

On the computation of the Apéry set of numerical monoids and affine semigroups

Guadalupe Márquez Campos,
Work with Ignacio Ojeda and José María Tornero

Departamento de Álgebra, Universidad de Sevilla

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- 1 Introduction to Groebner bases.
- 2 Numerical monoids.
- 3 The Groebner correspondence.
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 $\forall f, g \in B$ the remainder of dividing the polynomial $S(f, g)$ (Syzygy) by B is zero.

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- The embedding dimension: $e(S)$ is the minimal number of generators of S .

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Established by the following:

$$\begin{aligned} \mathcal{G} : S &\longrightarrow \overline{E} \cap \{x = 0\} \subset \mathbb{Z}_{\geq 0}^{k+1} \\ m &\longmapsto \exp(N_B(x^m)) \end{aligned}$$

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Lemma

With the previous notation, for all $s \in S$

$$Ap(S, a) = \{s \in S \mid s - a \notin S\}.$$

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1: We define the binomial ideal:

$$I_S = \langle y_1 - x^{a_1}, y_2 - x^{a_2}, y_3 - x^{a_3}, \dots, y_k - x^{a_k} \rangle \subset \mathbb{Q}[x, y_1, \dots, y_k].$$

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$$\begin{pmatrix} 1 & 0 & 0 & \dots & \overbrace{0}^{(j+1)} & \dots & 0 \\ 0 & a_1 & a_2 & \dots & 0 & \dots & a_k \\ 0 & 1 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 1 & \dots & 0 \end{pmatrix}$$

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$$\Delta_{\sigma_j}(S, a_j) := \left\{ N \in \mathbb{Z}_{\geq 0} \mid \exp \left(N_j \left(x^N \right) \right) \in \{x = y_j = 0\} \cap \overline{E(I)} \right\}$$

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Theorem

Under the previous conditions $Ap(S, a_j) = \Delta_{\sigma_j}(S, a_j)$.

Proof.

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there are $x_1, \dots, x_k \in \mathbb{Z}_{\geq 0}$ such that

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$$\exp(N_j(x^n)) \in \{x = 0\} \cap \{y_j = 0\}.$$

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$\implies \exists \alpha_1, \dots, \alpha_k \in \mathbb{Z}_{\geq 0}$ such that

$$n = \sum_{i=1, i \neq j}^k a_i \alpha_i + a_j(\alpha_j + 1) \longrightarrow (\alpha_1, \dots, (\alpha_j + 1), \dots, \alpha_k) \in \mathbb{Z}_{\geq 0}^k$$

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which yields,

$$\sum_{i=1, i \neq j}^k a_i(\alpha_i - \gamma_i) + a_j(\alpha_j + 1) = 0. \quad (*)$$

Proof

So by the definition of σ_j , we know that

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Let $S = \langle 7, 8, 9, 13 \rangle$. We have.

$$I_S = \langle y_1 - x^7, y_2 - x^8, y_3 - x^9, y_4 - x^{13} \rangle \subset \mathbb{Q}[x, y_1, y_2, y_3],$$

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For this monoid we have

$$Ap(S, 13) = \{0, 7, 8, 9, 14, 15, 16, 17, 18, 23, 24, 25, 32\}.$$

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In both cases $Ap(S, 13) = \Delta_{\sigma_j}(S, 13)$.

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$$\prec_\Lambda = \begin{cases} \text{monomial arbitrary order } \prec_x & \text{for } x \\ S\text{-graded reversed lex } \prec_y & \text{for } y \mid y_j \prec_y y_i \end{cases}$$

$\forall j \in \{1, \dots, k-n\} \text{ y } \forall j \in \{k-n+1, \dots, k\}$.

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Noted $\{z_1, \dots, z_n\} \longleftarrow \{y_{k-n+1}, \dots, y_k\}$.

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Algorithm: Affine case.

Let I_S the kernel of the ring homomorphism

$$\begin{aligned}\tilde{\phi} : \mathbb{Q}[x_1, \dots, x_d, y_1, \dots, y_k] &\longrightarrow \mathbb{Q}[x_1, \dots, x_d] \\ y_j &\longmapsto \mathbf{x}^{\mathbf{a}_j} := x_1^{a_{1j}} \cdots x_d^{a_{dj}} \\ x_i &\longmapsto x_i\end{aligned}$$

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Theorem

$$\mathcal{Q}_{\prec_\Lambda}(S) = Ap(S, \Lambda).$$

- 1 Introduction to Groebner bases.
- 2 Numerical monoids.
- 3 The Groebner correspondence.
- 4 An algorithm to the computation of the Apéry set.
- 5 Computation of the type set.

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$$t(S) = \# \bigcup_{n \in S} \left\{ \max_{\leq_S} Ap(S, n) \right\}.$$

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$T(S)$: **Type set.**

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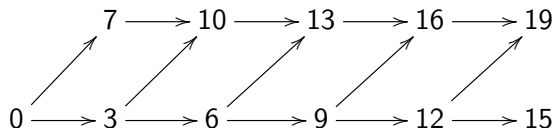
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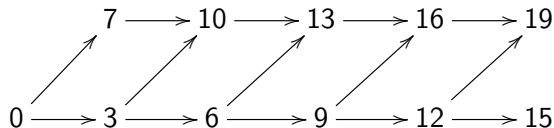


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$$Ap(S, 13) = \{0, 7, 8, 9, 14, 15, 16, 17, 18, 23, 24, 25, 32\}.$$

$$T(S) = \{32\}, \quad S \text{ is symmetrical}$$

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0	0	0	0	1	2	3	4
	y2->	0	0	7	14	21	28
		1	8	15	22	29	36
		2	16	23	30	37	44
		3	24	31	38	45	52
		4	32	39	46	53	60
		5	40	47	54	61	68

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x	y4	y3	y1		
0	0	1	0	1	2
	y2->	0	9	16	23
		1	17	24	31
		2	25	32	39
		3	33	40	47
		4	41	48	55
		5	49	56	63

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	y3	y1		
	2	0	1	2
y2->	0	18	25	32
	1	26	33	40
	2	34	41	48
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	3	42	49	56

$$\partial_{\sigma_1}(S, 13) = \{ 14, 18, 23, 25, 32 \}.$$

Algorithm. Example.

Set $\Delta_{\sigma_3}(S, 13)$:

x	y4	y3	y1				
0	0	0	0	1	2	3	4
	y2->	0	0	7	14	21	28
		1	8	15	22	29	36
		2	16	23	30	37	44
		3	24	31	38	45	52
		4	32	39	46	53	60

x	y4	y3	y1				
0	0	1	0	1	2	3	4
	y2->	0	9	16	23	30	37
		1	17	24	31	38	45
		2	25	32	39	46	53
		3	33	40	47	54	61
		4	41	48	55	62	69

	y3	y1			
	2	0	1	2	3
y2->	0	18	25	32	39
	1	26	33	40	47
	2	34	41	48	55
	3	42	49	56	63

Algorithm. Example.

Set $\Delta_{\sigma_3}(S, 13)$:

x	y4	y3	y1				
0	0	0	0	1	2	3	4
	y2->	0	0	7	14	21	28
		1	8	15	22	29	36
		2	16	23	30	37	44
		3	24	31	38	45	52
		4	32	39	46	53	60

x	y4	y3	y1				
0	0	1	0	1	2	3	4
	y2->	0	9	16	23	30	37
		1	17	24	31	38	45
		2	25	32	39	46	53
		3	33	40	47	54	61
		4	41	48	55	62	69

	y3	y1			
	2	0	1	2	3
y2->	0	18	25	32	39
	1	26	33	40	47
	2	34	41	48	55
	3	42	49	56	63

$$\partial_{\sigma_1}(S, 13) = \{ 24, 32 \}$$

Algorithm. Example.

Set $\Delta_{\sigma_3}(S, 13)$:

x	y4	y3	y1				
0	0	0	0	1	2	3	4
	y2->	0	0	7	14	21	28
		1	8	15	22	29	36
		2	16	23	30	37	44
		3	24	31	38	45	52
		4	32	39	46	53	60

x	y4	y3	y1				
0	0	1	0	1	2	3	4
	y2->	0	9	16	23	30	37
		1	17	24	31	38	45
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		3	33	40	47	54	61
		4	41	48	55	62	69

	y3	y1			
	2	0	1	2	3
y2->	0	18	25	32	39
	1	26	33	40	47
	2	34	41	48	55
	3	42	49	56	63

$$\partial_{\sigma_1}(S, 13) = \{ 24, 32 \} \Rightarrow$$

Algorithm. Example.

Set $\Delta_{\sigma_3}(S, 13)$:

x	y4	y3	y1				
0	0	0	0	1	2	3	4
	y2->	0	0	7	14	21	28
		1	8	15	22	29	36
		2	16	23	30	37	44
		3	24	31	38	45	52
		4	32	39	46	53	60

x	y4	y3	y1				
0	0	1	0	1	2	3	4
	y2->	0	9	16	23	30	37
		1	17	24	31	38	45
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		4	41	48	55	62	69

	y3	y1			
	2	0	1	2	3
y2->	0	18	25	32	39
	1	26	33	40	47
	2	34	41	48	55
	3	42	49	56	63

$$\partial_{\sigma_1}(S, 13) = \{24, 32\} \implies \partial_{\sigma_1}(S, 13) \cap \partial_{\sigma_3}(S, 13) = \{32\}.$$

Thank you very much!
Grazie tante!