Cyclotomic Numerical Semigroups and Graded Algebras

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International meeting on numerical semigroups

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Part I: Cyclotomic Numerical Semigroups

Part II: Algebraic point of view

Semigroup polynomial

Definition

The **semigroup polynomial** *of S is*

$$P_S(x) = 1 + (x-1) \sum_{g \in \mathbb{N} \setminus S} x^g$$

Example

$$S = \langle 3, 4 \rangle$$

$$\cdots \circ \; \circ \; \circ \; \bullet \; \circ \; \circ \; \bullet \; \bullet \; \circ \; \bullet \; \bullet \; \bullet \; \bullet \cdots$$

$$-1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$$

$$P_S(x) = 1 - x + x^3 - x^5 + x^6$$

Symmetric numerical semigroups

Definition

A numerical semigroup S is **symmetric** *if $x \in S \Leftrightarrow F(S) - x \notin S$*

Theorem

S is symmetric if and only if $P_S(x)$ is palindromic.

palindromic : $P_S(x) = x^d P_S(x^{-1}), \quad d = \deg P_S$

i.e. the coefficients reads
the same forward or backward

Example

$$S = \langle 3, 4 \rangle$$

$$P_S(x) = 1x^0 + (-1)x^1 + 0x^2 + 1x^3 + 0x^4 + (-1)x^5 + 1x^6$$

Cyclotomic numerical semigroups

Definition

A polynomial $f(x)$ *is* **cyclotomic** *if it is an irreducible factor of* $x^n - 1$ *for some* $n > 0$.

Definition

A numerical semigroup S *is* **cyclotomic** *if* $P_S(x)$ *is a product of cyclotomic polynomials.*

Example

$$S = \langle 3, 4 \rangle$$

$$\begin{aligned} P_S(x) &= 1 - x + x^3 - x^5 + x^6 \\ &= \underbrace{(1 - x + x^2)}_{\text{divides } x^6 - 1} \underbrace{(1 - x^2 + x^4)}_{\text{divides } x^{12} - 1} \end{aligned}$$

Complete intersection numerical semigroups

Let S, S_1, S_2 be n. s. and let $a_1 \in S_2$ and $a_2 \in S_1$ such that they are coprime and not minimal generators of their semigroups.

Definition

*S is a **gluing** of S_1 and S_2 if $S = a_1S_1 + a_2S_2$.*

Proposition

TFAE

- $S = a_1S_1 + a_2S_2$,
- $P_S(x) = P_{\langle a_1, a_2 \rangle}(x) P_{S_1}(x^{a_1}) P_{S_2}(x^{a_2})$.

Definition

*S is a **complete intersection** if*

- *$S = \mathbb{N}$, or*
- *S is the gluing of two complete intersection numerical semigroups.*

$$\text{complete intersection} \underset{?}{\Longrightarrow} \text{cyclotomic} \underset{\not\Longleftarrow}{\Longrightarrow} \text{symmetric}$$

Herrera-Poyatos, Moree and independently Sawhney, Stoner:

Theorem

$S_k = \langle k, k+1, \ldots, 2k-2 \rangle$ is symmetric but not cyclotomic for every $k \geq 5$.

Conjecture (Ciolan, García-Sánchez, Moree 2016)

$$\textit{complete intersection} \iff \textit{cyclotomic}$$

The conjecture is true for $F(S) \leq 70$ by a computation check.

Theorem (Herzog)

If $e(S) \leq 3$ then S is symmetric iff it is a complete intersection.

As a corollary, the conjecture is true for $e(S) \leq 3$.

Theorem (B., Herrera-Poyatos, Moree)

If $P_S(x)$ has at most 2 irreducible factors then S is cyclotomic iff it is a complete intersection.

An algebraic point of view

$S = \langle n_1, \dots, n_e \rangle$,

$k[S] = k[t^s : s \in S] \simeq \dfrac{k[x_1, \dots, x_e]}{I}$ graded by $\deg(x_i) = n_i$.

$$H(k[S], x) = \frac{\mathcal{K}(k[S], x)}{(1 - x^{n_1}) \dots (1 - x^{n_e})} = \frac{P_S(x)}{(1 - x)}$$

An algebraic point of view

$$R \simeq \frac{k[x_1, \dots, x_e]}{I} \text{ graded by } \deg(x_i) = n_i \in \mathbb{N}$$

$$H(R, x) = \frac{\mathcal{K}(R, x)}{(1 - x^{n_1}) \dots (1 - x^{n_e})} = \frac{N_R(x)}{D_R(x)}$$

Definition

A graded algebra R is **cyclotomic** *if $N_R(x)$ is a product of cyclotomic polynomials.*

Gorenstein algebras

Theorem (Kunz)

$k[S]$ is Gorenstein iff S is symmetric$P_S(x)$ is palindromic.

Theorem (Stanley)

A Cohen-Macaulay graded domain R is Gorenstein iff $N_R(x)$ is palindromic.

Corollary

If R is a cyclotomic Cohen-Macaulay graded domain then R is Gorenstein.

Complete intersections

Definition

A graded algebra $R \simeq k[x_1, \dots, x_e]/I$ is a **complete intersection** *if I is generated by a regular sequence.*

If R is a complete intersection, then

$$H(R, x) = \frac{(1 - x^{d_1}) \dots (1 - x^{d_m})}{(1 - x^{n_1}) \dots (1 - x^{n_e})} = \frac{N_R(x)}{D_R(x)}$$

Corollary

If R is a complete intersection, it is cyclotomic.

$$\text{complete intersection} \underset{\not\Longleftarrow}{\Longrightarrow} \text{cyclotomic} \underset{\not\Longleftarrow}{\overset{\left(\substack{\text{Cohen-Macaulay}\\ \text{domains}}\right)}{\Longrightarrow}} \text{Gorenstein}$$

Example (Stanley)

$R = k[x,y]/(x^3, xy, y^2)$ with $\deg(x) = \deg(y) = 1$. We have

$$H(R,t) = \frac{1 - 2t^2 + t^4}{(1-t)^2} = (1+t)^2.$$

R is cyclotomic, but not a complete intersection.

Koszul algebras

Definition

A graded algebra $R \simeq k[x_1, \ldots, x_e]/I$ *with* $\deg(x_i) = 1$ *is* **Koszul** *if the minimal free resolution of* k *as an* R*-module is linear (i.e.* $\beta_{i,j}^R(k) = 0$ *whenever* $i \neq j$*).*

I has a Gröbner basis of quadrics $\Longrightarrow$ R is Koszul $\Longrightarrow$ I is generated by quadrics

Theorem (B., D'Alì)

A Koszul algebra R *is cyclotomic iff it is a complete intersection.*

Thank you for your attention!