The generalized Gorenstein property and numerical semigroup rings obtained by gluing

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September 6, 2018

Joint work with Shiro Goto

Setting

- $H_1 = \langle a_1, a_2, \ldots, a_\ell \rangle$, $H_2 = \langle b_1, b_2, \ldots, b_m \rangle$, $\mathrm{gcd}(a) = \mathrm{gcd}(b) = 1$.
- $\alpha_1 \in H_1 \setminus \{a_1, \ldots, a_\ell\}$, $\alpha_2 \in H_2 \setminus \{b_1, \ldots, b_m\}$, $\mathrm{gcd}(\alpha_1, \alpha_2) = 1$.
- $H = \alpha_2 H_1 + \alpha_1 H_2 = \langle \alpha_2 a_1, \ldots, \alpha_2 a_\ell, \alpha_1 b_1, \ldots, \alpha_1 b_m \rangle$ *a gluing of* H_1 *and* H_2.
- $V = k[[t]]$ *the formal power series ring over a field* k.
- $R = k[[H]] = k[[t^h \mid h \in H]] \subseteq V$,
  $R_1 = k[[H_1]] = k[[t^h \mid h \in H_1]] \subseteq V$, *and*
  $R_2 = k[[H_2]] = k[[t^h \mid h \in H_2]] \subseteq V$.

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Problem 1

Explore the relation between the structure of H *(resp.* R*) and the structures of* H_1 *and* H_2 *(resp.* R_1 *and* R_2*).*

Motivation

Fact 2 (Delorme, Rosales)

- *R is complete intersection $\Leftrightarrow$ R_1 and R_2 are complete intersection*
- *R is Gorenstein (i.e. H is symmetric) $\Leftrightarrow$ R_1 and R_2 are Gorenstein*

Reamrk

It is also well-known that a numerical semigroup $H \neq \mathbb{N}$ is complete intersection if and only if H is a gluing of two complete intersection numerical semigroups.

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(Starting point of this study, cf. Nari, Numata)

How about the almost Gorenstein property?

Theorem 3 (Nari)

R can not be almost Gorenstein, if R is not Gorenstein.

Example 4

- Let $H_1 = \langle 3, 4, 5 \rangle$, $H_2 = \langle 2, 3 \rangle$, $\alpha_1 = 6$, $\alpha_2 = 5$.
- Then $H = \langle 15, 20, 25, 12, 18 \rangle$ and this is not almost symmetric, because $\mathrm{PF}(H) = \{41, 46\}$.

However,this H is still "good".

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However,this H is still "good".

How is H "good" ?
$\Rightarrow$ *We need the definition of "generalized Gorenstein rings".*

History of almost Gorenstein (AG) rings

- **1997** *Barucci-Fröberg :*
  introduced for analytically unramified 1-dimensional local ring.
- **2013** *Goto-M.-Phuong :*
  extended for arbitrary 1-dimensional Cohen-Macaulay(CM) local ring.
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When I and my colleagues studied about AGness of local/graded rings, we often met rings which is not AG but still seems "good".

Generalized Gorenstein rings

Definition 5 (Goto-Kumashiro, in preparation)

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with the canonical module K_R. We say that R is a generalized Gorenstein local (GGL) ring, if either

- R is Gorenstein, or
- R is not Gorenstein but

$$\exists 0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$$

such that C is an Ulrich R-module with respect to some $\mathfrak{m}$-primary ideal $\mathfrak{a}$ and $\varphi \otimes R/\mathfrak{a}$ is injective.

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R is AG $\Leftrightarrow$ R is GGL ($\mathfrak{a} = \mathfrak{m}$).

Generalized Gorenstein NS rings

Let H a numerical semigroup and $R = k[[H]]$ where k is a field.

- Then R is a CM local ring with $\dim R = 1$ and $\mathfrak{m} = (t^h \mid 0 < h \in H)$.
- Put $K = \sum_{p \in \mathrm{PF}(H)} R{\cdot}t^{\mathrm{F}(H)-p} \cong \mathrm{K}_R$ the canonical module of R such that $R \subseteq K \subseteq V = k[[t]]$.
- Set $S = R[K]$ and $\mathfrak{c} = R : S \subseteq R$.
- With this notation, R is GGL $\Leftrightarrow$ K/R can be controlled.

Theorem 6 (GK)

Let $\mathrm{PF}(H) = \{p_1 < p_2 < \cdots < p_r\}$ *and suppose* $r \geq 2$. *TFAE:*

1. *R is a GGL ring (in 1-dimensional case, automatically $\mathfrak{a} = \mathfrak{c}$).*
2. *$R/\mathfrak{c}$ is a Gorenstein ring and $p_i + p_{r-i} = p_r + x$ for all $1 \leq i \leq r-1$, where $t^x \in (\mathfrak{c} : \mathfrak{m}) \setminus \mathfrak{c}$.*

Since $\mathfrak{c} : \mathfrak{m}/\mathfrak{c}$ is a k-vector space and $R/\mathfrak{c}$ is Gorenstein, x as in Theorem 6 is uniquely determined.

The easiest example

Example 7

Let $H = \langle 3, 7, 8 \rangle$. Then $R = k[[H]]$ is a GGL ring but not AG.

- We easily get that $\mathrm{PF}(H) = \{4, 5\}$, $K = R + R{\cdot}t$, and $S = R[K] = k[[t]] = V$.
- Then $\mathfrak{c} = R : S = (t^6, t^7, t^8)$ and hence $\mathfrak{c} : \mathfrak{m} = \mathfrak{c} + (t^3)$.
- Since $4 + 4 = 5 + 3$, R is GGL by Theorem 6.

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Theorem 8 (GK)

Any 1-dimensional CM local ring of multiplicity ≤ 3 is GGL.

GGL NS rings with embedding dimension 3.

Suppose $H = \langle a, b, c \rangle$ and H is not symmetric. Then it is well-known (by [Herzog]) that

$$R = k[[H]] \cong k[[x, y, z]]/\, \mathrm{I}_2 \begin{pmatrix} x^{\alpha} & y^{\beta} & z^{\gamma} \\ y^{\beta'} & z^{\gamma'} & x^{\alpha'} \end{pmatrix}$$

for some $\alpha, \beta, \gamma, \alpha', \beta', \gamma' > 0$.

Theorem 9 (GK)

TFAE:

1. *R is a GGL ring.*
2. *Either*
   1. $\alpha \geq \alpha'$, $\beta \geq \beta'$, *and* $\gamma \geq \gamma'$ *or*
   2. $\alpha \leq \alpha'$, $\beta \leq \beta'$, *and* $\gamma \leq \gamma'$.

Recall that

- $H_1 = \langle a_1, a_2, \dots, a_\ell \rangle$, $H_2 = \langle b_1, b_2, \dots, b_m \rangle$
- $\alpha_1 \in H_1 \setminus \{a_1, \dots, a_\ell\}$, $\alpha_2 \in H_2 \setminus \{b_1, \dots, b_m\}$
- $H = \alpha_2 H_1 + \alpha_1 H_2 = \langle \alpha_2 a_1, \dots, \alpha_2 a_\ell, \alpha_1 b_1, \dots, \alpha_1 b_m \rangle$
- $R = k[[H]]$, $R_1 = k[[H_1]]$, $R_2 = k[[H_2]]$

Main Theorem

TFAE:

1. *R is a GGL ring.*
2. *One of R_1 and R_2 is Gorenstein and another one is GGL.*

To prove the main theorem, we need some preparative lemmas.

Lemma 10 (Nari)

Let $\mathrm{PF}(H_1) = \{p_1 < p_2 < \cdots < p_r\}$ *and* $\mathrm{PF}(H_2) = \{q_1 < q_2 < \cdots < q_s\}$. *Then*

$$\mathrm{PF}(H) = \{\alpha_2 p_i + \alpha_1 q_j + \alpha_1\alpha_2 \mid 1 \leq i \leq r, 1 \leq j \leq s\}.$$

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Idea of a proof (different from Nari's proof).

- The minimal graded free resolution of $k[H]$ is completely computed by Gimenez and Srinivasan, by using the graded minimal free resolutions of $k[H_1]$ and $k[H_2]$.

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Idea of a proof (different from Nari's proof).

- The minimal graded free resolution of $k[H]$ is completely computed by Gimenez and Srinivasan, by using the graded minimal free resolutions of $k[H_1]$ and $k[H_2]$.
- Thanks to their result, we can compute the pseudo-Frobenius numbers of H by checking the grading of the resolution. □

Lemma 11

Let $\mathrm{PF}(H) = \{\xi_1 < \xi_2 < \cdots < \xi_u\}$ *and suppose that* $\xi_i + \xi_{u-i}$ *is constant for* $1 \leq i \leq u-1$. *Then* H_1 *or* H_2 *is symmetric.*
In particular, if R *is GGL, then* R_1 *or* R_2 *is Gorenstein.*

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Proof.

- Let $\mathrm{PF}(H_1) = \{p_1 < \cdots < p_r\}$ and $\mathrm{PF}(H_2) = \{q_1 < \cdots < q_s\}$. Suppose $r, s > 1$.
- Then $\mathrm{PF}(H) = \{\alpha_2 p_i + \alpha_1 q_j + \alpha_1\alpha_2 \mid 1 \le i \le r, 1 \le j \le s\}$.
- We may assume $\alpha_2 p_r + \alpha_1 q_{s-1} + \alpha_1\alpha_2 > \alpha_2 p_{r-1} + \alpha_1 q_s + \alpha_1\alpha_2$.
- By our assumption,
  $(\alpha_2 p_1 + \alpha_1 q_1 + \alpha_1\alpha_2) + (\alpha_2 p_r + \alpha_1 q_{s-1} + \alpha_1\alpha_2) =$
  $(\alpha_2 p_1 + \alpha_1 q_s + \alpha_1\alpha_2) + (\alpha_2 p_i + \alpha_1 q_j + \alpha_1\alpha_2)$ for some i and j.
- This implies $\alpha_1((q_s - q_{s-1}) + (q_j - q_1)) = \alpha_2(p_r - p_i) > 0$.
- Because $\gcd(\alpha_1, \alpha_2) = 1$, we can write $p_r = \alpha_1 x + p_i$, $\exists x > 0$. □

Sketch of Proof of Main Theorem

Main Theorem

TFAE:

1. *R is a GGL ring.*
2. *One of R_1 and R_2 is Gorenstein and another one is GGL.*

- *We may assume R_2 is Gorenstein. Let $q = \mathrm{F}(H_2)$.*
- *Let $\mathrm{PF}(H_1) = \{p_1 < \cdots < p_r\}$ and $\xi_i = \alpha_2 p_i + \alpha_1 q + \alpha_1\alpha_2$.*
- *Then $\mathrm{PF}(H) = \{\xi_1 < \cdots < \xi_r\}$.*
- *Easy: $p_i + p_{r-i} = p_r + x \ (\forall i) \Leftrightarrow \xi_i + \xi_{r-i} = \xi_r + y \ (\forall i)$ where $y = \alpha_2 x + \alpha_1 q + \alpha_1\alpha_2$.*
- *Therefore, what we have to prove is the following.*

Lemma 12

Suppose H_2 *is symmetric. Let*

- $K = \sum_{p \in \mathrm{PF}(H)} R \cdot t^{\mathrm{F}(H)-p}$, $K_1 = \sum_{p \in \mathrm{PF}(H_1)} R_1 \cdot t^{\mathrm{F}(H_1)-p}$,
- $S = R[K]$, $S_1 = R_1[K_1]$,
- $\mathfrak{c} = R : S$, $\mathfrak{c}_1 = R_1 : S_1$.

Then $R/\mathfrak{c}$ *is Gorenstein if and only if* $R_1/\mathfrak{c}_1$ *is Gorenstein. When this is the case, let* $x \in H_1$ *such that* $\mathfrak{c}_1 : \mathfrak{m}_1 = \mathfrak{c}_1 + (t^x)$, *then* $\mathfrak{c} : \mathfrak{m} = \mathfrak{c} + (t^y)$ *where* $y = \alpha_2 x + \alpha_1 \mathrm{F}(H_2) + \alpha_1\alpha_2$.

(Recall: GK)

Let $\mathrm{PF}(H) = \{p_1 < p_2, \ldots, p_r\}$ *and suppose* $r \geq 2$. *TFAE:*

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Back to the first example

- $H = \langle 15, 20, 25, 12, 18 \rangle$ *is gluing of* $\langle 3, 4, 5 \rangle$ *and* $\langle 2, 3 \rangle$. $\mathrm{PF}(H) = \{41, 46\}$. *Hence* $K = R + Rt^5$.

0	1	2	3	4	5	6	7	8	9	10	11
12	13	14	15	16	17	18	19	20	21	22	23
24	25	26	27	28	29	30	31	32	33	34	35
36	37	38	39	40	41	42	43	44	45	46	47
48	49	50	51	52	53	54	55	56	57	58	59

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- *We get* $K^2 = K^3$. *This is equivalent to* $\mathfrak{c} = R : S = R : K$ *(by GK).*
- *Hence* $\mathfrak{c} = R : K = R : t^5 \Rightarrow \mathfrak{c} = (t^{15}, t^{20}, t^{25})$.
- *Notice that for* $H_1 = \langle 3, 4, 5\rangle$, $\mathfrak{c}_1 = (t^3, t^4, t^5)$.

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- *We get* $K^2 = K^3$.*This is equivalent to* $\mathfrak{c} = R : S = R : K$ *(by GK).*
- *Hence* $\mathfrak{c} = R : K = R : t^5 \Rightarrow \mathfrak{c} = (t^{15}, t^{20}, t^{25})$.
- *Then* $\mathfrak{c} : \mathfrak{m} = \mathfrak{c} + (t^{36})$. *To check this, we need to consider* 18 *and* 36*, because other numbers* $+12$ *is not in* $\mathfrak{c}$.
- *By* $41 + 41 = 46 + 36$, R *is GGL.*

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That's all of my talk.
Thank you for your attention.