On the computation of the Apéry set of numerical monoids and affine semigroups

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Index

Index

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(Fixed an order on the variables.)
A set of polynomials $B = \{f_1, ..., f_s\} \subset I$, is a Groebner basis of I $\Longleftrightarrow$ $\forall f, g \in B$ the remainder of dividing the polynomial $S(f, g)$ (Syzygy) by B is zero.

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- The embedding dimension: $e(S)$ is the minimal number of generators of S.

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$$Ap(S, a) = \{s \in S \mid s - a \notin S\}.$$

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1: We define the binomial ideal:

$$I_S = \langle y_1 - x^{a_1}, y_2 - x^{a_2}, y_3 - x^{a_3}, ..., y_k - x^{a_k} \rangle \subset \mathbb{Q}[x, y_1, ..., y_k].$$

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$$\begin{array}{c}
\begin{array}{ccccccc} & & & & \overbrace{}^{(j+1)} & & \end{array} \\
\left(\begin{array}{ccccccc}
1 & 0 & 0 & \dots & 0 & \dots & 0 \\
0 & a_1 & a_2 & \dots & 0 & \dots & a_k \\
0 & 1 & 0 & \dots & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
0 & 0 & 0 & \dots & 0 & \dots & 1 \\
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$$\Delta_{\sigma_j}(S, a_j) := \left\{ N \in \mathbb{Z}_{\geq 0} \mid exp\left(N_j\left(x^N\right)\right) \in \{x = y_j = 0\} \bigcap \overline{E(I)} \right\}$$

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Theorem

Under the previous conditions $Ap(S, a_j) = \Delta_{\sigma_j}(S, a_j)$.

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And therefore

$$\gamma_j = 0$$

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  $\Longrightarrow \exists \alpha_1, ..., \alpha_k \in \mathbb{Z}_{\geq 0}$ such that

$$n = \sum_{i=1, i \neq j}^{k} a_i \alpha_i + a_j(\alpha_j + 1) \longrightarrow (\alpha_1, ..., (\alpha_j + 1), ..., \alpha_k) \in \mathbb{Z}_{\geq 0}^{k}$$

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which yields,

$$\sum_{i=1, i\neq j}^{k} a_i(\alpha_i - \gamma_i) + a_j(\alpha_j + 1) = 0. \qquad (*)$$

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Let $S = \langle 7, 8, 9, 13 \rangle$. We have.

$$I_S = \left\langle y_1 - x^7, y_2 - x^8, y_3 - x^9, y_4 - x^{13} \right\rangle \subset \mathbb{Q}[x, y_1, y_2, y_3],$$

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For this monoid we have

$$Ap(S, 13) = \{0,\ 7,\ 8,\ 9,\ 14,\ 15,\ 16,\ 17,\ 18,\ 23,\ 24,\ 25,\ 32\}.$$

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In both cases $Ap(S,13) = \Delta_{\sigma_j}(S,13)$.

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$$\prec_\Lambda = \begin{cases} \text{monomial arbitrary order } \ \prec_x & \text{for } x \\ \\ S\text{–graded reversed lex } \ \prec_y & \text{for } y \mid y_j \prec_y y_i \end{cases}$$

$\forall j \in \{1, \dots, k-n\}$ y $\forall j \in \{k-n+1, \dots, k\}$.

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$\mathcal{B}_\Lambda$, the reducided Groebner basis of I_S w.r.t. $\prec_\Lambda$.
N_Λ, for the normal form operator with respect to this basis.
Noted $\{z_1,\ldots,z_n\} \longleftarrow \{y_{k-n+1},\ldots,y_k\}$.
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Theorem

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Index

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$T(S)$: **Type set.**

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Let $S = \langle 7, 8, 9, 13 \rangle$ the previous example.

$$Ap(S, 13) = \{0,\ 7,\ 8,\ 9,\ 14,\ 15,\ 16,\ 17,\ 18,\ 23,\ 24,\ 25,\ 32\}.$$

$$T(S) = \{32\}, \quad S \text{ is symmetrical}$$

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Set $\Delta_{\sigma_1}(S, 13)$:

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Set $\Delta_{\sigma_1}(S,13)$:

x	y4	y3	y1				
0	0	0	0	1	2	3	4
	y2->	0	***0***	***7***	***14***	*21*	*28*
		1	***8***	***15***	*22*	*29*	*36*
		2	***16***	***23***	*30*	*37*	*44*
		3	***24***	*31*	*38*	*45*	*52*
		4	***32***	*39*	*46*	*53*	*60*
		5	*40*	*47*	*54*	*61*	*68*

Algorithm. Example.

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0	0	0	0	1	2	3	4
	y2->	0	***0***	***7***	***14***	*21*	*28*
		1	***8***	***15***	*22*	*29*	*36*
		2	***16***	***23***	*30*	*37*	*44*
		3	***24***	*31*	*38*	*45*	*52*
		4	***32***	*39*	*46*	*53*	*60*
		5	*40*	*47*	*54*	*61*	*68*

x	y4	y3	y1		
0	0	1	0	1	2
	y2->	0	***9***	16	23
		1	***17***	24	31
		2	***25***	32	39
		3	33	40	47
		4	41	48	55
		5	*49*	*56*	*63*

Algorithm. Example.

Set $\Delta_{\sigma_1}(S, 13)$:

x	y4	y3	y1				
0	0	0	0	1	2	3	4
	y2->	0	0	7	14	21	28
		1	8	15	22	29	36
		2	16	23	30	37	44
		3	24	31	38	45	52
		4	32	39	46	53	60
		5	40	47	54	61	68

x	y4	y3	y1		
0	0	1	0	1	2
	y2->	0	9	16	23
		1	17	24	31
		2	25	32	39
		3	33	40	47
		4	41	48	55
		5	49	56	63

	y3	y1		
	2	0	1	2
y2->	0	18	25	32
	1	26	33	40
	2	34	41	48
	3	42	49	56

Algorithm. Example.

Set $\Delta_{\sigma_1}(S, 13)$:

x	y4	y3	y1				
0	0	0	0	1	2	3	4
	y2->	0	0	7	14	21	28
		1	8	15	22	29	36
		2	16	23	30	37	44
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		4	32	39	46	53	60
		5	40	47	54	61	68

x	y4	y3	y1		
0	0	1	0	1	2
	y2->	0	9	16	23
		1	17	24	31
		2	25	32	39
		3	33	40	47
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	y3	y1		
	2	0	1	2
y2->	0	18	25	32
	1	26	33	40
	2	34	41	48
	3	42	49	56

$$\partial_{\sigma_1}(S, 13) = \{\ 14,\ 18,\ 23,\ 25,\ 32\ \}.$$

Algorithm. Example.

Set $\Delta_{\sigma_3}(S, 13)$:

x	y4	y3	y1				
0	0	0	0	1	2	3	4
	y2->	0	***0***	***7***	***14***	21	28
		1	***8***	***15***	22	29	36
		2	16	23	30	37	44
		3	24	31	38	45	52
		4	32	39	46	53	60

x	y4	y3	y1				
0	0	1	0	1	2	3	4
	y2->	0	***9***	***16***	***23***	30	37
		1	***17***	***24***	31	38	45
		2	25	32	39	46	53
		3	33	40	47	54	61
		4	41	48	55	62	69

	y3	y1			
	2	*0*	*1*	*2*	*3*
y2->	*0*	***18***	***25***	***32***	39
	1	26	33	40	47
	2	34	41	48	55
	3	42	49	56	63

Algorithm. Example.

Set $\Delta_{\sigma_3}(S, 13)$:

x	y4	y3	y1				
0	0	0	0	1	2	3	4
	y2->	0	***0***	***7***	***14***	21	28
		1	***8***	***15***	22	29	36
		2	16	23	30	37	44
		3	24	31	38	45	52
		4	32	39	46	53	60

x	y4	y3	y1				
0	0	1	0	1	2	3	4
	y2->	0	***9***	***16***	***23***	30	37
		1	***17***	***24***	31	38	45
		2	25	32	39	46	53
		3	33	40	47	54	61
		4	41	48	55	62	69

	y3	*y1*			
	2	*0*	*1*	*2*	*3*
y2->	*0*	***18***	***25***	***32***	39
	1	26	33	40	47
	2	34	41	48	55
	3	42	49	56	63

$\partial_{\sigma_1}(S, 13) = \{\ 24,\ \ 32\ \}$

Algorithm. Example.

Set $\Delta_{\sigma_3}(S, 13)$:

x	y4	y3	y1				
0	0	0	0	1	2	3	4
	y2->	0	***0***	***7***	***14***	21	28
		1	***8***	***15***	22	29	36
		2	16	23	30	37	44
		3	24	31	38	45	52
		4	32	39	46	53	60

x	y4	y3	y1				
0	0	1	0	1	2	3	4
	y2->	0	***9***	***16***	***23***	30	37
		1	***17***	***24***	31	38	45
		2	25	32	39	46	53
		3	33	40	47	54	61
		4	41	48	55	62	69

	y3	y1			
	2	*0*	*1*	*2*	*3*
y2->	*0*	***18***	***25***	***32***	39
	1	26	33	40	47
	2	34	41	48	55
	3	42	49	56	63

$\partial_{\sigma_1}(S, 13) = \{\ 24,\ \ 32\ \} \Longrightarrow$

Set $\Delta_{\sigma_3}(S,13)$:

x	y4	y3	y1				
0	0	0	0	1	2	3	4
	y2->	0	***0***	***7***	***14***	21	28
		1	***8***	***15***	22	29	36
		2	16	23	30	37	44
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	y3	y1			
	2	*0*	*1*	*2*	*3*
y2->	*0*	***18***	***25***	***32***	39
	1	26	33	40	47
	2	34	41	48	55
	3	42	49	56	63

$$\partial_{\sigma_1}(S,13) = \{\ 24,\ \ 32\ \} \Longrightarrow \partial_{\sigma_1}(S,13) \cap \partial_{\sigma_3}(S,13) = \{32\}.$$

Thank you very much!
Grazie tante!