On the Abhyankar-Moh inequality

Evelia García Barroso

La Laguna University, Tenerife

September, 2014

In this talk we present some results of

- R.D. Barrolleta, E. García Barroso and A. Płoski, *On the Abhyankar-Moh inequality*, arXiv:1407.0176.
- E. García Barroso, J. Gwoździewicz and A. Płoski, *Semigroups corresponding to branches at infinity of coordinate lines in the affine plane,* arXiv:1407.0514.

Introduction

We study semigroups of integers appearing in connection with the Abhyankar-Moh inequality which is the main tool in proving the famous embedding line theorem.

Introduction

We study semigroups of integers appearing in connection with the Abhyankar-Moh inequality which is the main tool in proving the famous embedding line theorem.

Since the Abhyankar-Moh inequality can be stated in terms of semigroups associated with the branch at infinity of a plane algebraic curve it is natural to consider the semigroups for which such an inequality holds.

First definitions

A subset G of $\mathbf{N}$ is a *semigroup* if it contains 0 and it is closed under addition.

Let G be a nonzero semigroup and let $n \in G$, $n > 0$.
There exists a unique sequence $(v_0, \ldots, v_h)$ such that

- $v_0 = n$,
- $v_k = \min(G \backslash v_0\mathbf{N} + \cdots + v_{k-1}\mathbf{N})$ for $1 \leq k \leq h$ and
- $G = v_0\mathbf{N} + \cdots + v_h\mathbf{N}$.

We call the sequence $(v_0, \ldots, v_h)$ the *n-minimal system of generators of G*.

If $n = \min(G \backslash \{0\})$ then we say that $(v_0, \ldots, v_h)$ is the *minimal system of generators of G*.

Characteristic sequences

A sequence of positive integers $(\overline{b}_0, \ldots, \overline{b}_h)$ will be called a *characteristic sequence* if satisfies

- Set $e_k = \gcd(\overline{b}_0, \ldots, \overline{b}_k)$ for $0 \leq k \leq h$. Then $e_k < e_{k-1}$ for $1 \leq k \leq h$ and $e_h = 1$.
- $e_{k-1}\overline{b}_k < e_k\overline{b}_{k+1}$ for $1 \leq k \leq h-1$.

Put $n_k = \frac{e_{k-1}}{e_k}$ for $1 \leq k \leq h$. Therefore $n_k > 1$ for $1 \leq k \leq h$ and $n_h = e_{h-1}$.

Examples

If $h = 0$ there is exactly one characteristic sequence $(\overline{b}_0) = (1)$.
If $h = 1$ then the sequence $(\overline{b}_0, \overline{b}_1)$ is a characteristic sequence if and only if $\gcd(\overline{b}_0, \overline{b}_1) = 1$.

Characteristic sequences

Proposition

Let $G = \overline{b}_0\mathbf{N} + \cdots + \overline{b}_h\mathbf{N}$, *where* $(\overline{b}_0, \ldots, \overline{b}_h)$ *is a characteristic sequence. Then*

1. *the sequence* $(\overline{b}_0, \ldots, \overline{b}_h)$ *is the* $\overline{b}_0$*-minimal system of generators of the semigroup* G.
2. $\min(G\backslash\{0\}) = \min(\overline{b}_0, \overline{b}_1)$.
3. *The minimal system of generators of* G *is* $(\overline{b}_0, \ldots, \overline{b}_h)$ *if* $\overline{b}_0 < \overline{b}_1$, $(\overline{b}_1, \overline{b}_0, \overline{b}_2, \ldots, \overline{b}_h)$ *if* $\overline{b}_0 > \overline{b}_1$ *and* $\overline{b}_0 \not\equiv 0 \pmod{\overline{b}_1}$ *and* $(\overline{b}_1, \overline{b}_2, \ldots, \overline{b}_h)$ *if* $\overline{b}_0 \equiv 0 \pmod{\overline{b}_1}$.
4. *Let* $c = \sum_{k=1}^{h}(n_k - 1)\overline{b}_k - \overline{b}_0 + 1$. *Then* c *is the conductor of* G, *that is the smallest element of* G *such that all integers bigger than or equal to it are in* G.

Abhyankar-Moh semigroups

A semigroup $G \subseteq \mathbf{N}$ will be called an *Abhyankar-Moh semigroup of degree $n > 1$* if it is generated by a characteristic sequence $(\overline{b}_0 = n, \overline{b}_1, \dots, \overline{b}_h)$, satisfying the Abhyankar-Moh inequality

$$\text{(AM)} \qquad e_{h-1}\overline{b}_h < n^2.$$

Abhyankar-Moh semigroups

Let $G \subseteq \mathbf{N}$ be a semigroup generated by a characteristic sequence, which minimal system of generators is $(\overline{\beta}_0, \ldots, \overline{\beta}_g)$.

Proposition

G is an Abhyankar-Moh semigroup of degree $n > 1$ if and only if $\epsilon_{g-1}\overline{\beta}_g < n^2$ and $n = \overline{\beta}_1$ or $n = l\overline{\beta}_0$, where l is an integer such that $1 < l < \overline{\beta}_1/\overline{\beta}_0$ and $\epsilon_{g-1} = \gcd(\overline{\beta}_0, \ldots, \overline{\beta}_{g-1})$.

Abhyankar-Moh semigroups

Theorem (Barrolleta-GB-Płoski)

Let G be an Abhyankar-Moh semigroup of degree $n > 1$ and let c be the conductor of G. Then $c \leq (n-1)(n-2)$.

Moreover if G is generated by the characteristic sequence $(\overline{b}_0 = n, \overline{b}_1, \dots, \overline{b}_h)$ satisfying (AM) then $c = (n-1)(n-2)$ if and only if $\overline{b}_k = \frac{n^2}{e_{k-1}} - e_k$ for $1 \leq k \leq h$, where $e_k = \gcd(\overline{b}_0, \dots, \overline{b}_k)$.

Abhyankar-Moh semigroups

Let $n > 1$ be an integer. A sequence of integers $(e_0, \ldots, e_h)$ will be called a *sequence of divisors of n* if e_k divides e_{k-1} for $1 \leq k \leq h$ and $n = e_0 > e_1 > \cdots > e_{h-1} > e_h = 1$.

Lemma

If $(e_0, \ldots, e_h)$ is a sequence of divisors of $n > 1$ then the sequence

$$\left(n, n - e_1, \frac{n^2}{e_1} - e_2, \ldots, \frac{n^2}{e_{k-1}} - e_k, \ldots, \frac{n^2}{e_{h-1}} - 1\right) \tag{2.1}$$

is a characteristic sequence satisfying the Abhyankar-Moh inequality (AM).

Let $G(e_0, \ldots, e_h)$ be the semigroup generated by the sequence (2.1).

Abhyankar-Moh semigroups

Proposition (Barrolleta-GB-Płoski)

A semigroup $G \subseteq \mathbf{N}$ is an Abhyankar-Moh semigroup of degree $n > 1$ with $c = (n-1)(n-2)$ if and only if $G = G(e_0, \dots, e_h)$ where $(e_0, e_1, \dots, e_h)$ is a sequence of divisors of n.

Abhyankar-Moh semigroups

Corollary

Let G be an Abhyankar-Moh semigroup of degree $n > 1$ with $c = (n-1)(n-2)$ and let $n' = \min(G \setminus \{0\})$. Then $n - n'$ divides n.

Abhyankar-Moh semigroups

Corollary

Let G be an Abhyankar-Moh semigroup of degree $n > 1$ with $c = (n-1)(n-2)$ and let $n' = \min(G \backslash \{0\})$. Then $n - n'$ divides n.

Corollary

Let G be an Abhyankar-Moh semigroup of degree $n > 1$ with $c = (n-1)(n-2)$ and let $(\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g)$ be the minimal system of generators of the semigroup G. Then $n = \overline{\beta}_1$ or $n = 2\overline{\beta}_0$.

If $n = \overline{\beta}_1$ then $G = G(n, \epsilon_1, \ldots, \epsilon_g)$.

If $n = 2\overline{\beta}_0$ then $G = G(n, \epsilon_0, \ldots, \epsilon_g)$.

Plane curves with one branch at infinity

Let **K** be an algebraically closed field of arbitrary characteristic.

A projective plane curve C defined over **K** *has one branch at infinity* if there is a line (line at infinity) intersecting C in only one point O, and C has only one branch centered at this point. In what follows we denote by n the degree of C, by n' the multiplicity of C at O and we put $d := \gcd(n, n')$.

We call C *permissible* if $d \not\equiv 0$ (mod char **K**).

Plane curves with one branch at infinity

Theorem (Abhyankar-Moh inequality)

Assume that C is a permissible curve of degree $n > 1$. Then the semigroup G_O of the unique branch at infinity of C is an Abhyankar-Moh semigroup of degree n.

- Abhyankar, S.S.; Moh, T.T. Embeddings of the line in the plane. *J. reine angew. Math.* **276** (1975), 148-166. (0-characteristic).
- García Barroso, E. R., Płoski, A. An approach to plane algebroid branches. Revista Matemática Complutense (2014). doi: 10.1007/s13163-014-0155-5. First published online: July 29, 2014. (any characteristic).

Plane curves with one branch at infinity

Theorem (Abhyankar-Moh Embedding Line Theorem)

Assume that C is a rational projective irreducible curve of degree $n > 1$ with one branch at infinity and such that the center of the branch at infinity O is the unique singular point of C. Suppose that C is permissible and let n' be the multiplicity of C at O. Then $n - n'$ divides n.

Plane curves with one branch at infinity

Theorem (Abhyankar-Moh Embedding Line Theorem)

Assume that C is a rational projective irreducible curve of degree $n > 1$ with one branch at infinity and such that the center of the branch at infinity O is the unique singular point of C. Suppose that C is permissible and let n' be the multiplicity of C at O. Then $n - n'$ divides n.

Proof [Barrolleta-Gb-Płoski]

By Theorem (Abhyankar-Moh inequality) the semigroup G_O of the branch at infinity is an Abhyankar-Moh semigroup of degree n. Let c be the conductor of the semigroup G_O. Using the Noether formula for the genus of projective plane curve we get $c = (n-1)(n-2)$. Then the theorem follows from Corollary.

Response to Teissier's question on maximal contact

Let $\overline{\beta}_0 = n', \overline{\beta}_1, \cdots$ be the minimal system of generators of the semigroup G_O.

- From the first characterization of A-M semigroups if follows that the line at infinity L has maximal contact with C, that is intersects C with multiplicity $\overline{\beta}_1$ if and only if $n \not\equiv 0 \pmod{n'}$.

Response to Teissier's question on maximal contact

Let $\overline{\beta}_0 = n', \overline{\beta}_1, \cdots$ be the minimal system of generators of the semigroup G_O.

- From the first characterization of A-M semigroups if follows that the line at infinity L has maximal contact with C, that is intersects C with multiplicity $\overline{\beta}_1$ if and only if $n \not\equiv 0 \pmod{n'}$.

  What happens if $n \equiv 0 \pmod{n'}$?

Response to Teissier's question on maximal contact

Let $\overline{\beta}_0 = n', \overline{\beta}_1, \cdots$ be the minimal system of generators of the semigroup G_O.

- From the first characterization of A-M semigroups if follows that the line at infinity L has maximal contact with C, that is intersects C with multiplicity $\overline{\beta}_1$ if and only if $n \not\equiv 0 \pmod{n'}$.

  What happens if $n \equiv 0$ (mod n')?

- Using the main result on the approximate roots in [GB-Płoski] one proves that if $n \equiv 0 \pmod{n'}$) then there is an irreducible curve C' of degree n/n' intersecting C with multiplicity $\overline{\beta}_1$.

Response to Teissier's question on maximal contact

Let $\overline{\beta}_0 = n', \overline{\beta}_1, \cdots$ be the minimal system of generators of the semigroup G_O.

- From the first characterization of A-M semigroups if follows that the line at infinity L has maximal contact with C, that is intersects C with multiplicity $\overline{\beta}_1$ if and only if $n \not\equiv 0 \pmod{n'}$.

  What happens if $n \equiv 0$ (mod n')?

- Using the main result on the approximate roots in [GB-Płoski] one proves that if $n \equiv 0 \pmod{n'}$) then there is an irreducible curve C' of degree n/n' intersecting C with multiplicity $\overline{\beta}_1$.
- In particular, if C is rational then by last Corollary we get $n/n' = 2$ (if $n \equiv 0 \pmod{n'}$) and C' is a nonsingular curve of degree 2.

Geometrically characterization of Abhyankar-Moh semigroups with maximum conductor

An affine curve $\Gamma \subseteq \mathbf{K}^2$ is a *coordinate line* if there is a polynomial automorphism $F : \mathbf{K}^2 \longrightarrow \mathbf{K}^2$ such that $F(\Gamma) = \{0\} \times \mathbf{K}$.

Geometrically characterization of Abhyankar-Moh semigroups with maximum conductor

An affine curve $\Gamma \subseteq \mathbf{K}^2$ is a *coordinate line* if there is a polynomial automorphism $F : \mathbf{K}^2 \longrightarrow \mathbf{K}^2$ such that $F(\Gamma) = \{0\} \times \mathbf{K}$.

Theorem (GB-Gwoździewicz-Płoski)

Let $G \subseteq \mathbf{N}$ be a semigroup with conductor c. Then the following two conditions are equivalent:

(I) *$G \in AM(n)$ and $c = (n-1)(n-2)$,*

(II) *there exists a coordinate line $\Gamma \subseteq \mathbf{K}^2$ ($\mathrm{char}\,\mathbf{K}$ is arbitrary !) with a unique branch at infinity γ such that $G(\gamma) = G$.*