Almost Gorenstien rings

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History

- **1997** The notion of almost Gorenstein rings was introduced by Valentina Barucci-Ralf Fröberg (analytically unramified case) with a result about the Gorenstein property of $\mathfrak{m}:\mathfrak{m} = \{\alpha \in Q(R) \mid \alpha\mathfrak{m} \subseteq \mathfrak{m}\}$.
- **2009** A counterexample for a result about $\mathfrak{m}:\mathfrak{m}$ was given by Barucci. (But their result is true !)
- **2013** A new definition of almost Gorenstein rings of dimension one was given and repair the proof of the Gorenstein property of $\mathfrak{m}:\mathfrak{m}$.

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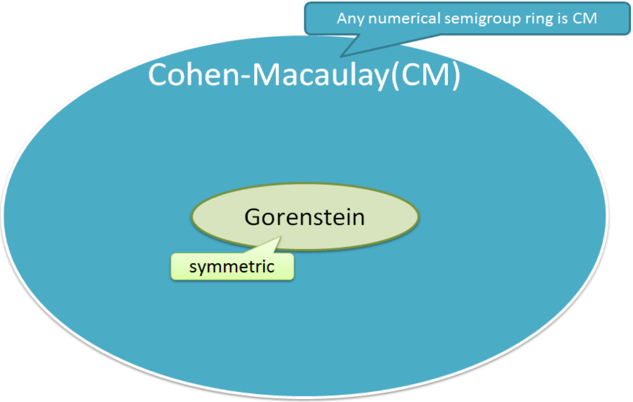
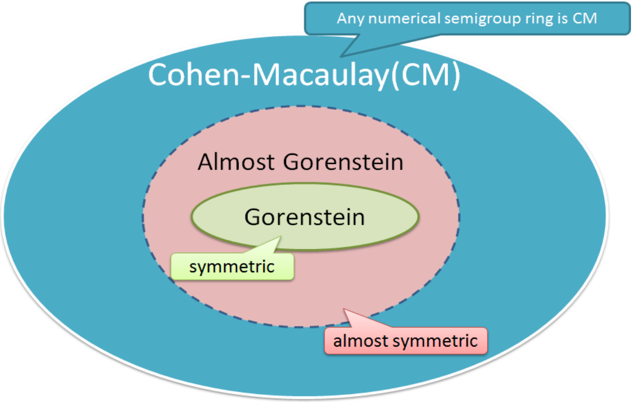
Classes of local rings

regular ⇒ complete-intersection ⇒ Gorenstein
⇒ Cohen-Macaulay ⇒ Buchsbaum



Classes of local rings

regular $\Rightarrow$ complete-intersection $\Rightarrow$ Gorenstein $\Rightarrow$ almost Gorenstein $\Rightarrow$ Cohen-Macaulay $\Rightarrow$ Buchsbaum



1. Preliminaries (Hilbert coefficients, existence of canonical ideals)
2. Definition of almost Gorenstein rings
3. The Gorenstein property of $\mathfrak{m} : \mathfrak{m}$
4. Almost Gorenstein rings obtained by idealization
5. Recent researches

$k[[H]] = k[[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}]] \subseteq k[[t]]$ for $H = \langle a_1, a_2, \ldots, a_\ell \rangle$.

§1. Preliminaries

Let $(R, \mathfrak{m})$ a CM local ring, $\dim R = 1$, I an $\mathfrak{m}$-primary ideal
$\Rightarrow \exists\, \mathrm{e}_0(I), \mathrm{e}_1(I) \in \mathbb{Z}$ such that

$$\ell_R(R/I^{n+1}) = \mathrm{e}_0(I)\binom{n+1}{1} - \mathrm{e}_1(I) \quad (\forall n \gg 0).$$

We call $\mathrm{e}_0(I)$ the multiplicity of R w.r.t. I and $\mathrm{e}_1(I)$ the first Hilbert coefficient of R w.r.t. I.

How to compute $\mathrm{e}_0(I)$ and $\mathrm{e}_1(I)$?

- Assume $\exists a \in I$ such that $Q = (a)$ is a reduction of I (i.e., $\exists n \geq 0,\ I^{n+1} = QI^n$)
- For any $n > 0$, put $\frac{I^n}{a^n} = \{\frac{x}{a^n} \mid x \in I^n\} \subseteq \mathrm{Q}(R)$
- Let $S = R[\frac{I}{a}] \subseteq \mathrm{Q}(R)$.
  $\Rightarrow S = \bigcup_{n>0} \frac{I^n}{a^n} = \frac{I^r}{a^r}$
  where $r = \mathrm{red}_Q(I) = \min\{n \geq 0 \mid I^{n+1} = QI^n\}$.
- Hence

$$\begin{aligned}\ell_R(R/I^{n+1}) &= \ell_R(R/Q^{n+1}) - \ell_R(I^{n+1}/Q^{n+1}) \\ &= \underbrace{\ell_R(R/Q)}_{\mathrm{e}_0(I)}\binom{n+1}{1} - \underbrace{\ell_R(S/R)}_{\mathrm{e}_1(I)} \quad \text{if } n \geq r-1\end{aligned}$$

Theorem

$e_0(I) = \ell_R(R/Q)$, $e_1(I) = \ell_R(S/R)$.

Corollary

$$\mu_R(I/Q) \leq \ell_R(I/Q) \leq e_1(I)$$

1. $\mu_R(I/Q) = \ell_R(I/Q) \iff \mathfrak{m}I \subseteq Q$ (*i.e.* $\mathfrak{m}I = \mathfrak{m}Q$)
2. $\ell_R(I/Q) = e_1(I) \iff I^2 = QI$ (*i.e.* $\mathrm{red}_Q(I) \leq 1$)

The case $H = \langle 3, 4, 5 \rangle$

Example

Let $H = \langle 3, 4, 5 \rangle$ and $R = k[[H]] = k[[t^3, t^4, t^5]]$ (k a field). Take $I = (t^3, t^4)$ and $Q = (t^3)$, then Q is a reduction of I. In fact, $I^3 = QI^2$. Hence $S = \frac{I^2}{t^6}$.

For $\mathrm{e}_0(I)$:

0	1	2
3	4	5
6	7	8
9	10	11
...		

R

0	1	2
3	4	5
6	7	8
9	10	11
...		

Q

0	1	2
3	4	5
6	7	8
9	10	11
...		

R/Q

$\Rightarrow \mathrm{e}_0(I) = \ell_R(R/Q) = 3$

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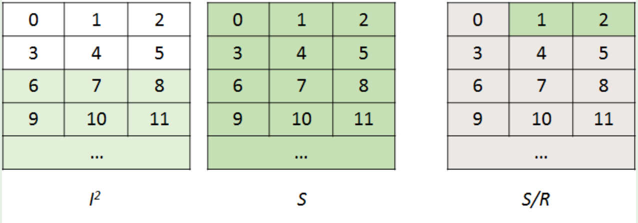
For $e_1(I)$:



$\Rightarrow e_1(I) = \ell_R(S/R) = 2$

Existence of canonical ideals

- Let K_R denote the canonical module of R.
- $\exists\, \mathrm{K}_R \Leftrightarrow R \cong$ a Gorenstein ring $/ \sim$.

Definition

We say that $I \subsetneq R$ is a *canonical ideal* of R if $I \cong \mathrm{K}_R$.

When $\exists I \subsetneq R$ a canonical ideal?

Theorem (Herzog-Kunz)

TFAE

1. $\exists I \subsetneq R$ *a canonical ideal of* R.
2. $\mathrm{Q}(\widehat{R})$ *is a Gorenstein ring.*

Hence if R *is analytically unramified then* $\exists I$ *a canonical ideal of* R.

Corollary

Suppose that $\mathrm{Q}(\widehat{R})$ *is Gorenstein. If* $|R/\mathfrak{m}| = \infty$, *then* $R \subseteq \exists K \subseteq \overline{R}$ *such that* $K \cong \mathrm{K}_R$ *where* $\overline{R}$ *is the integral closure of* R.

Proof.

$\exists a \in I$ such that $Q = (a)$ is a reduction of I. Put $K = \frac{I}{a}$ □

Example

Let $R = k[[X, Y, Z]]/(X, Y) \cap (Y, Z) \cap (Z, X)$. Then
$I = (x + y, y + z) \cong \mathrm{K}_R$.
If $k = \mathbb{Z}/(2)$, then $\forall a \in I$, (a) is *not* a reduction of I.

Proposition

Let $k = R/\mathfrak{m}$ *and* $\widetilde{k}/k$ *an extension of fields. then* $\exists \varphi : (R, \mathfrak{m}) \to (\widetilde{R}, \widetilde{\mathfrak{m}})$ *a flat homomorphism of local rings such that*

1. $\widetilde{\mathfrak{m}} = \mathfrak{m}\widetilde{R}$
2. $\widetilde{R}/\widetilde{\mathfrak{m}} \cong \widetilde{k}$ *as k-algebras.*

Moreover we have the following

(a) $Q(\widehat{\widetilde{R}})$ *is Gorenstein* $\Leftrightarrow Q(\widehat{R})$ *is Gorenstein. In this case,* $\forall I$ *a canonical ideal of* R, $I\widetilde{R}$ *is a canonical ideal of* $\widetilde{R}$ *and* $\mathrm{e}_1(I\widetilde{R}) = \mathrm{e}_1(I)$.

(b) $m : m$ *is Gorenstein* $\Leftrightarrow \widetilde{m} : \widetilde{m}$ *is Gorenstein.*

Problem

? R *is analytically unramified* $\Longleftrightarrow \widetilde{R}$ *is analytically unramified*

§2. Definition of almost Gorenstein rings

Definition

We say that R is an almost Gorenstein ring, if

1. $Q(\widehat{R})$ is Gorenstein. Hence $\exists I \subsetneq R$ a canonical ideal of R.
2. $e_1(I) \leq r(R)$ (the Cohen-Macaulay type of R) $= \mu_R(I)$.

Remark

Let $I, J \subsetneq R$ canonical ideals, then $e_1(I) = e_1(J)$.

R is Gorenstein $\Rightarrow r(R) = 1$ and I is a parameter ideal. Hence $e_1(I) = 0 \leq r(R)$. Thus R is almost Gorenstein

Examples of almost Gorenstein rings

Example

1. $R = k[[t^3, t^4, t^5]] \subseteq k[[t]]$ ($\mathrm{r}(R) = 2$; an integral domain)
2. $R = k[[X, Y, Z]]/(X, Y) \cap (Y, Z) \cap (Z, X)$ ($\mathrm{r}(R) = 2$; a reduced ring)
3. $R = k[[X, Y, Z, W]]/(Y^2, Z^2, W^2, YW, ZW, XW - YZ)$ ($\mathrm{r}(R) = 3$; not a reduced ring)
4. Let $3 \le a \in \mathbb{Z}$ and $R = k[[t^a, t^{a+1}, t^{a^2-a-1}]]$.
   Let I be a canonical ideal of $R \Rightarrow \mathrm{e}_1(I) = \frac{a(a-1)}{2} - 1$, $\mathrm{r}(R) = 2$.
   Hence R is an almost Gorenstein ring $\Leftrightarrow a = 3$. On the other hand, $R \cong k[[x, y, z]]/\mathrm{I}_2\begin{pmatrix} x & y^{a-2} & z \\ y & z & x^{a-1} \end{pmatrix}$. Hence, by Nari-Numata-Watanabe, R is almost Gorenstein $\Leftrightarrow a - 2 = 1$.

Settings

$\overline{R}$
|
S
|
K
|
R

- $R \subseteq \exists K \subseteq \overline{R}$ an R-submodule such that $K \cong \mathrm{K}_R$.
- Choose a NZD $a \in \mathfrak{m}$ such that $I = aK \subsetneq R$.
  Hence $Q = (a)$ is a reduction of I.
- $S = R[\frac{I}{a}] = R[K]$.
- $\mathfrak{c} = R : S := \{\alpha \in Q(R) \mid \alpha S \subseteq R\} \subseteq R$.

Definition (BF)

R is an almost Gorenstein ring (in the sense of [BF]) if $\mathfrak{m}K \subseteq R$.

Lemma

$$\mathrm{r}(R) - 1 = \mu_R(I/Q) \le \ell_R(I/Q) \le \mathrm{e}_1(I) = \ell_R(I/Q) + \ell_R(R/\mathfrak{c})$$

Proof

- $\mathrm{e}_1(I) = \ell_R(S/R) = \ell_R(S/K) + \ell_R(K/R)$.
- Since $\ell_R(K/R) = \ell_R(I/Q)$, it is enough to show that $\ell_R(S/K) = \ell_R(R/\mathfrak{c})$
- $K : S = K : KS = (K : K) : S = R : S = \mathfrak{c}$.
- $\ell_R(S/K) = \ell_R(K : K/K : S) = \ell_R(R/\mathfrak{c})$.

Characterization of Gorenstein rings

Theorem

TFAE

1. *R is a Gorenstein ring.*
2. $K = R$.
3. $K = S$.
4. $R = S$.
5. $\ell_R(S/R) = \ell_R(R/\mathfrak{c})$.
6. $I^2 = QI$.
7. $\mathrm{e}_1(I) = 0$.
8. $\mathrm{e}_1(I) = \mathrm{r}(R) - 1$.

Characterization of almost Gorenstein rings

Theorem

R is an almost Goresntein ring $\Longleftrightarrow \mathfrak{m}K \subseteq R$ (*i.e.* $\mathfrak{m}I \subseteq Q$)
When this is the case, $\mathfrak{m}S \subseteq R$.

This means two definitions of almost Gorenstein property coincide.

$$\begin{array}{ccccc} \mathrm{r}(R)-1 & \leq & \ell_R(I/Q) & \leq & \mathrm{e}_1(I) \\ & \uparrow & & \uparrow & \\ & \mathfrak{m}I \subseteq Q & & I^2 = QI & \\ & \Updownarrow & & \Updownarrow & \\ & \text{almost Gorenstein} & & \text{Gorenstein} & \end{array}$$

Proof

Theorem

R is an almost Goresntein ring $\iff \mathfrak{m}K \subseteq R$ (*i.e.* $\mathfrak{m}I \subseteq Q$)
When this is the case, $\mathfrak{m}S \subseteq R$

$$\mathrm{r}(R) - 1 \leq \ell_R(I/Q) \leq \mathrm{e}_1(I)$$

- $\Rightarrow$ is easy.
- $\Leftarrow$ We may assume R is not Gorenstein.
- Put $J = Q :_R \mathfrak{m}$. Then $I \subseteq J$ and $J^2 = QJ$.
- We have $R \subseteq S = R[\frac{I}{a}] \subseteq R[\frac{J}{a}] = \frac{J}{a}$.
- Hence $\mathrm{e}_1(I) = \ell_R(S/R) \leq \ell_R(R[\frac{J}{a}]/R) = \ell_R(J/Q) = \mathrm{r}(R)$.

Corollary

TFAE

1. *R is almost Gorenstein but not Gorenstein.*
2. $\mathrm{e}_1(I) = \mathrm{r}(R)$.
3. $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_R(R/I) + 1$ *(Sally's equality).*
4. $S = K : \mathfrak{m}$.
5. $\ell_R(I^2/QI) = 1$.
6. $\mathfrak{m} : \mathfrak{m} = S$ *and R is not a DVR.*

When this is the case,

(a) $\mathrm{red}_Q(I) = 2$.

(b) *Put $G = \mathrm{gr}_I(R)$ and $M = \mathfrak{m}G + G_+$. Then G is Buchsbaum and $\mathrm{H}^0_M(G) = [\mathrm{H}^0_M(G)]_0 \cong R/\mathfrak{m}$. Hence $\mathbb{I}(G) = 1$.*

Proof of (1) ⇔ (3)

1. R is almost Gorenstein but not Gorenstein.
3. $e_1(I) = e_0(I) - \ell_R(R/I) + 1$ (Sally's equality).

$$\begin{aligned} e_1(I) - e_0(I) &= (\ell_R(R/\mathfrak{c}) + \ell_R(I/Q)) - \ell_R(R/Q) \\ &= \ell_R(R/\mathfrak{c}) - \ell_R(R/I). \end{aligned}$$

- Hence $e_1(I) = e_0(I) - \ell_R(R/I) + \ell_R(R/\mathfrak{c})$.
- $(3) \iff \mathfrak{m} = \mathfrak{c}(= S : R) \iff S \neq R$ and $\mathfrak{m}S \subseteq R \iff (1)$

1. $e_1(I) = r(R) - 1 \Longleftrightarrow R$ is Gorenstein.
2. $e_1(I) = r(R) \Longleftrightarrow R$ is almost Gorenstein but not Gorenstein.

Problem

When $e_1(I) = r(R) + 1$*?*

Theorem

$e_1(I) \neq r(R) + 1$

1. $e_1(I) = r(R) - 1 \iff R$ is Gorenstein.
2. $e_1(I) = r(R) \iff R$ is almost Gorenstein but not Gorenstein.

Problem

When $e_1(I) = r(R) + 1$*?*

Theorem

$e_1(I) \neq r(R) + 1$

§3. The Gorenstein property of $\mathfrak{m} : \mathfrak{m}$

Theorem (Barucci-Fröberg)

TFAE

1. $A = \mathfrak{m} : \mathfrak{m}$ *is a Gorenstein ring.*
2. R *is an almost Gorenstein ring and* $v(R) = \mathrm{e}(R)$.
   $v(R)$ *the embedding dimension of* R,
   $\mathrm{e}(R)$ *the multiplicity of* R

When $R = k[[H]]$ is a numerical semigroup ring of $H = \langle a_1, a_2, \ldots, a_\ell \rangle$, $v(R) = \ell$ and $\mathrm{e}(R) = \min\{a_i \mid 1 \leq i \leq \ell\}$.

How to prove the theorem

- We may assume $R/\mathfrak{m}$ is algebraically closed.
- Thanks to this assumption, we get the following claim.

Claim

$\ell_A(X) = \ell_R(X)$ *for* $\forall X$ *A-modules.*

- Then Barucci and Fröberg's argument works well.

Problem (again)

$$? \ R \textit{ is analytically unramified} \iff \widetilde{R} \textit{ is analytically unramified}$$

We now have no proof in special cases, but we can prove in full generality.

§4. Almost Gorenstein rings obtained by idealization

Theorem

TFAE

1. $R \ltimes \mathfrak{m}$ *is an almost Gorenstein ring.*
2. R *is an almost Gorenstein ring.*

When this is the case, $v(R \ltimes \mathfrak{m}) = 2v(R)$.

Example

$\forall n \geq 0$, put

$$R_n = \begin{cases} R & (n = 0) \\ R \ltimes \mathfrak{m} & (n = 1) \\ [R_{n-1}]_1 & (n \geq 2). \end{cases}$$

1. If R is Gorenstein, then R_n is almost Gorenstein ($\forall n \geq 0$).
2. R_n is not Gorenstein ($\forall n \geq 2$).

Example

$$k[[X, Y, Z, W]]/(Y^2, Z^2, W^2, YW, ZW, XW - YZ)$$
$$\cong k[[X, Y]]/(Y^2) \ltimes (X, Y)/(Y^2)$$

Recent researches

GTT Shiro Goto, Ryo Takahashi, and Naoki Taniguchi gave a possible definition of **higher-dimensional** or **graded** almost Gorenstein rings in terms of $C = \mathrm{Coker}(0 \to R \to \mathrm{K}_R)$.
(to appear in Journal of Pure and Applied Algebra, arXiv:1403.3599)

MM Satoshi Murai and I consider the graded almost Gorenstein property for Stanley-Reisner rings following the definition by [GTT]. (Preprint, arXiv:1405.7438).