Ulrich Ideals

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This is a survey talk on the papers;

[GOTWY] S. Goto, Kazuho Ozeki, Ryo Takahashi, K. watanabe and K. Yoshida: Ulrich ideals and modules, Mathematical Proceedings of the Cambridge Philosophical Society, **156** (2014), 137–166.

[Nu] Takahiro Numata, Ulrich ideals of certain Gorenstein numerical semigroup rings, submitted.

As a related topic,

[GIW] S. Goto, S. Iai and K. Watanabe, Good ideals in Gorenstein local rings, Trans. A.M.S. 353 (2001), 2309-2346

Apology. I have worked on commutative algebra and singularity theory and after writing the paper

• K. Watanabe, Some examples of one dimensional Gorenstein domains, Nagoya Math. J. **49** (1973), 101-109,

I have not studied numerical semigroups until a few years

ago and I think there are lots of results I don't know. So,

I appreciate if you inform me the results I don't know.

In my talk, $(A, \mathfrak{m})$ is a d-dimensional local domain (or $A = \oplus_{n \geq 0} A_n$ be a graded domain with $A_0 = k$, a field), which is Cohen-Macaulay. Mainly I talk on the semigroup ring $k[[H]] = k[[t^a \mid a \in H]] \subset k[[t]]$ or $k[H] = k[t^a \mid a \in H]$, for a numerical semigroup H, which is Cohen-Macaulay of dimension 1. Let I be an (homogenous) $\mathfrak{m}$ primary ideal of A.

Defnition Let $(A, \mathfrak{m}), I$ be as above.

(1) (minimal reduction) An ideal $Q \subset I$ is a *minimal reduction* of I, if Q is generated by d elements and $I^{r+1} = QI^r$ for some r. (if $A = k[H]$ and $I = (t^a, t^b, \ldots, t^c)$ with $a < b < \ldots < c$, then $Q = (t^a)$) We will always assume that $I \neq Q$.

(2) (stable ideal) I is called *stable* if $I^2 = IQ$ for a minimal reduction Q of I. Assume $d = 1$. If $I^2 = QI$ and if we put $Q = (a)$, then

$$B_I = a^{-1} I = \{\frac{x}{a} \mid x \in I\}$$

is a ring containing A and $B_I/A \cong I/Q$. In this manner, a stable ideal corresponds to an overring of A contained in the integral closure of A.

(3) (Ulrich ideal) I is an *Ulrich ideal* if (i) $I^2 = QI$ for some (any) minimal reduction Q and (ii) I/I^2 is a free A/I-module.

(4) (Ulrich module) M is an *Ulrich A–module with respect to I*, if (1) M is a maximal Cohen-Macaulay A-module, (2) $\ell_A(M/IM) = \text{rank} M \cdot e(I)$ and (3) M/IM is A/I-free. ($e(I)$ is the multiplicity of I; $e(I) = \ell_A(A/Q)$).

Related definitions;

Let A, I, Q be as above.

(1) **Defnition** [GIW] I is good $\Longleftrightarrow I^2 = QI$ and $Q : I = I$. (Hence I is Ulrich $\Longrightarrow I$ is good $\Longrightarrow I$ is stable.)

(2) In case $\dim A = 1$, I is good iff I is stable and $A : B_I = I$. In this manner, there is one to one correspondence between *some* over rings of A and good ideals of A.

(3) **Thoerem.**[GIW] If $\dim A = 1$ and Gorenstein, then there is one to one correspondence between the set of good ideals of A and over rings B of A such that $A \subset B \subset K$ (total quotient ring of A) and B is finite over A.

(4) **Question** If $\dim A = 1$ and A is not Gorenstein, what is the characterization of B_I with I a good ideal?

(5) **Example.** If $\dim A = 1$, the conductor ideal $\mathfrak{c}$ is a good ideal.

Theorem. Let I be a stable ideal of A and $\mu(I)$ be the minimal number of generators of I.

(1) I is Ulrich ideal if and only if $e(I) = \ell_A(A/Q) = (\mu(I) - d + 1)\ell_A(A/I)$. ($\leq$ is always true)

(2) If I is Ulrich, then $\mathrm{type}(A) \geq (\mu(I) - 1)\mathrm{type}(A/I)$.

Corollary. Assume $d = 1$, A is Gorenstein and I is a good ideal. Then I is Ulrich iff $\mu(I) = 2$.

We want to determine homogeneous good (reps. Ulrich) ideals of $k[H]$.

Examples. (1) Let $H = \langle 4, 5, 6 \rangle$.

There is a sequence of semigroups

$\langle 4, 5, 6 \rangle \subset \langle 4, 5, 6, 7 \rangle \subset \langle 2, 5 \rangle \subset \langle 2, 3 \rangle \subset \mathbb{N}$.

The corresponding decreasing sequence of ideals ($I = A : B$) is

$A \supset \mathfrak{m} \supset I_1 = (4, 6) \supset I_2 = (6, 8, 9) \supset \mathfrak{c} = (8, \rightarrow)$,

where $\mathfrak{m}$ is not stable, I_1 is Ulrich and $I_3, \mathfrak{c}$ are good but not Ulrich.

(2) We will see that if $H = \langle a, b \rangle$ with a, b odd, then $A = k[H]$ has no homogeneous Ulrich ideal.

(3) [Taniguchi] Let $H = \langle 3, 7 \rangle$ and $A = k[[H]] = k[[t^3, t^7]]$. Then $I_c = (t^7 - ct^6, t^9)$ is Ulrich for $c \in k, c \neq 0$. Conversely, if I is an Ulrich ideal of A, then $I = I_c$ for some $c \neq 0$. On the other hand, $A = k[[t^3, t^5]]$ has no Ulrich udeals.

(4) K. Yoshida constructed Ulrich ideals of $k[[t^a, t^b]]$, where a, b odd and $b \geq 2a + 1$.

"Classical" Ulrich modules.

In 1984 B. Ulrich investigated the Maximal Cohen-Macaulay (MCM) modules over CM local domain A with equality

$\mu(M) = \text{rank}(M)e(A)$,

where $\mu(M)$ denotes the number of minimal generators of M and $e(A)$ is the multiplicity of A. We have $\leq$ always and thus Ulrich module is a MCM with most numbers of generators. Afterward, such modules are called "Ulrich modules". Even algebraic geometers (e.g. R. Hartshorne) studies "Ulrich Bundles" (vector bundles over projective varieties, the graded module associated to the bundle is an Umrich module.)

In [GOTWY] we investigated a relative version of this and we showed, for example, higher syzygy modules of an Ulrich ideal are Ulrich modules with respect to I. The classic Ulrich modules are Ulrich modules over $\mathfrak{m}$ in our language.

The theory of Ulrich ideals and good ideals have very rich results in dimension 2

Theorem.[GOTWY] Let $(A, \mathfrak{m})$ be a *rational singularity* of dimension 2. (In this case, every integrally closed ideal is stable [Lipman].)

(1) I is good if I is represented by minimal resolution of A. Hence the set of good ideals forms a semigroup and there are countable number of good ideals. (For example, if $A = k[X^r, X^{r-1}Y, \ldots, Y^r] \subset k[X, Y]$, the only good ideals on A are powers of $\mathfrak{m}$).

(2) The Ulrich ideals of A are completely classified using the geometric data of minimal resolution of A and each A has finite number of Ulrich ideals.

If $\dim A = 2$ and A is not a rational singularity, the situation is quite different.

Theorem.[Okuma-W-Yoshida]

(1) If $A \cong k[[X,Y,Z]]/(X^3+Y^3+Z^3)$ and I is an Ulrich ideal of A, then $\ell_A(A/I) = 2$ and the set of Ulrich ideals of A corresponds to the points of elliptic curve $\{X^3+Y^3+Z^3=0\} \subset \mathbb{P}^2_k$. (The same holds for any simple elliptic singularity of multiplicity 3.)

(2) If A is a 2-dimensional normal Gorenstein ring with $p_g(A) = 1$ and multiplicity ≥ 5, then A has no Ulrich ideal.

(3) If A is normal and Gorenstein, there are infinitely many good ideals on A.

A remark on Gluing (cf. [RG] Numerical Semigroups).

Let H_1, H_2 be numerical semigroups, $m_i \in H_i$ $(i = 1,2)$, which are not one of the minimal generators and assume $(m_1, m_2) = 1$. Pur $H = \langle n_2 H_1, n_1 H_2 \rangle$ and call H a *gluing* of H_1 and H_2.

Remark. [Nari] In this case, H is not almost symmetric unless H is symmetric.

Remark. There is a natural flat ring homomorphism $k[H_i] \to k[H]$. If I is an Ulrich ideal of $k[H_1]$, say, then $Ik[H]$ is an Ulrich ideal of $k[H]$. In this case, we say "$Ik[H]$ is lifted from $k[H_1]$.

Ulrich ideals of symmetric semigroups.

Theorem. Let H be a symmetric semigroup and $A = k[H] \subset k[t]$ be its semigroup ring. Then $I = (t^a, t^b)$ $(a < b)$ is an Ulrich ideal of A if and only if $b - a \notin H$, $2(b-a) \in H$ and $\langle H, b-a. \rangle$ is symmetric. Conversely, if we take $x \in \mathbb{N}$ such that $x \notin H, 2x \in H$ and $\langle H, bx. \rangle$ is symmetric, then $I = (t^a, t^b)$ is an Ulrich ideal of A, where $a = min\{h \in H \mid x + h \in H\}$.

Ulrich ideals of $A = k[t^a, t^b]$.

Let $A = k[H]$. We denote by χ_A^g the set of Ulrich ideals generated by homogeneous elements.

We can determine χ_A^g completely, since 3 generated symmetric semigroups are complete intersections and obtained by gluing.

Theorem. Let $A = k[t^a, t^b]$.

(1) If a, b are odd, then A has no Ulrich ideals.

(2) If $a = 2d$ and $b = 2l + 1$, then $\chi_A^g = \{(t^{ia}, t^{db}) \mid 1 \leq i \leq l\}$.

Ulrich ideals of 3 generated C.I.'s.

Theorem [Numata] Let $H = \langle a, b, c \rangle$ be a symmetric numerical semigroup and assume that $H = \langle d \langle a', b' \rangle, c \rangle$. We set $R = k[[H]]$, $H_1 = \langle a', b' \rangle$ and $R_1 = k[[H_1]]$. Then the following assertions hold true.

(1) If d and c are odd, then $\#\chi_R^g = \#\chi_{R_1}^g$ and every Ulrich ideal of R is lifted from R_1. In particular,

(2) If a, b, c are odd, then $\chi_R^g = \emptyset$.

In the following, we assume that a' and b' are odd (equivalently, $\chi_{R_1}^g = \emptyset$).

(3) If d is odd and c is even, then

(i) $\chi_R^g \neq \emptyset$ if and only if $H + \langle c/2 \rangle$ is symmetric.

(ii) if $\chi_R^g \neq \emptyset$, then $\#\chi_R^g = (d-1)/2$.

(4) If d is even and c is odd, then $H + \langle da'/2 \rangle$ or $H + \langle db'/2 \rangle$ is symmetric. In particular, $\chi_R^g \neq \emptyset$.

Remark. Ulrich ideals of $k[H]$, where H is symmetric and genteel by ≥ 4 elements or H is not symmetric is widely open. Also we don't know much about good ideals of $k[H]$ if H is not symmetric.

Thank you very much!