Numerical duplication of a numerical semigroup

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Based on:

- V. Barucci, M. D'Anna, F. Strazzanti, *A family of quotients of the Rees Algebra*, Communications in Algebra **43** (2015), no. 1, 130–142.
- M. D'Anna, F. Strazzanti, *The numerical duplication of a numerical semigroup*, Semigroup Forum **87** (2013), no. 1, 149-160.
- F. Strazzanti, *One half of almost symmetric numerical semigroups*, to appear in Semigroup Forum.

Idealization and amalgamated duplication

Let R be a commutative ring with identity and let M be an R-module.

The **idealization** of R with respect to M is defined as $R \oplus M$ endowed with the multiplication $(r, m)(s, n) = (rs, rn + sm)$ and it is denoted by $R \ltimes M$.

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These constructions have several properties in common, but $R \ltimes I$ is never reduced, while $R \bowtie I$ is reduced if R is.

Let $R[It] = \oplus_{n \geq 0} I^n t^n$ be the Rees algebra associated with R and I. For any $a, b \in R$ we define

$$R(I)_{a,b} := \frac{R[It]}{(I^2(t^2 + at + b))} \subseteq \frac{R[t]}{(t^2 + at + b)}$$

where $(I^2(t^2 + at + b)) = (t^2 + at + b)R[t] \cap R[It]$.

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There are the following isomorphisms:

- $R(I)_{0,0} = \frac{R[It]}{(I^2t^2)} \cong R \ltimes I$;
- $R(I)_{-1,0} = \frac{R[It]}{(I^2(t^2-t))} \cong R \bowtie I$.

Hence idealization and amalgamated duplication are members of this family, but there are also other rings.

Let S be a numerical semigroup, E an ideal of S and $b \in S$ an odd integer. The **numerical duplication** of S with respect to E and b is

$$S \bowtie^b E = 2 \cdot S \cup (2 \cdot E + b),$$

where $2 \cdot X = \{2x \mid x \in X\}$.

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Theorem

Let $R = k[[S]]$ be a numerical semigroup ring, let $b = X^m \in R$, with m odd, and let I be a proper ideal of R. Then $R(I)_{0,-b}$ is isomorphic to the semigroup ring $k[[T]]$, where $T = S \bowtie^m v(I)$.

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Theorem

Let R be an algebroid branch and let I be a proper ideal of R; let $b \in R$, such that $m = v(b)$ is odd. Then $R(I)_{0,-b}$ an algebroid branch and its value semigroup is $v(R) \bowtie^m v(I)$.

Properties

We will use this notation:

- $m(E)$ is the smallest element of E;
- $f(E)$ is the greatest element not in E;
- $g(E) = |(\mathbb{Z} \setminus E) \cap \{m(E), m(E)+1, \ldots, f(E)\}|$;
- $t(S)$ is the type of S.

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The following properties hold for $S \bowtie^b E$:

(1) $f(S \bowtie^b E) = 2f(E) + b$;

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(3) $S \bowtie^b E$ is symmetric if and only if E is a canonical ideal of S;

(4) $t(S \bowtie^b E) = |((M(S) - M(S)) \cap (E - E)) \setminus S| + |(E - M(S)) \setminus E|$, where $M(S) = S \setminus \{0\}$.

In particular $t(S \bowtie^b E)$ does not depend on b.

We set $\widetilde{E} = E - f(E) + f(S)$ and denote the standard canonical ideal of S by $K(S)$, i.e. $K(S) = \{x \in \mathbb{Z} \mid f(S) - x \notin S\}$.

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If $S \bowtie^b E$ is almost symmetric, the type of the numerical duplication is

$$t(S \bowtie^b E) = 2|(E - M(S)) \setminus E| - 1 = 2|K(S) \setminus \widetilde{E}| + 1.$$

In particular, $t(S \bowtie^b E)$ is an odd integer and $1 \leq t(S \bowtie^b E) \leq 2t(S) + 1$.

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In particular, $t(S \bowtie^b E)$ is an odd integer and $1 \leq t(S \bowtie^b E) \leq 2t(S) + 1$.

Moreover for any odd integer x such that $1 \leq x \leq 2t(S) + 1$, there exist infinitely many ideals $E \subseteq S$ such that $S \bowtie^b E$ is almost symmetric with type x.

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By definition S is one half of $S \bowtie^b E$ and then we get the next corollary

Corollary

Every numerical semigroup S is one half of infinitely many almost symmetric numerical semigroups of type x, where x is an odd integer such that $1 \leq x \leq 2t(S) + 1$.

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$$T = \langle 9, 10, 14, 15\rangle = \{0, 9, 10, 14, 15, 18, 19, 20, 23, 24, 25, 27, 28, 29, 30, 32 \rightarrow\},$$

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The point is to choose $b \in S$.
We must have $2 \cdot E + b = \{9, 15, 19, 23, 25, 27, 29, 33, 35, 37 \ldots\}$, so we have

$E = \{2, 5, 7, 9, 10, 11, 12, 14 \rightarrow\}$	if $b = 5$
$E = \{1, 4, 6, 8, 9, 10, 11, 13 \rightarrow\}$	if $b = 7$
$E = \{0, 3, 5, 7, 8, 9, 10, 12 \rightarrow\}$	if $b = 9$
E contains a negative element	if $b > 9$

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in any case E is not contained in S and then E is not a proper ideal of S.

Suppose that E is a relative ideal of S such that $E + E + b \subseteq S$. Then the numerical duplication is still a numerical semigroup. Moreover

Proposition

Every numerical semigroup T can be realized as numerical duplication $S \bowtie^b E$, where $S = \frac{T}{2}$, b is an odd element of S and E is a relative ideal of S such that $b + E + E \subseteq S$.

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Proposition

Let S be a numerical semigroup. The family of all symmetric doubles of S is

$$\mathcal{D}(S) = \{S \bowtie^b K(S) \mid K(S) + K(S) + b \subseteq S\}$$

Almost symmetric semigroups with odd type

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However $S \bowtie^b E = \{0, 8, 9, 10, 11, 12, 13, 16 \rightarrow\}$ is not almost symmetric because $1 \in K(S)$ and $1 + 13 \notin M(S \bowtie^b E)$.

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Theorem

A numerical semigroup $T = S \bowtie^b E$ is almost symmetric with odd type if and only if the following properties hold:
(1) $f(T) = 2f(E) + b$;
(2) $K(S) - (M(S) - M(S)) \subseteq \widetilde{E} \subseteq K(S)$;
(3) $K(S) - \widetilde{E}$ *is a numerical semigroup;*
(4) $E + K(S) + f(E) - f(S) + b \subseteq M(S)$.

Theorem

Suppose that $2f(S) > 2f(E) + b$*. Then the numerical duplication* $T := S \bowtie^b E$ *is almost symmetric (with even type) if and only if the following properties hold:*
(i) *S is almost symmetric;*
(ii) $M(S) - E \subseteq (E - M(S)) + b$*;*
(iii) $K(S) \subseteq E - E$*.*

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Theorem (Rosales)

A numerical semigroup different from $\mathbb{N}$ is one half of a pseudo-symmetric numerical semigroup if and only if it is either symmetric or pseudo-symmetric.

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Corollary

A numerical semigroup S different from $\mathbb{N}$ is one half of an almost symmetric numerical semigroup T with even type if and only if it is almost symmetric. In this case the type of S is less than or equal to the type of T.

THANK YOU!