [3], we provide a solution that generalizes those given in [1], which is based on *a stratification of the set of atoms of the set of solutions* by successive elimination of atoms living in extremal rays.

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Daniel Marín Aragón, Universidad de Cádiz, *How to characterize affine $\mathcal{C}$-semigroups* arXiv

Let $\mathcal{C} \subset \mathbb{N}$ be a finitely generated integer cone and $S \subset \mathcal{C}$ be an affine semigroup such that the real cones generated by both of them are equal. The semigroup S is called $\mathcal{C}$-semigroup if $\mathcal{C} \setminus S$ is a finite set. We characterize this kind of semigroups from their minimal generating sets, and we give an algorithm to check if an affine semigroup is a $\mathcal{C}$-semigroup and, in this case, to compute its set of gaps. We also study the embedding dimension of $\mathcal{C}$-semigroups obtaining a lower bound for it, and introduce some families of $\mathcal{C}$-semigroups whose embedding dimension reaches our bound. We also present a method to obtain a decomposition of a $\mathcal{C}$-semigroup into irreducible $\mathcal{C}$-semigroups.

Julio J. Moyano-Fernández, Universitat Jaume I de Castellón, *An extension of the Wilf conjecture to semimodules over a numerical semigroup* arXiv

The aim of this talk is to give an extension of the Wilf conjecture to semimodules over a numerical semigroup. The key point is the introduction of a new invariant that we call the Wilf function of a numerical semigroup; this can be also defined for semimodules over the semigroup. More precisely, we will show that the natural generalization of the Wilf conjecture to semimodules over the semigroup does not work unless we use the Wilf function of the semimodule. In this direction, we propose several questions regarding the possible generalizations of Wilf-type inequalities for semimodules. We will finish the talk by considering the case in which we solved the above mentioned extension, namely the case of a semigroup with two generators and the Wilf functions associated to its gaps.

This is a joint work with *Patricio Almirón*.

María Ángeles Moreno, Universidad de Cádiz, *Counting the ideals with given genus of a numerical semigroup*

If S is a numerical semigroup, denote by $\mathrm{g}(S)$ the genus of S. A numerical semigroup T is an $\mathrm{I}(S)$-*semigroup* if $T\backslash\{0\}$ is an ideal of S. If $k \in \mathbb{N}$, then we denote by $\mathrm{i}(S,k)$ the number of $\mathrm{I}(S)$-semigroups with genus $\mathrm{g}(S)+k$.

In this work we conjecture that $\mathrm{i}(S,a) \leq \mathrm{i}(S,b)$ if $a \leq b$, and we show that there is a term from which this sequence becomes stationary. That is, there exists $k_S \in \mathbb{N}$ such that $\mathrm{i}(S,k_S) = \mathrm{i}(S,k_S+h)$ for all $h \in \mathbb{N}$. Moreover, we prove that the conjecture is true for ordinary numerical semigroups, that is, numerical semigroups which the form $\{0,m,\rightarrow\}$ for some positive integer. Additionally, we calculate the term from which the sequence becomes stationary.

Ignacio Ojeda, Universidad de Extremadura, *Generalized repunit numerical semigroups* arXiv OA

In this talk, we introduce the numerical semigroups generated by $\{a_1,a_2,\ldots\} \subset \mathbb{N}$ such that a_1 is the repunit number in base $b>1$ of length $n>3$ and $a_i - a_{i-1} = a\,b^{i-2}$, for every $i \geq 2$, where a is a positive integer relatively prime with a_1. These numerical semigroups generalize the repunit numerical semigroups among many others. We show that they have interesting properties such as being homogeneous and Wilf. We solve the Frobenius problem for this family, by giving a closed formula for the Frobenius number in terms of a, b and n, and compute other usual invariants such as the Apéry sets, the genus or the type.

Moreover, we prove that the toric ideals associated to generalize the repunit numerical semigroups are determinantal and, they have unique minimal system of generators if and only if $a < b-1$.

This is a joint work with *M. B. Branco* and *I. Colaço*.

Jorge Ramírez-Alfonsín, Université Montpellier 2, *On perfect squares and primes in numerical semigroups* arXiv

Let S be a numerical semigroup. We are interested in studying the behavior of integers, enjoying some specific property, with respect to S. Is there always a prime number belonging to S? What is the largest perfect square not belonging to S? In this talk, we present partial answers to these questions. We will also discuss some related problems.

This talk is based on joint works with *J. Chappelon* and with *M. Skałba*.

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Mesut Şahin, Hacettepe University, *Vanishing Ideals and Codes on Toric Varieties*

Motivated by applications to the theory of error-correcting codes, we give an algorithmic method for computing a generating set for the ideal generated by β-graded polynomials vanishing on a subset of a simplicial complete toric variety X over a finite field $\mathbb{F}_q$, parameterized by rational functions, where β is a $d \times r$ matrix whose columns generate a subsemigroup $\mathbb{N}\beta$ of $\mathbb{N}^d$. We also give a method for computing the vanishing ideal of the set of $\mathbb{F}_q$-rational points of X. We talk about some of its algebraic invariants related to basic parameters of the corresponding evaluation code. When $\beta = [w_1 \cdots w_r]$ is a row matrix corresponding to a numerical semigroup $\mathbb{N}\beta = \langle w_1,\ldots,w_r\rangle$, X is a weighted projective space and generators of its vanishing ideal is related to the generators of the defining (toric) ideal of the numerical semigroup rings corresponding to semigroups generated by subsets of $\{w_1,\ldots,w_r\}$.

Indranath Sengupta, Indian Institute of Technology Gandhinagar, *Projective Closure of Numerical Semigroup Rings*

We know examples of families of Numercial Semigroups (equivalently affine monomial curves) such that the Betti numbers have no upper bound for the family. Bresinsky's examples fit into this category. An obvious question is whether the Betti sequence of the projective closure of such families exhibit the same behaviour or not. More generally, it can be asked what properties of the affine monomial curves are preserved after taking the projective closure. We will discuss some results and some examples in this talk.

This work is a part of an ongoing project with my students *Pranjal Srivastava*, *Joydip Saha*, *Om Prakash Bhradwaj* and *Kriti Goel*.

Dumitru Stamate, University of Bucharest, *Ulrich elements in affine semigroups* arXiv

The almost symmetric numerical semigroups served as motivation to introduce the class of almost Gorenstein rings for whom a rich theory is emerging. We distinguish the Ulrich elements as special elements in an affine semigroup which are relevant for deciding if the semigroup ring $K[H]$ is almost Gorenstein in a multigraded sense. We provide algebraic and combinatorial criteria for finding Ulrich elements in normal simplicial affine semigroups, with a focus on dimension two.

This is joint work with *Juergen Herzog* and *Raheleh Jafari*, see arXiv:1909.06846.

Francesco Strazzanti, Università di Torino, *Almost canonical ideals and GAS numerical semigroups* arXiv arXiv

The aim of this talk is to introduce the notions of *almost canonical ideal* and *Generalized Almost Symmetric* (GAS) *numerical semigroup*. The former generalizes canonical ideals of a numerical semigroup, while the latter is a generalization of both almost symmetric and 2-AGL semigroups. We will see how they allow us to generalize some known results concerning the numerical semigroup $M(S)-M(S)$.

Moreover, we will show how the GAS property relates with other similar notions and study its behaviour with respect to some semigroup constructions.

Finally, we will discuss a generalization in the context of Cohen-Macaulay one-dimensional local rings.

This is joint work with *Marco D'Anna*.

Danny Troia, Università di Catania, *The ideal duplication* OA

In this paper we present and study the ideal duplication, a new construction within the class of the relative ideals of a numerical semigroup S, that, under specific assumptions, produces a relative ideal of the numerical duplication $S \bowtie^b E$. We prove that every relative ideal of the numerical duplication can be uniquely written as the ideal duplication of two relative ideals of S; this allows us to better understand how the basic operations of the class of the relative ideals of $S \bowtie^b E$ work. In particular, we characterize the ideals E such that $S \bowtie^b E$ is nearly Gorenstein.

References

1. D. Troia, The ideal duplication. Semigroup Forum 103, 641–660 (2021) https://doi.org/10.1007/s00233-021-10201-1.

Andrew J. Warren, Sam Houston, *On the Periodicity of the Catenary Sequence of a Numerical Monoid with Embedding Dimension Three*

Let S be a numerical monoid with embedding dimension three with three different Betti elements. Suppose further that each Betti element attains a distinct catenary degree. We compute the set of catenary degrees achieved by S. It is well known that the sequence $\{c(n)\}_{n\in S}$ is eventually periodic, and we show the period length for this sequence. We close by providing insight into the beginning point of the periodic behavior of $\{c(n)\}_{n\in S}$.

Kei-ichi Watanabe, Nihon University and Meiji University, *Inverse polynomials of numerical semigroup rings* arXiv

This is a joint work with *Kazufumi Eto* (Nippon Institute of Technology).

Let $H \subset \mathbb{N}$ be a numerical semigroup ring and $k[H]$ be its semigroup ring over any field k. If $H = \langle n_1, \ldots, n_e \rangle$, we express $k[H]$ as $k[H] = k[x_1, \ldots, x_e]/I_H$ and we want to express $k[H]/(t^h)$ by "Inverse polynomials" of Macaulay.

Namely, if we put $S = k[x_1, \ldots, x_e]$ and $\mathfrak{M} = (x_1, \ldots, x_e)$, then the injective envelope $\mathbb{E} := \mathbb{E}_S(S/\mathfrak{M})$ is expressed by the *inverse polynomial*.

$$\mathbb{E} = k[X_1, \ldots, X_e],$$

where we think $\mathbb{E}$ an S module by

$$(x_1^{a_1}\cdots x_e^{a_e})\cdot(X_1^{b_1}\cdots X_e^{b_e}) = \begin{cases} X_1^{b_1-a_1}\cdots X_e^{b_e-a_e} & (\forall i, b_i \geq a_i) \\ 0 & (\text{otherwise}) \end{cases}$$

Then if S/I is an Artinian Gorenstein ring, then $\mathrm{Ann}_S(I) \subset \mathbb{E}$ is generated by a single element ϕ_I as an S module. Namely, there is one-to-one correspondence between Artinian factor rings of S and elements of $\mathbb{E}$ (up to constant multiplication).

For $h \in H$, we say the expression $h = \sum_{i=1}^{e} a_i n_i$ with $a_i \in \mathbb{N}$ a factorization of h in H. Then we attach an inverse polynomial

$$J_{H,h} = \sum X_1^{a_1}\cdots X_e^{a_e},$$

where $\mathbf{a} = (a_1,\ldots,a_e)$ runs all the factorization $h = \sum_{i=1}^{e} a_i e_i$ of h in H.

We study the defining ideal of a numerical semigroup ring $k[H]$ using the inverse polynomial attached to the Artinian ring $k[H]/(t^h)$ for $h \in H_+$. I believe this method to express by inverse polynomials is very powerful and can be used to many purposes. At the present, we apply this method for the following cases.

1. To give a criterion for H to be symmetric or almost symmetric.
2. Charcterizations of symmetric numerical semigroups of small multiplicity.
3. new proof of Bresinsky's Theorem for symmetric semigroups generated by 4 elements.
4. Characterization of H, for which $k[\![H]\!]/(f)$ is a stretched Artinian ring for some $f \in \mathfrak{m} \subset k[\![H]\!]$. (This part is a joint work with *K. Eto*, *N. Matsuoka* and *T. Numata*.)

For example, we can show;

Theorem Let $H = \langle n_1,\ldots,n_e\rangle$. Then for any $h \in H_+$, we have

$$(\dagger)\quad \dim_k(S/\mathrm{Ann}_S(J_{H,\mathrm{F}(H)+h})) \leq h - (\mathrm{type}(H) - 1).$$

Moreover, the following conditions are equivalent.

1. H is almost symmetric
2. For all $h \in H_+$, we have equality in $(\dagger)$.

Moreover, if the equality holds for some $h \in H_+ + H_+$, then H is almost symmetric.

Daniel Windisch, Graz University of Technology, *Class groups, prime divisors and factorizations in numerical semigroup algebras* arXiv arXiv

Let K be a field and S a numerical semigroup. The ring $K[S]$, called the semigroup algebra of S over K, consists of all polynomials over K with exponents in S. We describe the class group of $K[S]$ and give sufficient conditions for $K[S]$ to contain (infinitely many) prime divisors in all classes. As a corollary, we reobtain a theorem by Pollack on prime elements in arithmetic progressions of algebraic function fields. Furthermore, we apply the results to study the factorization behaviour of $K[S]$ and generalize them to affine semigroups S.

This talk is based on joint work with *G.W. Chang* and *V. Fadinger*.