The Frobenius number for sequences of binomial coefficients

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Frobenius coin problem

- Given relatively prime positive integers $a_1, \ldots, a_n$, $n \geq 2$, find a formula to compute the largest integer that is not representable as a non-negative integer linear combination of $a_1, \ldots, a_n$.
- $\mathrm{F}(a_1, \ldots, a_n)$ (the Frobenius number of the set $\{a_1, \ldots, a_n\}$) denotes the solution of the previous problem.
- $\mathrm{F}(a_1, a_2) = a_1 a_2 - a_1 - a_2$.
- Frobenius problem is open for $n \geq 3$.
  - Curtis: it is impossible to find a polynomial formula that computes the Frobenius number if $n = 3$.
  - Ramírez Alfonsín: the problem is NP-hard for n variables.

Particular cases

- Arithmetic and almost arithmetic sequences (A. Brauer; M. Lewin; J. B. Roberts; E. S. Selmer),
- Fibonacci sequences (J. M. Marín, J. L. Ramírez Alfonsín and M. P. Revuelta),
- geometric sequences (D. C. Ong and V. Ponomarenko),
- Mersenne, repunit, and Thabit sequences (J. C. Rosales, M. B. Branco, D. Torrão),
- squares and cubes sequences (M. Lepilov, J. O'Rourke and I. Swanson; A. Moscariello),
- ...

Particular cases: motivations (i)

- Brauer:

$$F(n, n+1, \ldots, n+k-1) = \left(\left\lfloor \frac{n-2}{k-1} \right\rfloor + 1\right) n - 1.$$

- Baker (conjecture):

  If T_n is the nth triangular (or triangle) number, then

$$F(T_n, T_{n+1}, T_{n+2}) = \begin{cases} \frac{6n^3+18n^2+12n-8}{8} & \text{if } n \text{ is even;} \\ \frac{6n^3+12n^2-6n-20}{8} & \text{if } n \text{ is odd.} \end{cases}$$

  Equivalently,

$$F(T_n, T_{n+1}, T_{n+2}) = \frac{6n^3 + 3(5+(-1)^n)n^2 + 3(1+3(-1)^n)n - (14-6(-1)^n)}{8}.$$

Particular cases: motivations (ii)

- Sequences of binomial coefficients (or combinatorial numbers).
  - $\{n, n+1, \ldots, n+k-1\} = \left\{\binom{n}{1}, \binom{n+1}{1}, \ldots, \binom{n+k-1}{1}\right\}$
  - $\{\mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2}\} = \left\{\binom{n+1}{2}, \binom{n+2}{2}, \binom{n+3}{2}\right\}$

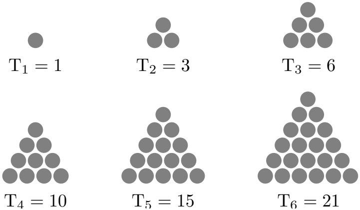

Figure: First six triangular numbers.

Purposes

- Prove the Baker's conjecture.
- Find formulas for other binomial coefficients sequences.
  - Tetrahedral (or triangular pyramidal) numbers: $\mathrm{TH}_n = \binom{n+2}{3}$.

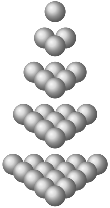

Figure: Tetrahedral number TH_5 (by layers).

  - Pentatope (or 4-Hypertetrahedral or 4-dimensional triangular pyramidal) numbers: $\mathrm{P}_n = \binom{n+3}{4}$.

Tools (i)

- Johnson's formula: if a_1, a_2, a_3 are relatively prime numbers and $\gcd\{a_1, a_2\} = d$, then

$$\mathrm{F}(a_1, a_2, a_3) = d\,\mathrm{F}\left(\frac{a_1}{d}, \frac{a_2}{d}, a_3\right) + (d-1)a_3.$$

- Generalization by Brauer and Shockley: if $a_1, \ldots, a_n$ are relatively prime numbers and $d = \gcd\{a_1, \ldots, a_{n-1}\}$, then

$$\mathrm{F}(a_1, \ldots, a_n) = d\,\mathrm{F}\left(\frac{a_1}{d}, \ldots, \frac{a_{n-1}}{d}, a_n\right) + (d-1)a_n.$$

- Telescopic sequences.

Tools (ii)

Definition

- Let $(a_1,\ldots,a_n)$ be a sequence of positive integers such that $\gcd\{a_1,\ldots,a_n\}=1$ (where $n\geq 2$).
- Let $d_i=\gcd\{a_1,\ldots,a_i\}$ for $i=1,\ldots,n$.
- Then $(a_1,\ldots,a_n)$ is a telescopic sequence if, for each $i=2,\ldots,n$, $\frac{a_i}{d_i}$ is representable as a non-negative integer linear combination of $\frac{a_1}{d_{i-1}},\ldots,\frac{a_{i-1}}{d_{i-1}}$.

Remark

- If $(a_1,\ldots,a_n)$ is a telescopic sequence, then
  - $\left(\frac{a_1}{d_i},\ldots,\frac{a_i}{d_i}\right)$ is also a telescopic sequence for $i=2,\ldots,n-1$;
  - $\mathrm{F}(a_1,\ldots,a_n)=d_{n-1}\,\mathrm{F}\left(\frac{a_1}{d_{n-1}},\ldots,\frac{a_{n-1}}{d_{n-1}}\right)+(d_{n-1}-1)a_n$.

Examples

Example: (8,14,19)

- $d_1 = \gcd\{8\} = 8$; $\quad d_2 = \gcd\{8,14\} = 2$; $\quad d_3 = \gcd\{8,14,19\} = 1$.
- $\frac{14}{2} = 7 \cdot \frac{8}{8}$; $\quad \frac{19}{1} = 3 \cdot \frac{8}{2} + 1 \cdot \frac{14}{2}$.
- $(8,14,19)$ is a telescopic sequence.
- $\mathrm{F}(8,14,19) = 2 \cdot \mathrm{F}\left(\frac{8}{2}, \frac{14}{2}\right) + (2-1) \cdot 19 = 2 \cdot \mathrm{F}(4,7) + 19 = 2 \cdot 17 + 19 = 53$.

Example: (8,14,17)

- $d_1 = \gcd\{8\} = 8$; $\quad d_2 = \gcd\{8,14\} = 2$; $\quad d_3 = \gcd\{8,14,17\} = 1$.
- $\frac{14}{2} = 7 \cdot \frac{8}{8}$; $\quad \frac{17}{1} \neq \alpha_1 \cdot \frac{8}{2} + \alpha_2 \cdot \frac{14}{2}$, for all $\alpha_1, \alpha_2 \in \mathbb{N} = \{0,1,2,\ldots\}$.
- $(8,14,17)$ is not a telescopic sequence.
- $\mathrm{F}(8,14,17) = ?$

Triangular numbers sequences

$$\mathrm{T}_n = \binom{n+1}{2} = \frac{n(n+1)}{2}, \quad n \geq 1.$$

Lemma

- $\gcd\{\mathrm{T}_n, \mathrm{T}_{n+1}\} = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd;} \\ n+1 & \text{if } n \text{ is even.} \end{cases}$

Lemma

- $\gcd\{\mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2}\} = 1.$

Proof

* $\gcd\{\mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2}\} = \gcd\left\{\gcd\{\mathrm{T}_n, \mathrm{T}_{n+1}\}, \gcd\{\mathrm{T}_{n+1}, \mathrm{T}_{n+2}\}\right\}.$

Triangular numbers sequences

Proposition

- $(\mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2})$ and $(\mathrm{T}_{n+2}, \mathrm{T}_{n+1}, \mathrm{T}_n)$ are telescopic sequences.

Proof

- Let n be an odd integer.
- $d_1 = \gcd\{\mathrm{T}_n\} = \frac{n(n+1)}{2}$; $\quad d_2 = \gcd\{\mathrm{T}_n, \mathrm{T}_{n+1}\} = \frac{n+1}{2}$;

  $d_3 = \gcd\{\mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2}\} = 1$.
- $\frac{\mathrm{T}_{n+1}}{\frac{n+1}{2}} = (n+2) \cdot \frac{\mathrm{T}_n}{\frac{n(n+1)}{2}}$; $\quad \frac{\mathrm{T}_{n+2}}{1} = 0 \cdot \frac{\mathrm{T}_n}{\frac{n+1}{2}} + \frac{n+3}{2} \cdot \frac{\mathrm{T}_{n+1}}{\frac{n+1}{2}}$.
- $(\mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2})$ is a telescopic sequence if n is odd.
- The proofs for the other three cases are similar.

Triangular numbers sequences

Proposition

- $\mathrm{F}(\mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2}) = \begin{cases} \frac{3n^3+6n^2-3n-10}{4} & \text{if } n \text{ is odd;} \\ \frac{3n^3+9n^2+6n-4}{4} & \text{if } n \text{ is even.} \end{cases}$

Proposition

- $\mathrm{F}(\mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2}) = \left\lfloor \frac{n}{2} \right\rfloor (\mathrm{T}_n + \mathrm{T}_{n+1} + \mathrm{T}_{n+2} - 1) - 1.$

Triangular numbers sequences: obstacles

- Two consecutive triangular numbers are not relatively prime.
- If $n = 4$ or $n \geq 6$, then

$$\left(\binom{n+1}{2},\binom{n+2}{2},\binom{n+3}{2},\binom{n+4}{2}\right)$$

  is not a telescopic sequence (of four consecutive relatively prime triangular numbers).

  * In fact, none of its permutations is telescopic.

Tetrahedral numbers sequences

$$\mathrm{TH}_n = \binom{n+2}{3} = \frac{n(n+1)(n+2)}{6}, \quad n \geq 1.$$

Lemma

- $\gcd\{\mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}, \mathrm{TH}_{n+3}\} = 1.$

Proof

- $(\mathrm{TH}_{n+1} - \mathrm{TH}_n, \mathrm{TH}_{n+2} - \mathrm{TH}_{n+1}, \mathrm{TH}_{n+3} - \mathrm{TH}_{n+2}) = (\mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2}).$
- Fact
  - Let $(a_1, a_2, \ldots, a_n)$ be a sequence of positive integers.
  - Let $d_1 = \gcd\{a_1, a_2, \ldots, a_n\}$ and $d_2 = \gcd\{a_2 - a_1, \ldots, a_n - a_{n-1}\}$.
  - Then $d_1 | d_2$. In particular, if $d_2 = 1$, then $d_1 = 1$.

Tetrahedral numbers sequences

Lemma

- $$\gcd\{\mathrm{TH}_n, \mathrm{TH}_{n+1}\} = \begin{cases} (6k+1)(3k+1) & \text{if } n = 6k; \\ (3k+1)(2k+1) & \text{if } n = 6k+1; \\ (2k+1)(3k+2) & \text{if } n = 6k+2; \\ (3k+2)(6k+5) & \text{if } n = 6k+3; \\ (6k+5)(k+1) & \text{if } n = 6k+4; \\ (k+1)(6k+7) & \text{if } n = 6k+5. \end{cases}$$

Lemma

- $$\gcd\{\mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}\} = \begin{cases} 3k+1 & \text{if } n = 6k; \\ 2k+1 & \text{if } n = 6k+1; \\ 3k+2 & \text{if } n = 6k+2; \\ 6k+5 & \text{if } n = 6k+3; \\ k+1 & \text{if } n = 6k+4; \\ 6k+7 & \text{if } n = 6k+5. \end{cases}$$

Tetrahedral numbers sequences

Proposition

- $(\mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}, \mathrm{TH}_{n+3})$ is telescopic if and only if $n \equiv r \bmod 6$ with $r \in \{0,1,2,3\}$.

Proposition

- $(\mathrm{TH}_{n+3}, \mathrm{TH}_{n+2}, \mathrm{TH}_{n+1}, \mathrm{TH}_n)$ is telescopic if and only if $n \equiv r \bmod 6$ with $r \in \{4,5\}$.

Proposition

- $$\mathrm{F}(\mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}) = \begin{cases} \frac{n-3}{3}\mathrm{TH}_{n+1} + n\mathrm{TH}_{n+2} + \frac{n}{2}\mathrm{TH}_{n+3} - \mathrm{TH}_n & \text{if } n = 6k; \\ (n-1)\mathrm{TH}_{n+1} + \frac{n-1}{2}\mathrm{TH}_{n+2} + \frac{n-1}{3}\mathrm{TH}_{n+3} - \mathrm{TH}_n & \text{if } n = 6k+1; \\ (n-1)\mathrm{TH}_{n+1} + \frac{n-2}{3}\mathrm{TH}_{n+2} + \frac{n}{2}\mathrm{TH}_{n+3} - \mathrm{TH}_n & \text{if } n = 6k+2; \\ \frac{n-3}{3}\mathrm{TH}_{n+1} + \frac{n-1}{2}\mathrm{TH}_{n+2} + (n+1)\mathrm{TH}_{n+3} - \mathrm{TH}_n & \text{if } n = 6k+3; \\ \frac{n+2}{3}\mathrm{TH}_{n+2} + \frac{n+2}{2}\mathrm{TH}_{n+1} + (n+2)\mathrm{TH}_n - \mathrm{TH}_{n+3} & \text{if } n = 6k+4; \\ (n+4)\mathrm{TH}_{n+2} + \frac{n+1}{3}\mathrm{TH}_{n+1} + \frac{n+1}{2}\mathrm{TH}_n - \mathrm{TH}_{n+3} & \text{if } n = 6k+5. \end{cases}$$

Tetrahedral numbers sequences

Proposition

- $$\mathrm{F}(\mathrm{TH}_n,\mathrm{TH}_{n+1},\mathrm{TH}_{n+2})=\begin{cases} \frac{1}{36}(11n^4+90n^3+265n^2+258n-36) & \text{if } n=6k; \\ \frac{1}{36}(11n^4+70n^3+133n^2-22n-228) & \text{if } n=6k+1; \\ \frac{1}{36}(11n^4+74n^3+169n^2+82n-132) & \text{if } n=6k+2; \\ \frac{1}{36}(11n^4+102n^3+373n^2+570n+252) & \text{if } n=6k+3; \\ \frac{1}{36}(11n^4+70n^3+133n^2-22n-228) & \text{if } n=6k+4; \\ \frac{1}{36}(11n^4+98n^3+349n^2+526n+228) & \text{if } n=6k+5. \end{cases}$$

Proposition

- $$\mathrm{F}(\mathrm{TH}_n,\mathrm{TH}_{n+1},\mathrm{TH}_{n+2})=\begin{cases} 396k^4+540k^3+265k^2+43k-1 & \text{if } n=6k; \\ 396k^4+684k^3+409k^2+83k-1 & \text{if } n=6k+1; \\ 396k^4+972k^3+877k^2+333k+41 & \text{if } n=6k+2; \\ 396k^4+1404k^3+1885k^2+1125k+249 & \text{if } n=6k+3; \\ 396k^4+1476k^3+2029k^2+1203k+253 & \text{if } n=6k+4; \\ 396k^4+1908k^3+3469k^2+2811k+853 & \text{if } n=6k+5. \end{cases}$$

Pentatope numbers sequences

$$\mathrm{P}_n = \binom{n+3}{4} = \frac{n(n+1)(n+2)(n+3)}{24}, \quad n \geq 1.$$

Lemma

- $\gcd\{\mathrm{P}_n, \mathrm{P}_{n+1}, \mathrm{P}_{n+2}, \mathrm{P}_{n+3}, \mathrm{P}_{n+4}\} = 1$.

Proposition

- $(\mathrm{P}_n, \mathrm{P}_{n+1}, \mathrm{P}_{n+2}, \mathrm{P}_{n+3}, \mathrm{P}_{n+4})$ is telescopic if and only if $n \equiv r \bmod 6$ with $r \in \{0, 1, 2\}$ (and $n \geq 1$).
- $(\mathrm{P}_{n+4}, \mathrm{P}_{n+3}, \mathrm{P}_{n+2}, \mathrm{P}_{n+1}, \mathrm{P}_n)$ is telescopic if and only if $n \equiv r \bmod 6$ with $r \in \{3, 4, 5\}$ (and $n \geq 9$).
- If $n \in \{3, 4, 5\}$, then $(\mathrm{P}_n, \mathrm{P}_{n+1}, \mathrm{P}_{n+2}, \mathrm{P}_{n+3}, \mathrm{P}_{n+4})$ and $(\mathrm{P}_{n+4}, \mathrm{P}_{n+3}, \mathrm{P}_{n+2}, \mathrm{P}_{n+1}, \mathrm{P}_n)$ are telescopic.

General binomial coefficients sequences

Remark

- All possible permutations of

$$\left(\binom{12}{5},\binom{13}{5},\binom{14}{5},\binom{15}{5},\binom{16}{5},\binom{17}{5}\right)=$$

$$(792,1287,2002,3003,4368,6188)$$

are not telescopic.

Consequences on numerical semigroups

- Let $(\mathbb{N},+)$ be the additive monoid of non-negative integers.
- We say that S is a numerical semigroup if it is an additive submonoid of $\mathbb{N}$ such that $0 \in S$ and $\mathbb{N} \setminus S$ is a finite set.
- Let $X = \{x_1, \ldots, x_n\}$ be a non-empty subset of $\mathbb{N} \setminus \{0\}$. We define

$$\langle X \rangle = \{\lambda_1 x_1 + \cdots + \lambda_n x_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}.$$

- Let S be submonoid of $(\mathbb{N},+)$. Then there exists a unique X such that $S = \langle X \rangle$ and $S \neq \langle Y \rangle$ for any $Y \subsetneq X$.

  In such a case, X is the minimal system of generators of S.
- X is the minimal system of generators of a numerical semigroup S if and only if $\gcd(X) = 1$.

  The cardinality of X is the embedding dimension of S.

Free numerical semigroups

Definition

- S is a free numerical semigroup if there exists a telescopic sequence $(a_1,\ldots,a_n)$ such that $S=\langle a_1,\ldots,a_n\rangle$.

Example

- $\mathcal{T}_n=\langle \mathrm{T}_n,\mathrm{T}_{n+1},\mathrm{T}_{n+2}\rangle$, $n\geq 3$, are free numerical semigroups with embedding dimension equal to 3.
- $\mathcal{TH}_n=\langle \mathrm{TH}_n,\mathrm{TH}_{n+1},\mathrm{TH}_{n+2},\mathrm{TH}_{n+3}\rangle$, $n\geq 4$, are free numerical semigroups with embedding dimension equal to 4.

Preliminaries (i)

Definition

- Let S be a numerical semigroup.
- Let $\{n_1, \ldots, n_e\}$ be a minimal system of generators of S.
- Then we define

$$c_i^* = \min\left\{k \in \mathbb{N} \;\middle|\; kn_i \in \langle n_1, \ldots, n_{i-1} \rangle\right\} \text{ for every } i \in \{2, \ldots, e\}.$$

Lemma

- Let S be a numerical semigroup.
- Let $\{n_1, \ldots, n_e\}$ be the minimal system of generators of S.
- Then S is free (for the arrangement $\{n_1, \ldots, n_e\}$) if and only if $n_1 = c_2^* \cdots c_e^*$.

Preliminaries (ii)

Lemma

- Let S be a free numerical semigroup.
- Let $\{n_1, \ldots, n_e\}$ be the minimal system of generators of S.
- Assume that $c_i^* n_i = a_{i_1} n_1 + \cdots + a_{i_{i-1}} n_{i-1}$ for some $a_{i_1}, \ldots, a_{i_{i-1}} \in \mathbb{N}$.
- Then
  - $\left\{(c_i^* x_i, a_{i_1} x_1 + \cdots + a_{i_{i-1}} x_{i-1}) \mid i \in \{2, \ldots, e\}\right\}$ is a minimal presentation of S;
  - $c_i^* n_i$, $i \in \{2, \ldots, e\}$, are the Betti elements of S;
  - $\mathrm{Ap}(S, n_1)$ (that is, the Apéry set of n_1 in S) is equal to

    $$\left\{\lambda_2 n_2 + \cdots + \lambda_e n_e \mid \lambda_j \in \{0, \ldots, c_j^* - 1\} \text{ for all } j \in \{2, \ldots, e\}\right\}.$$

Triangular case (i)

$$\mathcal{T}_n = \langle \mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2} \rangle,\ n \geq 3.$$

Lemma

- $c_2^* = n$ and $c_3^* = \frac{n+1}{2}$ if n is odd;
- $c_2^* = \frac{n}{2}$ and $c_3^* = n+1$ if n is even.

Proposition

- A minimal presentation of $\mathcal{T}_n = \langle \mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2} \rangle$ is
  - $\left\{ \left(\frac{n+1}{2} x_3, \frac{n+3}{2} x_2 \right), \left(n x_2, (n+2) x_1 \right) \right\}$ if n is odd;
  - $\left\{ \left((n+1) x_3, (n+3) x_2 \right), \left(\frac{n}{2} x_2, \frac{n+2}{2} x_1 \right) \right\}$ if n is even.

Triangular case (ii)

Corollary

- The Betti elements of $\mathcal{T}_n$ are
  - $\frac{3}{2}\binom{n+3}{3}$ and $\frac{3}{1}\binom{n+2}{3}$ if n is odd;
  - $\frac{3}{1}\binom{n+3}{3}$ and $\frac{3}{2}\binom{n+2}{3}$ if n is even.

(Observe that they are particular multiples of tetrahedral numbers.)

Corollary

- The Apéry set $\mathrm{Ap}(\mathcal{T}_n, \mathrm{T}_n)$ is equal to
  - $\left\{ a\mathrm{T}_{n+1} + b\mathrm{T}_{n+2} \,\middle|\, a \in \{0,\ldots,n-1\},\ b \in \left\{0,\ldots,\frac{n-1}{2}\right\} \right\}$ if n is odd;
  - $\left\{ a\mathrm{T}_{n+1} + b\mathrm{T}_{n+2} \,\middle|\, a \in \left\{0,\ldots,\frac{n-2}{2}\right\},\ b \in \{0,\ldots,n\} \right\}$ if n is even.

Tetrahedral case (i)

$$\mathcal{TH}_n = \langle \mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}, \mathrm{TH}_{n+3} \rangle$$

Lemma

- Let $\mathcal{TH}_n = \langle \mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}, \mathrm{TH}_{n+3} \rangle$, where $n \equiv r \bmod 6$ with $r \in \{0,1,2,3\}$. Then
  - $c_2^* = \frac{n}{3}$, $c_3^* = n+1$, and $c_4^* = \frac{n+2}{2}$ if $n = 6k$;
  - $c_2^* = n$, $c_3^* = \frac{n+1}{2}$, and $c_4^* = \frac{n+2}{3}$ if $n = 6k+1$;
  - $c_2^* = n$, $c_3^* = \frac{n+1}{3}$, and $c_4^* = \frac{n+2}{2}$ if $n = 6k+2$;
  - $c_2^* = \frac{n}{3}$, $c_3^* = \frac{n+1}{2}$, and $c_4^* = n+2$ if $n = 6k+3$.

Lemma

- Let $\mathcal{TH}_n = \langle \mathrm{TH}_{n+3}, \mathrm{TH}_{n+2}, \mathrm{TH}_{n+1}, \mathrm{TH}_n \rangle$, where $n \equiv r \bmod 6$ with $r \in \{4,5\}$. Then
  - $c_2^* = \frac{n+5}{3}$, $c_3^* = \frac{n+4}{2}$, and $c_4^* = n+3$ if $n = 6k+4$;
  - $c_2^* = n+5$, $c_3^* = \frac{n+4}{3}$, and $c_4^* = \frac{n+3}{2}$ if $n = 6k+5$.

Tetrahedral case (ii)

Corollary

- The Betti elements of $\mathcal{TH}_n$ are
  - $\frac{4}{2}\binom{n+5}{4}$, $\frac{4}{1}\binom{n+4}{4}$ and $\frac{4}{3}\binom{n+3}{4}$ if $n=6k$;
  - $\frac{4}{3}\binom{n+5}{4}$, $\frac{4}{2}\binom{n+4}{4}$ and $\frac{4}{1}\binom{n+3}{4}$ if $n=6k+1$;
  - $\frac{4}{2}\binom{n+5}{4}$, $\frac{4}{3}\binom{n+4}{4}$ and $\frac{4}{1}\binom{n+3}{4}$ if $n=6k+2$;
  - $\frac{4}{1}\binom{n+5}{4}$, $\frac{4}{2}\binom{n+4}{4}$ and $\frac{4}{3}\binom{n+3}{4}$ if $n=6k+3$;
  - $\frac{4}{3}\binom{n+5}{4}$, $\frac{4}{2}\binom{n+4}{4}$ and $\frac{4}{1}\binom{n+3}{4}$ if $n=6k+4$;
  - $\frac{4}{1}\binom{n+5}{4}$, $\frac{4}{3}\binom{n+4}{4}$ and $\frac{4}{2}\binom{n+3}{4}$ if $n=6k+5$.

  (Observe that they are particular multiples of pentatope numbers.)

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Thank you very much for your attention!!