Counting modules over numerical semigroups with two generators.

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Reference

The talk is based on my joint work with Jan Uliczka

Lattice paths with given number of turns
and semimodules over numerical semigroups

published in Semigroup Forum 88(3) (2014), 631–646.

Outline

Our motivation was to gain a better understanding of our previous result on Hilbert depth.

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Review: fundamental couple

The crucial notion in the previous work was that of a *fundamental couple*: Let $\alpha, \beta > 0$ be coprime and let $G := \mathbb{N} \setminus \langle \alpha, \beta \rangle$.

An (α, β)–*fundamental couple* $[I, J]$ consists of two integer sequences $I = (i_k)_{k=0}^m$ and $J = (j_k)_{k=0}^m$, such that

(0) $i_0 = 0$.

(1) $i_1, \dots, i_m, j_1, \dots, j_{m-1} \in G$ and $j_0, j_m \leq \alpha\beta$.

(2)
$$\begin{array}{lllll} i_k \equiv j_k & \bmod \alpha & \text{and} & i_k < j_k & \text{for } k = 0, \dots, m; \\ j_k \equiv i_{k+1} & \bmod \beta & \text{and} & j_k > i_{k+1} & \text{for } k = 0, \dots, m-1; \\ j_m \equiv i_0 & \bmod \beta & \text{and} & j_m \geq i_0. & \end{array}$$

(3) $|i_k - i_\ell| \in G$ for $1 \leq k < \ell \leq m$.

Γ-lean sets

One of the problems considered in this talk will be the counting of sets of integers like those appearing in the first position of a fundamental couple. We coin a name for these sets:

Definition

Let Γ be a numerical semigroup. A set $\{x_0 = 0, x_1, \ldots, x_n\} \subseteq \mathbb{N}$ is called Γ-lean if $|x_i - x_j| \notin \Gamma$ for $0 \leq i < j \leq n$.

Γ-semimodules

A key notion in this talk will be that of a *module* over a numerical semigroup Γ:

Definition

A Γ-semimodule Δ is a non-empty subset of $\mathbb{N}$ such that $\Delta + \Gamma \subseteq \Delta$.

Note that a Γ-semimodule $\Delta \neq \Gamma, \mathbb{N}$ is not a semigroup.

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Two Γ-semimodules Δ, Δ' are called *isomorphic* if there is an integer n such that $x \mapsto x + n$ is a bijection from Δ to Δ'.

For every Γ-semimodule Δ there is a unique semimodule $\Delta^\circ \cong \Delta$ containing 0; such a Γ-semimodule is called *normalized*.

Generators of Γ-semimodules

A system of generators of a Γ-semimodule Δ is a subset $\mathcal{E}$ of Δ with

$$\bigcup_{x \in \mathcal{E}} (x + \Gamma) = \Delta.$$

It is called *minimal* if no proper subset of $\mathcal{E}$ generates Δ.

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Lemma

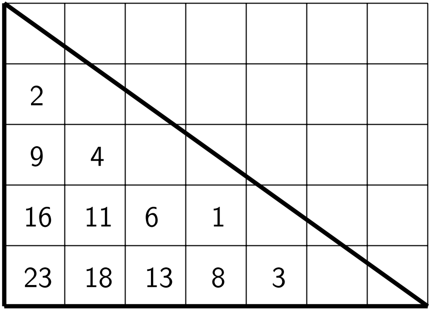
(i) Every Γ-semimodule Δ has a unique minimal system of generators.

(ii) The minimal system of generators of a normalized Γ-semimodule is Γ-lean, and conversely, every Γ-lean subset of $\mathbb{N}$ generates minimally a normalized Γ-semimodule.

Gaps of $\langle \alpha, \beta \rangle$ and lattice points

From now on we only consider semigroups $\Gamma = \langle \alpha, \beta \rangle$ with $\alpha < \beta$.

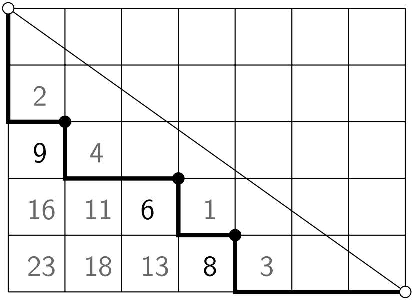
There is a map $G \to \mathbb{N}^2$, $\alpha\beta - a\alpha - b\beta \mapsto (a, b)$ which identifies a gap with a lattice point. Since $\alpha\beta - a\alpha - b\beta > 0$ the point lies inside the triangle with corners $(0,0), (\beta, 0), (0, \alpha)$.



Gaps of $\langle 5, 7 \rangle$

$\langle \alpha, \beta \rangle$-lean sets and lattice paths

An $\langle \alpha, \beta \rangle$-lean set yields a lattice path with steps downwards and to the right from $(0, \alpha)$ to $(\beta, 0)$ not crossing the diagonal, where the points identified with the gaps mark the turns from x-direction to y-direction. In the sequel those turns will be called ES-turns for short.



Lattice path for the $\langle 5, 7 \rangle$-lean set $\{0, 9, 6, 8\}$.

Counting of lattice paths

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The number of *all* lattice paths with r ES-turns from $(0, \alpha)$ to $(\beta, 0)$ is easily computed: The r turning points have x-coordinates in the range $\{1, \ldots, \beta - 1\}$ and also y-coordinates in the range $\{1, \ldots, \alpha - 1\}$. Since the sequence of coordinates has to be increasing resp. decreasing there are

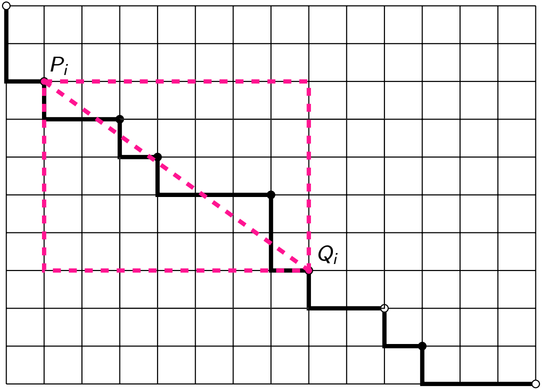
$$\binom{\beta - 1}{r} \binom{\alpha - 1}{r}$$

lattice paths.

Question: How many of these paths stay below the diagonal?

Cyclic permutation of a path

We extend the path with turning points $P_i = (x_i, y_i)$ beyond $(\beta, 0)$ with points $Q_i = (x_i + \beta, y_i - \alpha)$, thus amending a second copy of the original path. The cyclic permutations are the paths from P_i to Q_i. Among these there is exactly one staying below the diagonal.



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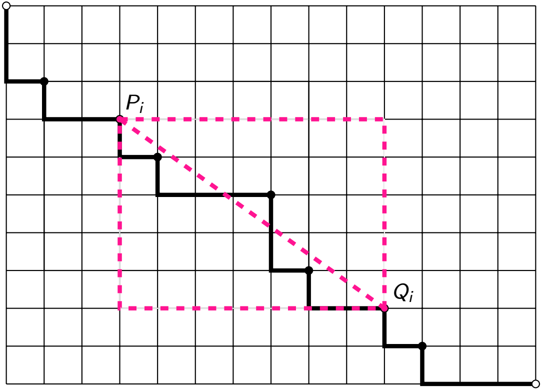

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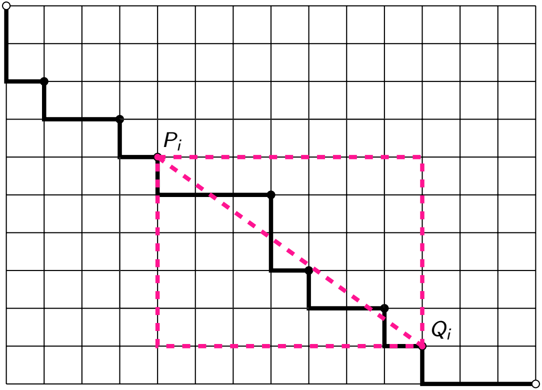

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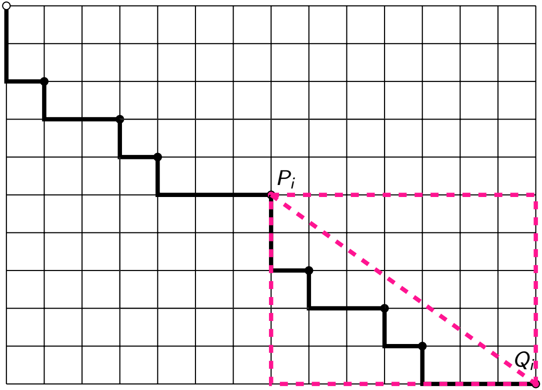

Cyclic permutation of a path: conclusion

Proposition

Let α and β be two coprime positive integers.

1. For every lattice path from $(0, \alpha)$ to $(\beta, 0)$ there is exactly one cyclic permutation staying below the diagonal.
2. The number of $\langle \alpha, \beta \rangle$-lean sets with r gaps equals the number of lattice paths with r ES-turns from $(0, \alpha)$ to $(\beta, 0)$ staying below the diagonal, and this number is given by

$$\frac{1}{r+1}\binom{\alpha-1}{r}\binom{\beta-1}{r}.$$

Counting of $\langle\alpha, \beta\rangle$-lean sets and lattice paths

Number of sets with given size

Theorem

Let $\alpha, \beta, r \in \mathbb{N}$ *with* $\gcd(\alpha, \beta) = 1$. *Then the following numbers*

- *the number of isomorphism classes of* $\langle\alpha, \beta\rangle$*-semimodules minimally generated by* $r + 1$ *elements*
- *the number of* $\langle\alpha, \beta\rangle$*-lean sets with* r *gaps*
- *the number of lattice paths with* r *ES-turns from* $(0, \alpha)$ *to* $(\beta, 0)$ *staying below the diagonal*

equal

$$L_{\alpha,\beta}(r) := \frac{1}{r+1}\binom{\alpha - 1}{r}\binom{\beta - 1}{r}.$$

Counting of $\langle\alpha,\beta\rangle$-lean sets and lattice paths

Total number

Using the Vandermonde convolution one gets a formula for $\sum_{r\geq 0} L_{\alpha,\beta}(r)$, recovering results of Bizley, resp. Beauville, Piontkowski, and Fantechi–Göttsche–van Straten:

Theorem

Let $\alpha,\beta\in\mathbb{N}$ be coprime. Then the following numbers

- *the number of isomorphism classes of $\langle\alpha,\beta\rangle$-semimodules*
- *the number of $\langle\alpha,\beta\rangle$-lean sets*
- *the number of lattice paths from $(0,\alpha)$ to $(\beta,0)$ staying below the diagonal*

equal

$$L_{\alpha,\beta} := \sum_{r\geq 0} L_{\alpha,\beta}(r) = \frac{1}{\alpha+\beta}\binom{\alpha+\beta}{\alpha}.$$

The sequence J and lattice paths

We consider now the second position of a fundamental couple.

Let $[I, J]$ be a fundamental couple with sequences $I = [i_0 = 0, \dots, i_n]$ and $J = [j_0, \dots, j_n]$. By definition, the elements $j_1, \dots, j_{n-1}$ are gaps of $\langle \alpha, \beta \rangle$ such that

$$j_k \equiv i_k \mod \alpha \quad \text{and } j_k \equiv i_{k+1} \mod \beta.$$

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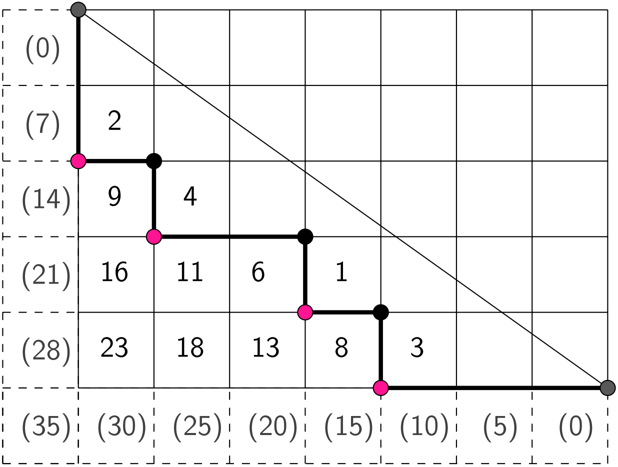
An inspection of the lattice path belonging to I shows that these gaps $j_1, \ldots, j_{n-1}$ correspond to the inner SE-turning points of the path. By extension of the labeling beyond the axis we can even identify j_0 and j_n with the remaining SE-turns.

The sequence J and lattice paths: example



$I = [0, 8, 6, 9]$ and $J = [15, 13, 16, 14]$.

Syzygies of $\langle\alpha, \beta\rangle$-semimodules

Next we explain the meaning of J in terms of $\langle\alpha, \beta\rangle$-semimodules: Every $\langle\alpha, \beta\rangle$-semimodule Δ yields another $\langle\alpha, \beta\rangle$-semimodule $\mathrm{Syz}(\Delta)$.

Definition

Let I be an $\langle\alpha, \beta\rangle$-lean set, and let Δ be the $\langle\alpha, \beta\rangle$-semimodule generated by I. The syzygy of Δ is the $\langle\alpha, \beta\rangle$-semimodule

$$\mathrm{Syz}(\Delta) := \bigcup_{\substack{i, i' \in I \\ i \neq i'}} \left((i + \langle\alpha, \beta\rangle) \cap (i' + \langle\alpha, \beta\rangle) \right).$$

The semimodule $\mathrm{Syz}(\Delta)$ consists of those elements in Δ which admit more than one presentation of the form $i + x$ with $i \in I, x \in \langle\alpha, \beta\rangle$.

Fundamental couples and syzygies

The connection between fundamental couples and syzygies is described in the following theorem:

Theorem

Let $[I, J]$ be an $\langle\alpha,\beta\rangle$-fundamental couple and let Δ be the $\langle\alpha,\beta\rangle$-semimodule generated by the elements of I. Then

$$\mathrm{Syz}(\Delta) = \bigcup_{0\leq k<m\leq n} \Big(i_k + \langle\alpha,\beta\rangle \cap i_m + \langle\alpha,\beta\rangle\Big) = \bigcup_{k=0}^{n} (j_k + \langle\alpha,\beta\rangle).$$

Relation between I and J

Consider the I and J-sets of $\langle 3,5\rangle$ of length 2:

$$\begin{array}{ccccc}
\{0,1\} & \leq & \{6,10\} & \longleftrightarrow & \{0,4\} \\
\{0,2\} & \leq & \{5,12\} & \longleftrightarrow & \{0,7\} \\
\{0,4\} & \leq & \{9,10\} & \longleftrightarrow & \{0,1\} \\
\{0,7\} & \leq & \{10,12\} & \longleftrightarrow & \{0,2\}
\end{array}$$

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We see *two* $\langle 3,5\rangle$-*orbits* of length 2:

$$\begin{array}{ccccc} \{0,1\} & \longleftrightarrow & \{0,4\} & \longleftrightarrow & \{0,1\} \\ \{0,2\} & \longleftrightarrow & \{0,7\} & \longleftrightarrow & \{0,2\} \end{array}$$

Take I and J-sets of $\langle 3,5\rangle$ of (maximal) length 3:

$$\begin{array}{ccccc} \{0,1,2\} & \leq & \{5,6,7\} & \longleftrightarrow & \{0,1,2\} \\ \{0,2,4\} & \leq & \{5,7,9\} & \longleftrightarrow & \{0,2,4\} \end{array}$$

Take I and J-sets of $\langle 3,5\rangle$ of (maximal) length 3:

$$\begin{array}{ccccc} \{0,1,2\} & \leq & \{5,6,7\} & \longleftrightarrow & \{0,1,2\} \\ \{0,2,4\} & \leq & \{5,7,9\} & \longleftrightarrow & \{0,2,4\} \end{array}$$

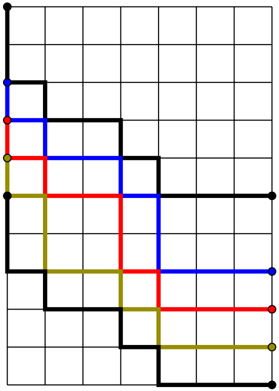
There are *two* $\langle 3,5\rangle$-*fixed points*:

$$\begin{array}{ccc} \{0,1,2\} & \longleftrightarrow & \{0,1,2\} \\ \{0,2,4\} & \longleftrightarrow & \{0,2,4\} \end{array}$$

Iterated syzygies and their orbits

The procedure of building a syzygy can be iterated; we set

$$\mathrm{Syz}^{\ell}(\Delta) := \mathrm{Syz}(\mathrm{Syz}^{\ell-1}(\Delta)), \quad \ell \geq 2.$$

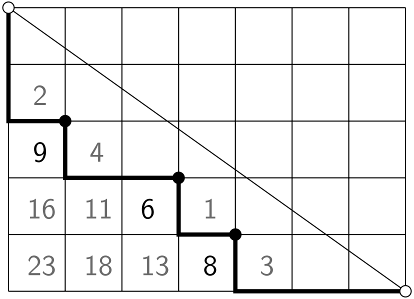


Since all semimodules $\mathrm{Syz}^{\ell}(\Delta)$ share the same number of generators, it is clear that this sequence must be periodic up to isomorphism.

The set of isomorphism classes appearing in such a sequence of syzygies will be called an orbit. We want to investigate which orbits occur.

Matrix description of a path

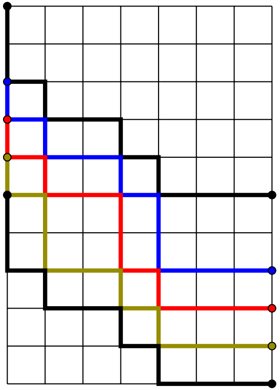
A lattice path with r ES-turns can also be described by a $2 \times (r+1)$-matrix where the i-th column contains the numbers of steps downwards and to the right the path takes between the $(i-1)$-th and the i-th ES-turning points. In our example:



$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$$

Syzygies and the matrix description

It is easily seen that taking the syzygy cyclically permutates the top row of the matrix by one position to the left:



$$\Delta \mapsto \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$$

$$\mathrm{Syz}(\Delta) \mapsto \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 3 \end{pmatrix}$$

$$\mathrm{Syz}^2(\Delta) \mapsto \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$$

$$\mathrm{Syz}^3(\Delta) \mapsto \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$$

Matrix pattern for elements of an orbit

Let Δ be an $\langle \alpha, \beta \rangle$-semimodule with $\mathrm{Syz}^{\ell}(\Delta) \cong \Delta$. Then the matrices

$$\begin{pmatrix} y_0 & y_1 & \dots & y_{n-1} \\ x_0 & x_1 & \dots & x_{n-1} \end{pmatrix} \text{ and } \begin{pmatrix} y_\ell & y_{\ell+1} & \dots & y_{\ell-2} & y_{\ell-1} \\ x_0 & x_1 & \dots & x_{n-1} & x_n \end{pmatrix}$$

for Δ resp. $\mathrm{Syz}^{\ell}(\Delta)$ have to be equal up to cyclic permutation of columns.

Some elementary number-theoretical arguments show that the matrix has to be of the form

$$\begin{pmatrix} [y_0 \ \dots\dots \ y_{m-1}] & \dots & [y_0 \ \dots\dots \ y_{m-1}] \\ [x_0 \ \dots \ x_{k-1}] & \dots & [x_0 \ \dots \ x_{k-1}] \end{pmatrix}$$

with $\gcd(k, m) = \ell$. Note that there are constraints for the numbers of blocks in the rows, since $\sum_i y_i = \alpha$ and $\sum_i x_i = \beta$.

Counting orbits

The matrix pattern leads—for instance—to a formula for the counting of fixed points of sets of given n = length of I:

Theorem

For any integer $n \leq \alpha$ with $n \mid \alpha\beta$ there are

$$\frac{1}{n}\binom{\frac{\alpha}{\gcd(\alpha,n)}-1}{\gcd(\beta,n)-1}\binom{\frac{\beta}{\gcd(\beta,n)}-1}{\gcd(\alpha,n)-1}$$

$\langle\alpha,\beta\rangle$-fixed points with n generators.

Remark: If $n \nmid \alpha\beta$, no fixed points occur.

BAHN(7,12);

Bahnen mit $|I| = 2$ (33 Mengen)
Fixpunkte: 3
2-Bahnen: 15
gesamt: 18

BAHN(7,12);

Bahnen mit $|I| = 2$ (33 Mengen)
Fixpunkte: 3
2-Bahnen: 15
gesamt: 18

Bahnen mit $|I| = 3$ (275 Mengen)
Fixpunkte: 5
2-Bahnen: 0
3-Bahnen: 90
gesamt: 95

BAHN(7,12);

Bahnen mit $|I| = 2$ (33 Mengen)
Fixpunkte: 3
2-Bahnen: 15
gesamt: 18

Bahnen mit $|I| = 3$ (275 Mengen)
Fixpunkte: 5
2-Bahnen: 0
3-Bahnen: 90
gesamt: 95

Bahnen mit $|I| = 4$ (825 Mengen)
Fixpunkte: 5
2-Bahnen: 10
3-Bahnen: 0
4-Bahnen: 200
gesamt: 215

Bahnen mit $|I| = 5$ (990 Mengen)

Fixpunkte: 0

2-Bahnen: 0

3-Bahnen: 0

4-Bahnen: 0

5-Bahnen: 198

gesamt: 198

Bahnen mit $|I| = 5$ (990 Mengen)
Fixpunkte: 0
2-Bahnen: 0
3-Bahnen: 0
4-Bahnen: 0
5-Bahnen: 198
gesamt: 198

Bahnen mit $|I| = 6$ (462 Mengen)
Fixpunkte: 1
2-Bahnen: 1
3-Bahnen: 3
4-Bahnen: 0
5-Bahnen: 0
6-Bahnen: 75
gesamt: 80

Bahnen mit $|I| = 7$ (66 Mengen)
Fixpunkte: 66
2-Bahnen: 0
3-Bahnen: 0
4-Bahnen: 0
5-Bahnen: 0
6-Bahnen: 0
7-Bahnen: 0
gesamt: 66